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SOLUTIONS TO A CLASS OF SCHRÖDINGER EQUATIONS

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ABSTRACT. We establish existence and multiplicity of solutions to a class of nonlinear Schrödinger equations with, e.g., "atomic" Hamiltonians, via critical point theory.

1. INTRODUCTION AND RESULTS

Consider the Schrödinger equation

$$-i\frac{d\varphi}{dt} = A\varphi + G_\varphi(x, \varphi), \tag{1.1}$$

where A stands for a Schrödinger operator in $L^2(\mathbb{R}^3)$, and $G : \mathbb{R}^3 \times \mathbb{C} \to \mathbb{R}$ a nonlinear coupling with $G(x, 0) = 0$ and $G(x, e^{i\theta t}u) = G(x, u)$. We are interested in existence and multiplicity of solutions of the form

$$\varphi(t, x) = e^{i\lambda t}u(x) \quad \text{with} \quad u \in H^1(\mathbb{R}^3).$$

Substituting it in (1.1) one sees that u satisfies

$$\begin{cases} Au - \lambda u + g(x, u) = 0, \\ u \in H^1(\mathbb{R}^3), \end{cases} \tag{S}$$

where $g(x, u) := G_u(x, u)$. In this paper we deal with the following Schrödinger operators:

- The Schrödinger operator with Rollnik potentials

$$A_1 := -\Delta + V(x),$$

that is, for any $\epsilon > 0$, there is an expression $V = V_1 + V_2 \in \mathcal{R} + (L^\infty)_\epsilon$ where $\mathcal{R}$ is the class of real functions W satisfying

$$\|W\|^2_{\mathcal{R}} := \int_{\mathbb{R}^3\times\mathbb{R}^3} \frac{|W(x)||W(y)|}{|x - y|^2} dxdy < \infty,$$

and $(L^\infty)_\epsilon$ indicates the class of real functions W satisfying $\|W\|_\infty < \epsilon$. Suppose that $V(x) \le -ar^{-2+\epsilon}$ for $r := |x| > R_0$ some R_0 and $a > 0, \epsilon > 0$.

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- The Hamiltonian of the helium atom

$$A_2 := -\Delta_1 - \Delta_2 - \frac{2}{|\mathbf{r}_1|} - \frac{2}{|\mathbf{r}_2|} + \frac{1}{|\mathbf{r}_1 - \mathbf{r}_2|}$$

where $\mathbf{r}_i = (x_1, x_2, x_3) \in \mathbb{R}^3$ and $\Delta_i = \sum_{k=1}^3 \frac{\partial^2}{\partial x_k^2}$.

- The general "atomic" Hamiltonian of the form

$$A_3 := \sum_{i=1}^m \Big(-\frac{\Delta_i}{2\mu_i} - \frac{m}{|\mathbf{r}_i|} \Big) + \sum_{i<j} \Big(\frac{\nabla_i \cdot \nabla_j}{M} + \frac{1}{|\mathbf{r}_i - \mathbf{r}_j|} \Big),$$

a system consisting of the nucleus of mass M and m electrons of masses $\mu_1, \cdots, \mu_m$ after the center of mass motion is removed.

Recall that the operator $A = A_i$ $(i = 1, 2, 3)$ is selfadjoint with $\mathcal{D}(A_i) \subseteq H^2(\mathbb{R}^3)$. Moreover, the bound state energies $\sigma(A_i) \cap [\lambda_0, \lambda_e)$ is an infinite set (cf. [13]), where $\lambda_0 := \inf \sigma(A)$, $\lambda_e := \inf \sigma_{ess}(A)$ ($\sigma(A)$ and $\sigma_{ess}(A)$ denote the spectrum and essential spectrum). Assume first that g satisfies the following superlinear hypotheses:

$(g1)$ $g(x, u) = o(|u|)$ uniformly with respect to x as $u \to 0$;

$(g2)$ there exist $\gamma > 2$, $\delta > 0$ such that $0 < \gamma G(x, u) \le u g(x, u)$ whenever $u \ne 0$ and $G(x, u) \ge \delta$ whenever $|u| = 1$;

$(g3)$ there are $p > 2$ and $c > 0$ such that

$$|g(x, u + v) - g(x, u)| \le c(1 + |u|^{p-2} + |v|^{p-2})|v|.$$

For obtaining multiple solutions we assume g is odd in u, that is,

$(g4)$ $g(x, -u) = -g(x, u)$.

In the following, for any $\lambda \le \lambda_e$, let $\mathcal{N}(\lambda)$ denote the number of all eigenvalues of A which are less than λ and counted in multiplicity.

Theorem 1. (i) *Assume* $(g1) - (g3)$. *Then* (S) *has no nontrivial solutions if* $\lambda \le \lambda_0$, *and has at least two nontrivial solutions if* $\lambda \in (\lambda_0,\ \lambda_e)$.

(ii) *Assume* $(g1) - (g3)$ *and* $(g4)$. *Then* (S) *has at least* $\mathcal{N}(\lambda)$ *pairs of solutions.*

Next consider the case where $g(x, u)$ is sublinear with respect to $u \in \mathbb{R}$. Assume

$(g5)$ $g(x, u)u \ge 0$, and there are $1 < \alpha \le \beta < 2$, $c_1, c_2 > 0$ such that $|g(x, u)| \le c_1 \left(|u|^{\alpha-1} + |u|^{\beta-1} \right)$ and $2G(x, u) - g(x, u)u \ge c_2 |u|^\alpha$.

Theorem 2. (i) *Assume* $(g5)$ *holds. Then* (S) *has no nontrivial solutions if* $\lambda \le \lambda_0$, *and has at least one nontrivial solution if* $\lambda \in (\lambda_0,\ \lambda_e)$.

(ii) *Assume* $(g4) - (g5)$ *hold. Then* (S) *has* $\mathcal{N}(\lambda)$ *pairs of solutions.*

A typical example satisfying the above hypotheses is $g(x, u) = |u|^{p-2}u$. With this nonlinearity, the above results conclude that the problem (S) has $\mathcal{N}(\lambda)$ pairs of solutions provided $p \in (1,\ 2) \cup (2,\ \infty)$ and $\lambda \in (\lambda_0, \lambda_e)$. Clearly, if $p = 2$, (S) has solutions if and only if $\lambda - 1$ is an eigenvalue of A.

The problems looking for stationary solutions to Schrödinger equations have been studied extensively via variational methods. We refer, e.g., to [10] for Hartree-Fock equations, [5] for Schrödinger operators of atoms and molecules, and [1], [3], [4], [6], [8], [12], [14] for the Schödinger operators with continuous (periodic etc.) potentials. We also mention that a similar problem of semilinear elliptic equations on bounded domains was studied in [2], [7]. The arguments used below extend to higher dimensions and more operators.

2. The superlinear case

Fix $\lambda < \lambda_e$ and set $A_\lambda := A - \lambda$. Let E^+ be the subspace of $L^2(\mathbb{R}^3)$ spanned by all eigenfunctions corresponding to negative eigenvalues of A_λ, and let $E^0 := \ker A_\lambda$. Then $\dim E^+ = \mathcal{N}(\lambda)$ and $L^2 = E^+ \oplus E^0 \oplus L^-, u = u^+ + u^0 + u^-$, where L^- denotes the orthogonal complement of $E^+ \oplus E^0$ in L^2. Let $E := \mathcal{D}(|A_\lambda|^{1/2})$ be the Hilbert space with the inner product

$$(u, v) := (|A_\lambda|^{1/2}u, |A_\lambda|^{1/2}v)_{L^2} + (u^0, v^0)_{L^2}$$

and norm $\|u\| = (u, u)^{1/2}$, where $(\cdot, \cdot)_{L^2}$ denotes the usual L^2-inner product. There is a decomposition

$$E = E^+ \oplus E^0 \oplus E^-, \quad u = u^+ + u^0 + u^-$$

with $E^- = L^- \cap E$, orthogonal with respect to both $(\cdot, \cdot)_{L^2}$ and $(\cdot, \cdot)$. Since $\mathcal{D}(A) \subseteq H^2(\mathbb{R}^3)$, E is embedded continuously in $H^1(\mathbb{R}^3)$, hence continuously in $L^t(\mathbb{R}^3)$ for all $t \in [2, 6]$ and compactly in $L^t_{loc}(\mathbb{R}^3)$ for all $t \in [1, 6)$.

Now, let E_p be the Banach space of the completion of $\mathcal{C}_0^\infty(\mathbb{R}^3)$ in the norm $\|u\|_p = \left(\|u\|^2 + |u|_p^2\right)^{1/2}$. Here and below by $|\cdot|_p$ we denote the usual L^p-norm. By the Sobolev embedding theorem $E_p = E$ if $p \in [2, 6]$. In general, E_p embeds continuously in L^s for all $s \in [p, 6]$ (resp. $s \in [2, p]$) if $p \in (1, 6)$ (resp. $p \geq 6$). It is clear that E_p possesses the direct sum decomposition

$$E_p = E^+ \oplus E^0 \oplus E_p^-, \quad u = u^+ + u^0 + u^-$$

where E_p^- is the completion of $\mathcal{C}_0^\infty(\mathbb{R}^3) \cap L^-$ under the norm $\|\cdot\|_p$. Throughout this section we assume $(g1) - (g3)$ and fix E_p with $p > 2$ from $(g3)$. Define

$$I(u) = I_\lambda(u) := \frac{1}{2}\left(\|u^+\|^2 - \|u^-\|^2\right) - \int_{\mathbb{R}^3} G(x, u).$$

Under the assumptions, $I \in \mathcal{C}^1(E_p, \mathbb{R})$ and solutions of (S) can be found as critical points of I. It follows from the assumptions that, for any $\varepsilon > 0$, there are $C_\varepsilon,\ C'_\varepsilon > 0$ such that

$$C_\varepsilon |u|^\gamma - \varepsilon |u|^2 \leq G(x, u) \leq \varepsilon |u|^2 + C'_\varepsilon |u|^p \quad \text{for all } (x, u). \tag{2.1}$$

Since $\dim(E^+ \oplus E^0) < \infty$ and $\gamma > 2$, the following lemma clearly holds.

Lemma 2.1. *I is bounded from above:* $\sup I(E) < \infty$. *In addition, there is $R > 0$ such that* $\sup I(E_p \backslash B_R) \leq 0$.

Using (2.1) one gets easily the following

Lemma 2.2. *If $\lambda > \lambda_0$, there is $\rho > 0$ such that $\mu := \inf I(S_\rho^+) > 0$ where $S_\rho^+ := \partial B_\rho \cap E^+$.*

Recall that a sequence $(u_n) \subset E_p$ is called a $(\mathrm{PS})_c$ sequence if $I(u_n) \to c$ and $I'(u_n) \to 0$. I is said to satisfy the $(\mathrm{PS})_c$ condition if any $(\mathrm{PS})_c$ sequence has a convergent subsequence.

Lemma 2.3. *I satisfies the* $(\mathrm{PS})_c$ *condition.*

Proof. Let $I(u_n) \to c$ and $I'(u_n) \to 0$. A standard argument shows that (u_n) is bounded (cf. [8] or [4], [15]). We can suppose, without loss of generality, that $u_n \rightharpoonup u$. Since $\dim(E^+ \oplus E^0) < \infty$, $u_n^0 + u_n^+ \to u^0 + u^+$. It remains to show that $u_n^- \to u^-$.

Set $w_n = u_n - u$. Then $w_n \rightharpoonup 0$ and $w_n(x) \to 0$ a.e. in x since E_p embeds compactly in L^2_{loc}. Observe that

$$(I'(u_n) - I'(u))\, w_n = \|w_n^+\|^2 - \|w_n^-\|^2 - \int_{\mathbb{R}^3} (g(x,u_n) - g(x,u))\, w_n,$$

or

$$\|w_n^-\|^2 + \int_{\mathbb{R}^3} (g(x,u_n) - g(x,u))\, w_n = o(1). \tag{2.2}$$

Write

$$\int_{\mathbb{R}^3} (g(x,u_n) - g(x,u))\, w_n = \int_{\mathbb{R}^3} g(x,w_n) w_n + \int_{\mathbb{R}^3} (g(x,w_n+u) - g(x,w_n) - g(x,u))\, w_n.$$

Fix $\varepsilon > 0$. By assumptions it is easy to verify that there is $c_\varepsilon > 0$ satisfying

$$|g(x,w_n+u) - g(x,w_n) - g(x,u)| \le \varepsilon |w_n|^{p-1} + c_\varepsilon \left(|u| + |u|^{p-1}\right).$$

Set $\Omega := \{x \in \mathbb{R}^3 : \ |u(x)| < 1\}, \Omega^c := \mathbb{R}^3 \backslash \Omega$, and

$$f_n^\varepsilon := \left(|g(x,w_n+u) - g(x,w_n) - g(x,u)| - \varepsilon |w_n|^{p-1}\right)^+,$$

where $(a)^+ := \max\{a,\ 0\}$. Then $f_n^\varepsilon \le 2c_\varepsilon |u|$ on Ω and $f_n^\varepsilon \le 2c_\varepsilon |u|^{p-1}$ on Ω^c. The Lebesgue theorem then implies $\int_\Omega (f_n^\varepsilon)^2 \to 0$ and $\int_{\Omega^c} (f_n^\varepsilon)^{p'} \to 0$ where $p' = p/(p-1)$. Therefore,

$$\begin{aligned} &\limsup_{n\to\infty} \int_\Omega |g(x,w_n+u) - g(x,w_n) - g(x,u)|\, |w_n| \\ &\le \limsup_{n\to\infty} \left(\left(\int_\Omega (f_n^\varepsilon)^2 \right)^{1/2} |w_n|_2 + \varepsilon |w_n|_p^p \right) \ \le \ c\varepsilon \end{aligned}$$

and

$$\begin{aligned} &\limsup_{n\to\infty} \int_{\Omega^c} |g(x,w_n+u) - g(x,w_n) - g(x,u)|\, |w_n| \\ &\le \limsup_{n\to\infty} \left(\left(\int_{\Omega^c} (f_n^\varepsilon)^{p'} \right)^{1/p'} |w_n|_p + \varepsilon |w_n|_p^p \right) \ \le \ c\varepsilon, \end{aligned}$$

where $c := \sup_n |w_n|_p^p$. We obtain

$$\int_{\mathbb{R}^3} (g(x,u_n) - g(x,u))\, w_n = \int_{\mathbb{R}^3} g(x,w_n) w_n + o(1).$$

Substituting in (2.2) yields

$$\|w_n^-\|^2 + \int_{\mathbb{R}^3} g(x,w_n) w_n = o(1).$$

Since $g(x,w_n)w_n \ge 0$, it follows that $u_n^- \to u^-$. □

Now we are ready to prove Theorem 1.

Proof of Theorem 1. (i) Assume $(g1), (g2)$ and $(g3)$ hold. Remark that if u is a nontrivial critical point of I, then

$$I(u) = I(u) - \frac{1}{2} I'(u)u \ge \frac{\gamma - 2}{2} \int_{\mathbb{R}^N} G(x,u) > 0. \tag{2.3}$$

If $\lambda \le \lambda_0$, then $\sup I(E) \le 0$, so I has no nontrivial critical points by (2.3).

Let $\lambda \in (\lambda_0, \lambda_e)$. By Lemma 2.1, I is bounded from above. In virtue of Lemma 2.3 I satisfies the $(\mathrm{PS})_c$. In addition, it follows from (2.1) that there exist $r, \varepsilon > 0$ such that

$$\sup I(\partial B_r \cap (E^- \oplus E^0)) \le -\varepsilon < \sup I(E^- \oplus E^0) = 0$$

and

$$0 = \inf I(B_r \cap E^+) < \varepsilon \le \inf I(\partial B_r \cap E^+).$$

Therefore I has at least two nontrivial critical points via the well-known "three solution theorem" (cf. [9]).

(ii) Assume $(g4)$ also holds. Recall that I is even and bounded from above, satisfies the $(\mathrm{PS})_c$, $I(0) = 0$, and $\inf I(\partial B_\rho \cap E^+) \ge \mu$. Now an application of a Clark's theorem (see [11, Theorem 9.1]) to $-I$ yields that I has at least $\mathcal{N}(\lambda)$ distinct pairs of critical points. □

3. The sublinear case

In this section we always assume that $(g5)$ holds. It follows that

$$c_1|u|^\alpha \le G(x,u) \le c_2\left(|u|^\alpha + |u|^\beta\right) \quad \text{for all } (x,u). \tag{3.1}$$

We consider the functional defined on E_α

$$J(u) = J_\lambda(u) := \int_{\mathbb{R}^3} G(x,u) + \frac{1}{2}\|u^-\|^2 - \frac{1}{2}\|u^+\|^2.$$

Then $J \in \mathcal{C}^1(E_\alpha, \mathbb{R})$ and critical points of J are solutions of (S).

If $\lambda \le \lambda_0$, $E^+ = \{0\}$ and clearly J has no nontrivial critical points. Therefore we fix $\lambda \in (\lambda_0, \lambda_e)$ from now on. Note that J is unbounded from either above or below.

Since $\dim(E^+ \oplus E^0) < \infty$ and $\beta < 2$, it follows from (3.1) that

(J_1) There is $R > 0$ such that $\sup J\,(E^+ \backslash B_R) \le 0$;

(J_2) There is $\rho > 0$ such that $\mu := \inf J(S_\rho) > 0$.

Let $\{e_n : n \in \mathbb{N}\}$ be a normal orthogonal base of E_α^-, $Y_n := \operatorname{span}\{e_1, \cdots, e_n\}$ and $E_n := E^+ \oplus E^0 \oplus Y_n$. Note that $\dim E_n < \infty$. Set $J_n := J|_{E_n}$, the restriction of J on E_n. A sequence $(u_n) \subset E_\alpha$, $u_n \in E_n$, is called a $(\mathrm{PS})_c^*$ sequence if $J(u_n) \to c$ and $J_n'(u_n) \to 0$. J is said to satisfy the $(\mathrm{PS})_c^*$ condition if any $(\mathrm{PS})_c^*$ sequence has a convergent subsequence.

Lemma 3.1. *J satisfies the* $(\mathrm{PS})_c^*$ *condition.*

Proof. Let $u_n \in E_n$ be such that $J(u_n) \to c$ and $\varepsilon_n = \|J_n'(u_n)\| \to 0$. It follows from $(g5)$ that

$$J(u_n) + \varepsilon_n\|u_n\|_\alpha \ge \int_{\mathbb{R}^3}\left(G(x,u_n) - \frac{1}{2}g(x,u_n)u_n\right) \ge c|u_n|_\alpha^\alpha,$$

hence

$$|u_n|_\alpha \le c\left(1 + \|u\|_\alpha^{1/\alpha}\right). \tag{3.2}$$

By the Hölder inequality

$$\int_{\mathbb{R}^3} |g(x,u_n)u_n^+| \le c\int_{\mathbb{R}^3}\left(|u_n|^{\alpha-1}+|u_n|^{\beta-1}\right)|u_n^+|$$
$$\le c\left(|u_n|_\alpha^{\alpha-1}|u_n^+|_\alpha + |u_n|_\beta^{\beta-1}|u_n^+|_\beta\right)$$

and so, since $\dim E^+ < \infty$,

$$\int_{\mathbb{R}^3} |g(x,u_n)u_n^+| \le c\left(\|u_n\|_\alpha^{\alpha-1}+\|u_n\|_\alpha^{\beta-1}\right)\|u_n^+\|.$$

Therefore,

$$\|u_n^+\|^2 = \int_{\mathbb{R}^3} g(x,u_n)u_n^+ - I_n'(u_n)u_n^+$$
$$\le c\left(1+\|u_n\|_\alpha^{\beta-1}\right)\|u_n^+\|$$

or

$$\|u_n^+\| \le c\left(1+\|u_n\|_\alpha^{\beta-1}\right). \tag{3.3}$$

Moreover, since

$$\|u_n^-\|^2 = 2I(u_n) + \|u_n^+\|^2 - 2\int_{\mathbb{R}^3} G(x,u_n)$$
$$\le c\left(1+\|u_n\|_\alpha^{2(\beta-1)}\right),$$

one has

$$\|u_n^-\| \le c\left(1+\|u_n\|_\alpha^{\beta-1}\right). \tag{3.4}$$

Hence, in virtue of (3.2)–(3.4), (u_n) is bounded. We can suppose $u_n \rightharpoonup u$ and $u_n^0 + u_n^+ \to u^0 + u^+$.

Let $P_n : E_\alpha \to E_n$ denote the projection. Then

$$J_n'(u_n)(u_n^- - P_n u^-) = \int_{\mathbb{R}^3} g(x,u_n)(u_n^- - P_n u^-) + (u_n^-,\ u_n^- - P_n u^-)$$
$$= \int_{\mathbb{R}^3} g(x,u_n)(u_n - u) + (u_n^-,\ u_n^- - u^-) + o(1)$$

or

$$\int_{\mathbb{R}^3} g(x,u_n)u_n = \int_{\mathbb{R}^3} g(x,u_n)u - \|u_n^-\|^2 + (u_n^-,\ u^-) + o(1).$$

Thus

$$\limsup_{n\to\infty}\int_{\mathbb{R}^3} g(x,u_n)u_n \le \int_{\mathbb{R}^3} g(x,u)u.$$

On the other hand, by a Fatou lemma,

$$\liminf_{n\to\infty}\int_{\mathbb{R}^3} g(x,u_n)u_n \ge \int_{\mathbb{R}^3} g(x,u)u.$$

Thus

$$\int_{\mathbb{R}^3} g(x,u_n)u_n \to \int_{\mathbb{R}^3} g(x,u)u.$$

We obtain $\|u_n^-\| \to \|u^-\|$, which implies $u_n^- \to u^-$. □

Similar to the above arguments one can prove the following lemma.

Lemma 3.2. *J and J_n satisfy the* $(\mathrm{PS})_c$ *condition.*

Proof of Theorem 2. It follows from (J_1) and (J_2) that J has the Mountain-Pass geometry [11], [15]. Since J satisfies the $(\mathrm{PS})_c$ condition for all c, J possesses a critical value $c \geq \mu$, hence conclusion (i) follows.

Next we prove conclusion (ii). We write $\mathrm{gen}(A) \in \mathbb{N} \cup \{0, \infty\}$ for the Krasnoselski genus of a symmetric subset A of E_α, that is, $\mathrm{gen}(A)$ is the least integer k such that there is an odd continuous map $A \to S^{k-1}$. If no such map exists, then $\mathrm{gen}(A) := \infty$. Let $n^0 := \dim E^0$ and set

$$\Gamma_k^n := \{A \subset E_n \backslash \{0\} : \ A \text{ closed, symmetric, } \mathrm{gen}(A) \geq n + n^0 + k\},$$

$$c_k^n := \sup_{A \in \Gamma_k^n} \inf_{u \in A} J(u).$$

If $A \in \Gamma_k^n$, then $A \cap E^+ \neq \emptyset$ (see [11]). Hence

$$c_k^n \leq J(E^+) < \infty.$$

Since $\mathrm{gen}(S_\rho \cap E_n) = n + n^0 + \mathcal{N}(\lambda)$, one has $S_\rho \cap E_n \in \Gamma_k^n$ for $k = 1, \cdots, \mathcal{N}(\lambda)$, and consequently by (J_2)

$$c_k^n \geq \inf \ J(S_\rho \cap E_n) \geq \mu.$$

Since J_n satisfies the $(\mathrm{PS})_c$ condition, c_k^n is a critical value of J_n for $k = 1, \cdots, \mathcal{N}(\lambda)$. Clearly $\Gamma_{k+1}^n \subset \Gamma_k^n$ and one has

$$c_k^n \geq c_{k+1}^n.$$

Therefore, letting $c_k^n \to c_k$ as $n \to \infty$ (along a subsequence), we have

$$c_{\mathcal{N}} \leq \cdots \leq c_1,$$

and, by $(\mathrm{PS})_c^*$, c_k is a critical value of J.

It remains to show that if $c = c_k = \cdots = c_{k+j}$, then $\mathrm{gen}(\mathcal{K}(c)) \geq j + 1$ where $\mathcal{K}(c) := \{u \in E_\alpha : \ J(u) = c, J'(u) = 0\}$. For this goal we note that, since $J(0) = 0$ while $c \geq \mu > 0$, $0 \notin \mathcal{K}(c)$. The continuity of genus implies that there is $\delta > 0$ such that $\mathrm{gen}(N_{3\delta}(\mathcal{K}(c))) = \mathrm{gen}(\mathcal{K}(c))$. Invoking $(\mathrm{PS})_c$ we have $\overline{\varepsilon}, \sigma > 0$ and $\overline{n} \in \mathbb{N}$ satisfying

$$\|J_n'(u)\| \geq \sigma \quad \text{for all } u \in (J_n)_{c-\overline{\varepsilon}}^{c+\overline{\varepsilon}} \setminus N_\delta^n, \ n \geq \overline{n},$$

where $N_\delta^n = N_\delta(\mathcal{K}(c)) \cap E_n$. By a standard deformation argument we choose $\varepsilon > 0$ and construct a sequence of odd homeomorphisms $h_n : E_n \to E_n$ which maps $(J_n)_{c-\varepsilon} \backslash N_\delta^n$ into $(J_n)_{c+\varepsilon}$ (see [15]). Now choose a $n \geq \overline{n}$ large so that

$$c - \varepsilon/2 < c_{k+j}^n \leq \cdots \leq c_k^n < c + \varepsilon/2.$$

Choose $A \in \Gamma_{k+j}^n$ with

$$\inf \ J(A) > c - \varepsilon/2.$$

If $\mathrm{gen}(\mathcal{K}(c)) \leq j$, then

$$\mathrm{gen}(\overline{A \backslash N_\delta^n}) \geq n + n^0 + k + j - \mathrm{gen}(\mathcal{K}(c)) \geq n + n^0 + k,$$

that is, $\overline{A \backslash N_\delta^n} \in \Gamma_k^n$. Therefore $h_n(\overline{A \backslash N_\delta^n}) \in \Gamma_k^n$ and, consequently,

$$c + \varepsilon \leq \inf \ J(h_n(\overline{A \backslash N_\delta^n})) \leq c_k^n \leq c + \varepsilon/2,$$

a contradiction. □

Remark 3.3. There is a three-body system with two-body potentials, $V = V_{01}(x) + V_{02}(y) + V_{12}(x-y)$, so that the number of bound states $E(\nu)$ of $A_\nu := -\Delta_x - \Delta_y + \nu V$ has the following property: There is an increasing sequence $0 < \nu_1 < \nu_2 < \cdots$ so that $E(\nu) = \infty$ if $\nu \in [\nu_1, \nu_2] \cup [\nu_3, \nu_4] \cup \cdots$ and $E(\nu) < \infty$ if $\nu \in \mathcal{T} := [0, \nu_1) \cup (\nu_2, \nu_3) \cup \cdots$; see [13]. All the previous existence and multiplicity results can be established for the equation (S_ν) obtained from (S) with A replaced by A_ν in parallel.

Theorem 3.4. *Let $\nu \in \mathcal{T}$. (a) Assume $(g1)-(g3)$. Then (S_ν) has no nontrivial solutions if $\lambda \le \lambda_0$, has at least two nontrivial solutions if $\lambda \in (\lambda_0,\ \lambda_e)$, and has $\mathcal{N}(\lambda)$ pairs solutions if $(g4)$ also holds. (b) Assume $(g5)$ holds. Then (S_ν) has no nontrivial solutions if $\lambda \le \lambda_0$, has at least one nontrivial solution if $\lambda \in (\lambda_0,\ \lambda_e)$, and has $\mathcal{N}(\lambda)$ pairs solutions if $(g4)$ also holds.*

References

[1] S. Alama and Y. Y. Li, *On "Multibump" bound states for certain semilinear elliptic equations*, Indiana J. Math. **44** (1992), 983–1026. MR **94d:**35044

[2] S. Alama and G. Tarantello, *On the solvability of a semilinear elliptic equation via an associated eigenvalue problem*, Math. Z. **221** (1996), 467–493. MR **97d:**35067

[3] A. Ambrosetti, M. Badiale and S. Cingolani, *Semiclassical states of nonlinear Schrödinger equations*, Arch. Rat. Mech. Anal. **140** (1997), 285–300. MR **98k:**35172

[4] T. Bartsch and Y. H. Ding, *On a nonlinear Schrödinger equation with periodic potential*, Math. Ann. **313** (1999), 15–37. MR **99m:**35218

[5] R. Benguria, H. Brézis and E. H. Lieb, *The Thomas-Fermi-von Weizäcker theory of atoms and molecules*, Comm. Math. Phys. **79** (1981), 167–180. MR **83m:**81114

[6] V. Coti-Zelati and P. H. Rabinowitz, *Homoclinic type solutions for a semilinear elliptic PDE on $\mathbb{R}^N$*, Comm. Pure Appl. Math. **45** (1992), 1217–1269. MR **93k:**35087

[7] M. A. Del Pino and P. Felmer, *Multiple solutions for a semilinear elliptic equation*, Trans. Amer. Math. Soc. **347** (1995), 4839–4853. MR **96c:**35062

[8] W. Kryszewski and A. Szulkin, *Generalized linking theorem with an application to semilinear Schrödinger equation*, Adv. Diff. Eq. **3** (1998), 441–472. CMP 2000:11

[9] S.J. Li and M. Willem, *Applications of local linking to critical point theory*, J. Math. Anal. Appl. **189** (1995), 6–32. MR **96a:**58045

[10] P. L. Lions, *Solutions of Hartree-Fock equations for Coulomb systems*, Comm. Math. Phys. **109** (1987), 33-97. MR **88e:**35170

[11] P.H. Rabinowitz, *Minimax Methods in Critical Point Theory with Applications to Differential Equations*, C. B. M. S. 65, Amer. Math. Soc., Providence, R. I., 1986. MR **87j:**58024

[12] P. H. Rabinowitz, *On a class of nonlinear Schrödinger equations*, ZAMP **43** (1992), 270–291. MR **93h:**35194

[13] M. Reed and B. Simon, *Methods of Mathematical Physics,* vol. I–IV, Academic Press, 1978. MR **58:**12429a; MR **58:**12429b; MR **80m:**81085; MR **58:**12429c

[14] C. Troestler and M. Willem, *Nontrivial solution of a semilinear Schrödinger equation*, Comm. Partial Diff. Eq. **21** (1996), 1431–1449. MR **98i:**35034

[15] M. Willem, *Minimax Methods*, Birkhäuser, Boston, 1996.

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