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THE LARGEST LINEAR SPACE OF OPERATORS SATISFYING THE DAUGAVET EQUATION IN L_1

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Abstract. We find the largest linear space of bounded linear operators on $L_1(\Omega)$ that, being restricted to any $L_1(A)$, $A \subset \Omega$, satisfy the Daugavet equation.

1. Introduction

Let (Ω, Σ, μ) be an arbitrary measure space without atoms of infinite measure. Also let $\Sigma^+ = \{A \in \Sigma : \mu(A) > 0\}$. If $A \in \Sigma^+$, $L_1(A)$ stands for the space of (classes of) real-valued μ-integrable functions supported on A. If T is a bounded linear operator on $L_1(\Omega)$ and $A \in \Sigma^+$, we denote by T_A the restriction of T onto $L_1(A)$. Finally, $\mathcal{L}(L_1(\Omega))$ denotes the space of all bounded linear operators on $L_1(\Omega)$.

The purpose of this note is to give an explicit description of the largest linear space $\mathcal{M}$ of operators $T \in \mathcal{L}(L_1(\Omega))$ satisfying the following identity:

$$\|Id_A + T_A\| = 1 + \|T_A\|, \tag{1}$$

for any set $A \in \Sigma^+$.

Originally, (1) was established by Daugavet for compact operators on $C[0,1]$ (see [2]). The case of L_1 was first treated by Lozanovskii in his paper [6], where he proved Daugavet's theorem for compact operators in $L_1[0,1]$ (see also [1]). Later, Holub generalized this result for all weakly compact operators on an arbitrary atomless $L_1(\Omega)$ (see [4]). Plichko and Popov in their work [7] found a still broader (in case of atomless μ) linear class of so-called narrow operators satisfying the Daugavet equation, and in fact their proof works for operators from $L_1(A)$ to $L_1(\Omega)$, whenever $A \in \Sigma^+$.

So, finding the largest class of such operators naturally completes this line of results.

We also refer the reader to papers [5] and [8] for recent developments and applications of the Daugavet theory.

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2. Main result

In the sequel it is convenient to denote $\Sigma_A^+ = \{B : B \subset A,\ B \in \Sigma^+\}$, whenever $A \in \Sigma^+$.

We define $\mathcal{M}$ as the set of all operators $T \in \mathcal{L}(L_1(\Omega))$ that meet the following condition:

$$\begin{aligned}&\text{For every } \varepsilon > 0 \text{ and } A \in \Sigma^+\\ &\text{there is a } B \in \Sigma_A^+ \text{ with } \mu(B) < \infty \text{ such that}\\ &\left\| \chi_B \cdot T\left(\frac{\chi_B}{\mu(B)}\right)\right\| < \varepsilon.\end{aligned} \tag{2}$$

This condition simply means that the operator T can shift sufficiently many functions from their supports.

Let us state our main result.

Theorem 1. *Every linear space of operators satisfying* (1) *for any $A \in \Sigma^+$ is contained in $\mathcal{M}$, and $\mathcal{M}$ itself is a closed linear space consisting of such operators.*

The main ingredient in the proof of this theorem is the following proposition.

Proposition 2. *For an operator $T \in \mathcal{L}(L_1(\Omega))$ the following conditions are equivalent:*

(i) *T and $-T$ satisfy* (1) *for all $A \in \Sigma^+$;*

(ii) *for every $\varepsilon > 0$ and $A \in \Sigma^+$ there is an $A' \in \Sigma_A^+$ such that if $B \in \Sigma_{A'}^+$, then we can find a $B' \in \Sigma_B^+$ with the following properties:*

a) $\left\|\frac{\chi_{B'}}{\mu(B')} - \frac{\chi_B}{\mu(B)}\right\| < \varepsilon$,

b) $\left\|\chi_{B'} \cdot T\left(\frac{\chi_{B'}}{\mu(B')}\right)\right\| < \varepsilon$;

(iii) *$T \in \mathcal{M}$.*

Proof. (i) implies (ii). We begin with the following observation.

Suppose $S : L_1(A) \mapsto L_1(\Omega)$ is a bounded linear operator. Then for any given $\varepsilon > 0$ there is a set $A_1 \in \Sigma_A^+$ with $\mu(A_1) < \infty$ such that for every non-negative function $f \in S(L_1(A_1))$ we have $\|Sf\| \geq \|S\| - \varepsilon$.

Indeed, we can assume that $\mu(A) < \infty$ and choose $g^* \in S(L_1^*(\Omega))$ so that $\|S^*g^*\| > \|S\| - \varepsilon$. Then, regarding S^*g^* as an element of $L_\infty(A)$ we find a set $A_1 \in \Sigma_A^+$ with $\theta S^*g^*(A_1) \subset (\|S\| - \varepsilon, \|S\|]$, where θ is a sign. Now, if $f \in S(L_1(A))$, $f \geq 0$ and $\operatorname{supp}(f) \subset A_1$, then $\|Sf\| > \theta g^*(Sf) = \theta S^*g^*(f) > \|S\| - \varepsilon$, from where the observation follows.

We know that $\|Id_A + T_A\| = 1 + \|T_A\|$. By scaling, without loss of generality we can and do assume that $\|T_A\| = 1$. So there is an $A_1 \in \Sigma_A^+$ with $\mu(A_1) < \infty$ such that

$$\left\|\frac{\chi_B}{\mu(B)} + T\left(\frac{\chi_B}{\mu(B)}\right)\right\| > 2 - \varepsilon, \tag{3}$$

whenever $B \in \Sigma_{A_1}^+$. We also know that $\|Id_{A_1} - T_{A_1}\| = 1 + \|T_{A_1}\| > 2 - \varepsilon$. Thus there exists an $A' \in \Sigma_{A_1}^+$ such that

$$\left\|\frac{\chi_B}{\mu(B)} - T\left(\frac{\chi_B}{\mu(B)}\right)\right\| > 2 - \varepsilon, \tag{4}$$

whenever $B \in \Sigma_{A'}^+$.

We prove that A' is the desired set.

To this end, let us fix $B \in \Sigma^+_{A'}$. It follows from (3), (4) and a theorem of Dor [3] that there are two disjoint measurable sets Ω_1 and Ω_2 in Ω such that

$$\int_{\Omega_1} \left| T\left(\frac{\chi_B}{\mu(B)}\right)\right|(t)d\mu(t) > (1-\varepsilon)^2 \tag{5}$$

and

$$\int_{\Omega_2} \frac{\chi_B}{\mu(B)}(t)d\mu(t) > (1-\varepsilon)^2.$$

The last inequality implies

$$\begin{aligned} \mu(B\cap\Omega_1) &= \mu(B)\int_{B\cap\Omega_1} \frac{\chi_B}{\mu(B)}(t)d\mu(t) < \mu(B)\int_{\Omega\setminus\Omega_2} \frac{\chi_B}{\mu(B)}(t)d\mu(t) \\ &< (1-(1-\varepsilon)^2)\mu(B) = (2\varepsilon-\varepsilon^2)\mu(B). \end{aligned} \tag{6}$$

Let us put $B' = B\setminus\Omega_1$ and show that B' meets conditions a) and b).

First,

$$\begin{aligned} \left\|\frac{\chi_{B'}}{\mu(B')} - \frac{\chi_B}{\mu(B)}\right\| &= \int_\Omega \left|\frac{\chi_{B'}}{\mu(B')} - \frac{\chi_{B'}}{\mu(B)} + \frac{\chi_{B'}}{\mu(B)} - \frac{\chi_B}{\mu(B)}\right|(t)d\mu(t) \\ &\le 1 - \frac{\mu(B')}{\mu(B)} + \frac{\mu(B\cap\Omega_1)}{\mu(B)} = 2\frac{\mu(B\cap\Omega_1)}{\mu(B)}, \end{aligned}$$

and taking into account (6), we obtain

$$\left\|\frac{\chi_{B'}}{\mu(B')} - \frac{\chi_B}{\mu(B)}\right\| < 2(2\varepsilon-\varepsilon^2). \tag{7}$$

Second, from (5), (7) and $\|T_A\| = 1$ it follows that

$$\begin{aligned} \left\|\chi_{B'}\cdot T\left(\frac{\chi_{B'}}{\mu(B')}\right)\right\| &= \int_{B'} \left|T\left(\frac{\chi_{B'}}{\mu(B')}\right)\right|(t)d\mu(t) \\ &< \int_{B'} \left|T\left(\frac{\chi_B}{\mu(B)}\right)\right|(t)d\mu(t) + 2(2\varepsilon-\varepsilon^2) \\ &\le \int_{\Omega\setminus\Omega_1} \left|T\left(\frac{\chi_B}{\mu(B)}\right)\right|(t)d\mu(t) + 2(2\varepsilon-\varepsilon^2) \\ &\le 3(2\varepsilon-\varepsilon^2). \end{aligned}$$

In view of the arbitrariness of ε, this gives the desired result.

It is obvious that (iii) follows from (ii).

Let us finally prove that (iii) implies (i). Since $\mathcal{M}$ is stable under scalar multiplication, it is sufficient to prove (1) only for T.

To this end, we fix an arbitrary $A \in \Sigma^+$ and as above for any given $\varepsilon > 0$ we find an $A' \in \Sigma^+_A$ with $\mu(A') < \infty$ such that for every $B \in \Sigma^+_{A'}$, $\left\|T\left(\frac{\chi_B}{\mu(B)}\right)\right\| > \|T_A\| - \varepsilon$. By condition (2), there is a $B_0 \in \Sigma^+_{A'}$ such that $\left\|\chi_{B_0}\cdot T\left(\frac{\chi_{B_0}}{\mu(B_0)}\right)\right\| < \varepsilon$. This means that $\frac{\chi_{B_0}}{\mu(B_0)}$ and $T\left(\frac{\chi_{B_0}}{\mu(B_0)}\right)$ are almost disjoint functions, and as a consequence we have the following estimate:

$$\begin{aligned}\|Id_A + T_A\| &\geq \left\| \frac{\chi_{B_0}}{\mu(B_0)} + T\left(\frac{\chi_{B_0}}{\mu(B_0)}\right) \right\| \\ &= \int_{B_0} \left| \frac{\chi_{B_0}}{\mu(B_0)} + T\left(\frac{\chi_{B_0}}{\mu(B_0)}\right) \right| (t)d\mu(t) + \int_{\Omega} \left| T\left(\frac{\chi_{B_0}}{\mu(B_0)}\right) \right| (t)d\mu(t) \\ &\quad - \int_{B_0} \left| T\left(\frac{\chi_{B_0}}{\mu(B_0)}\right) \right| (t)d\mu(t) \\ &> 1 - \varepsilon + \|T_A\| - \varepsilon - \varepsilon = 1 + \|T_A\| - 3\varepsilon.\end{aligned}$$

This finishes the proof. □

Now we are in a position to prove our main result.

Proof of Theorem 1. Proposition 2 implies that $\mathcal{M}$ consists of operators satisfying (1) for all $A \in \Sigma^+$, and that every linear space of such operators is contained in $\mathcal{M}$. $\mathcal{M}$ is obviously closed and stable under scaling. So, the only thing we have to prove is that if operators U and V belong to $\mathcal{M}$, then their sum belongs to $\mathcal{M}$ too. To show this, we check condition (ii) of Proposition 2 for $U + V$. Further on, we assume that $\|V\| \leq 1$.

Indeed, let $A \in \Sigma^+$ and $\varepsilon > 0$ be arbitrary. Applying Proposition 2 to the operator U we find a set $A' \in \Sigma^+_A$ as in condition (ii). Then, by the same proposition applied to V we find a set $A'' \in \Sigma^+_{A'}$ with the corresponding properties. To show that A'' is the required set, suppose $B \in \Sigma^+_{A''}$. By the choice of A'' there is a $B' \in \Sigma^+_B$ such that

$$\left\| \frac{\chi_{B'}}{\mu(B')} - \frac{\chi_B}{\mu(B)} \right\| < \frac{\varepsilon}{4} \tag{8}$$

and

$$\left\| \chi_{B'} \cdot V\left(\frac{\chi_{B'}}{\mu(B')}\right) \right\| < \frac{\varepsilon}{4}. \tag{9}$$

Since $B' \subset A'$, by the analogous property of A', there is a $B'' \in \Sigma^+_{B'}$ with

$$\left\| \frac{\chi_{B''}}{\mu(B'')} - \frac{\chi_{B'}}{\mu(B')} \right\| < \frac{\varepsilon}{4} \tag{10}$$

and

$$\left\| \chi_{B''} \cdot U\left(\frac{\chi_{B''}}{\mu(B'')}\right) \right\| < \frac{\varepsilon}{2}.$$

From (8) and (10) we get $\left\| \frac{\chi_{B''}}{\mu(B'')} - \frac{\chi_B}{\mu(B)} \right\| < \varepsilon$. So, if we prove that $\left\| \chi_{B''} \cdot V\left(\frac{\chi_{B''}}{\mu(B'')}\right) \right\| < \frac{\varepsilon}{2}$, then $\left\| \chi_{B''} \cdot (V + U)\left(\frac{\chi_{B''}}{\mu(B'')}\right) \right\| < \varepsilon$, and we are done. But this easily follows from (9), (10) and the facts that $\|V\| \leq 1$ and $B'' \subset B'$.

The proof is completed. □

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