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METRIC ENTROPY OF CONVEX HULLS IN TYPE p SPACES—THE CRITICAL CASE

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Abstract. Given a precompact subset A of a type p Banach space E, where $p \in (1,2]$, we prove that for every $\beta \in [0,1)$ and all $n \in \mathbb{N}$

$$\sup_{k\leq n} k^{1/p'}(\log k)^{\beta-1} e_k(\operatorname{aco} A) \ \leq\ c \ \sup_{k\leq n} k^{1/p'}(\log k)^{\beta} e_k(A)$$

holds, where $\operatorname{aco} A$ is the absolutely convex hull of A and $e_k(.)$ denotes the k^{th} dyadic entropy number. With this inequality we show in particular that for given A and $\beta \in (-\infty, 1)$ with $e_n(A) \leq n^{-1/p'}(\log n)^{-\beta}$ for all $n \in \mathbb{N}$ the inequality $e_n(\operatorname{aco} A) \leq c\, n^{-1/p'}(\log n)^{-\beta+1}$ holds true for all $n \in \mathbb{N}$. We also prove that this estimate is asymptotically optimal whenever E has no better type than p. For $\beta = 0$ this answers a question raised by Carl, Kyrezi, and Pajor which has been solved up to now only for the Hilbert space case by F. Gao.

1. Introduction and results

In the following, E shall always denote a Banach space and B_E its closed unit ball. For a bounded subset $A \subseteq E$ we define the *entropy numbers* of A to be

$$\varepsilon_n(A) := \inf\Big\{\varepsilon > 0 : \exists\, x_1,\dots,x_n \in A \text{ such that } A \subseteq \bigcup_{i=1}^{n}(x_i + \varepsilon B_E)\Big\}, \quad n \in \mathbb{N}.$$

Alternatively, one can consider the *covering numbers* of A, namely

$$N(\varepsilon, A) := \min\Big\{n \in \mathbb{N} : \exists\, x_1,\dots,x_n \in A \text{ such that } A \subseteq \bigcup_{i=1}^{n}(x_i + \varepsilon B_E)\Big\}, \quad \varepsilon > 0.$$

In the setting of our problems, we prefer to deal with $e_n(A) := \varepsilon_{2^{n-1}}(A)$, resp. $H(\varepsilon, A) := \log N(\varepsilon, A)$. We remember that if A is precompact, so is $\operatorname{aco} A$. Now, it is natural to ask for entropy estimates of $\operatorname{aco} A$ in terms of entropy numbers of A. Among others, the articles [Du], [BP], [CKP], [LL], [St1], [St2] and [Ga] are dedicated mainly to the study of this question with respect to different settings. For instance, asymptotically optimal estimates in type p spaces for the case of polynomially decaying $(e_n(A))$ were proved in [CKP]. Let us recall that a Banach

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space E is said to be of *type* p with $1 \le p \le 2$ iff there is a constant $c > 0$ such that for all $x_1, \dots, x_n \in E$ we have the estimate

$$\int_0^1 \Big\| \sum_{i=1}^n x_i r_i(t) \Big\| \, dt \ \le \ c \, \Big(\sum_{i=1}^n \|x_i\|^p \Big)^{1/p},$$

where (r_i) shall denote the Rademacher functions, i.e. $r_i(t) := \operatorname{sign}(\sin(2^i \pi t))$. The *type* p *constant* $\tau_p(E)$ is the smallest constant c satisfying the above inequality. We say that E is of *optimal type* p iff E is of type p, but of no greater type than p.

For $A \subseteq E$ bounded let us set $\|A\| := \sup_{x \in A} \|x\|$ and define $c_A := \frac{\|A\|}{e_1(A)}$. Then one of the main results of [CKP] reads as:

Theorem 1.1. *Let E be a Banach space of type $p \in (1, 2]$ and $A \subseteq E$ be precompact. Further, assume that there is a positive $\alpha \ne 1/p'$ such that for all $n \in \mathbb{N}$ we have $e_n(A) \le n^{-\alpha}$. Then for all $n \in \mathbb{N}$ it holds that*

$$e_n(\operatorname{aco} A) \ \le \ c_A \ c_{\alpha,p} \cdot \begin{cases} n^{-\alpha}, & \text{if } 0 < \alpha < 1/p', \\ n^{-1/p'} \big(\log(n+1)\big)^{1/p' - \alpha}, & \text{if } \alpha > 1/p', \end{cases}$$

where $c_{\alpha,p}$ depends on α, p and $\tau_p(E)$ only. These estimates are asymptotically optimal for some subsets $A \subset \ell_p$.

It is a strange phenomenon that the asymptotic behaviour of the absolutely convex hull drastically changes when the limit $1/p'$ is crossed. In particular, the asymptotic behaviour in the limit case $\alpha = 1/p'$ is an interesting problem raised in [CKP]. The following result due to Gao (cf. [Ga]) gives an answer to this question for Hilbert spaces.

Theorem 1.2. *Let H be a Hilbert space and $A \subset H$ be precompact. Then $e_n(A) \le n^{-1/2}$ for all $n \in \mathbb{N}$ implies*

$$e_n(\operatorname{aco} A) \ \le \ c \ n^{-1/2} \log(n+1)$$

for all $n \in \mathbb{N}$ and a suitable constant $c > 0$ independent of n. This estimate is asymptotically optimal.

It turns out that the techniques used by Gao also work in the more general setting of type p spaces. With the help of the ideas from [St1], Gao's estimate then becomes an inequality. Our first result is

Theorem 1.3. *Let E be a Banach space of type $p \in (1, 2]$ and $\beta \in [0, 1)$. Then there is a constant $c_{\beta,p} > 0$ such that for every precompact $A \subset E$ and all $n \ge 1$ we have*

$$\sup_{k \le n} k^{1/p'} \big(\log(k+1)\big)^{\beta - 1} e_k(\operatorname{aco} A) \le c_{\beta,p} \ c_A \sup_{k \le \frac{n}{(\log(n+1))^{p'}} + 1} k^{1/p'} \big(\log(k+1)\big)^{\beta} e_k(A),$$

with $c_{\beta,p}$ only depending on p, β and $\tau_p(E)$.

As a trivial consequence we can establish the following estimate.

Corollary 1.4. *Let E be a Banach space of type $p \in (1,2]$, $A \subset E$ be precompact, $\beta \in (-\infty, 1]$ and $f : [1,\infty) \to (0,\infty)$ be a positive and increasing function. Then*

$$e_n(A) \ \leq \ n^{-1/p'} \big(\log(n+1)\big)^{-\beta} f(n)$$

for all $n \in \mathbb{N}$ implies

$$e_n(\operatorname{aco} A) \ \leq \ c\, n^{-1/p'} \big(\log(n+1)\big)^{-\beta+1} f(n)$$

for every $n \in \mathbb{N}$ and a suitable constant $c > 0$ independent of n.

This estimate is asymptotically optimal under natural conditions on E and f as the next theorem shows.

Theorem 1.5. *Let E be an infinite dimensional Banach space of optimal type $p \in (1,2]$ and $\beta \in (-\infty, 1)$. Further, let $g : [1,\infty) \to (0,\infty)$ be an increasing function with $g(2x) \leq c\, g(x)$ for all $x \geq 1$ and a constant $c > 0$ independent of x. We define $f(x) := g(\log(x+3))$. Then there exist a precompact subset $A \subseteq E$ and constants $c_1, c_2 > 0$ such that for all $n \in \mathbb{N}$ we have*

$$e_n(A) \ \leq \ c_1\, n^{-1/p'} \big(\log(n+1)\big)^{-\beta} f(n) \tag{1}$$

and

$$e_n(\operatorname{aco} A) \ \geq \ c_2\, n^{-1/p'} \big(\log(n+1)\big)^{-\beta+1} f(n). \tag{2}$$

For the construction of these subsets with asymptotically optimal behaviour, we need the following two results. The first one is due to Schütt (cf. [Sch]).

Theorem 1.6. *For all $p \in (1,2]$ there exists a constant $c_p > 0$ such that for all integers n and N with $\log N \leq n \leq N$ we have*

$$e_n(\mathrm{id} : \ell_1^N \to \ell_p^N) \ \geq \ c_p \ \left(\frac{\log(N/n+1)}{n}\right)^{1/p'} .$$

The second result is a deep theorem due to Maurey and Pisier (see [MP, Théorème 2.1], or [MS, Theorem 13.1]).

Theorem 1.7. *Let E be an infinite dimensional Banach space of optimal type $p \in (1,2]$. Then for all $n \in \mathbb{N}$ there are subspaces $E_n \subseteq E$ and isomorphisms $T_n : \ell_p^n \to E_n$ with $\|T_n\|\|T_n^{-1}\| \leq 2$.*

2. Proof of the results

First, let us fix some notations and simple facts used in the following. For $A, B \subseteq E$ we write $A + B$ for the Minkowski sum. If A, B are symmetric, it holds that $(\operatorname{aco} A) + (\operatorname{aco} B) = \operatorname{aco}(A+B)$, while this is in general wrong for nonsymmetric sets.

For $x > 0$ we set $\lfloor x \rfloor$ to be the integer part of x. If E is a Banach space of type p and $Y_1, \ldots, Y_n$ are independent E-valued random variables with finite p^{th} moment, the inequality

$$\mathbb{E}\Big\|\sum_{i=1}^n (Y_i - \mathbb{E}Y_i)\Big\| \ \leq \ 4\ \tau_p(E) \ \Big(\sum_{i=1}^n \mathbb{E}\|Y_i\|^p\Big)^{1/p} \tag{3}$$

holds (cf. [MP] and [Ho]).

Due to technical reasons, we prefer to prove a statement similar to Theorem 1.3, only formulated for covering numbers first. For the proof we will combine techniques from [Ga] and [St1].

Proposition 2.1. *Let E be a Banach space of type $p \in (1,2]$ and $\alpha \in [0,p')$. Then there is a constant $c_{\alpha,p} > 0$ depending only on α, p and $\tau_p(E)$ such that for all precompact and symmetric $A \subset B_E$ and every $\varepsilon_0 \in (0,1]$ we have*

$$\sup_{\varepsilon \in [2\varepsilon_0, 2]} H(\varepsilon, \operatorname{aco} A)\ \varepsilon^{p'} (\log(3/\varepsilon))^{\alpha - p'} \ \le\ c_{\alpha,p} \sup_{\varepsilon \in [\varepsilon_0, 1]} H(\varepsilon, A)\ \varepsilon^{p'} (\log(3/\varepsilon))^{\alpha}.$$

Proof. For brevity's sake we first define

$$f(\varepsilon) := \varepsilon^{-p'} \big(\log(3/\varepsilon)\big)^{-\alpha} \quad \text{and} \quad g(\varepsilon) := \varepsilon^{-p'} \big(\log(3/\varepsilon)\big)^{p'-\alpha}.$$

We fix $\varepsilon_0 \in (0,1]$ and set

$$K(\varepsilon_0) := \sup_{\varepsilon \in [\varepsilon_0, 1]} \frac{H(\varepsilon, A)}{f(\varepsilon)}.$$

Since the latter is decreasing in ε_0 it suffices to show that

$$H(2\varepsilon_0, \operatorname{aco} A) \le c\ K(\varepsilon_0)\ g(2\varepsilon_0) \tag{4}$$

for some constant $c > 0$ depending only on α, p and $\tau_p(E)$. We can restrict ourselves to $e_1(A) = 1$ by a rescaling argument. Moreover, $A \subseteq B_E$ implies $H(1, \operatorname{aco} A) = 0$, and hence we have to show assertion (4) only for $\varepsilon_0 \in (0, 1/2]$.

Now, let us fix $n \in \mathbb{N}$ and $\gamma \in (1/2, 1]$ such that $\varepsilon_0 = \gamma\ 2^{-n}$. The definition of $K(\varepsilon_0)$ yields

$$H(\varepsilon, A) \ \le\ K(\varepsilon_0)\ f(\varepsilon) \quad \text{for all } \varepsilon \in [\varepsilon_0, 1/2].$$

In particular there exist $\gamma 2^{-k}$-nets N_k of A with cardinality

$$\big|N_k\big| \le \exp\big(K(\varepsilon_0)\ f(\gamma 2^{-k})\big), \quad k = 1, \dots, n.$$

We define $D_1 := N_1$ and

$$D_k := \{z \in N_k - N_{k-1} : \|z\| \le \gamma 2^{-k+1}\}, \quad k = 2, \dots, n.$$

Especially, we have $\big\|D_k\big\| \le 2^{-k+1}$ for every $k = 1, \dots, n$. Moreover, $\big|D_k\big| \le \big|N_k\big|\ \big|N_{k-1}\big| \le \exp\big(2K(\varepsilon_0)\ f(\gamma 2^{-k})\big)$ holds and hence for $D_k' := D_k \cup (-D_k) \cup \{0\}$ we obtain

$$\big|D_k'\big| \ \le 3\ \big|D_k\big| \ \le 3\ \exp\big(2K(\varepsilon_0)\ f(\gamma 2^{-k})\big). \tag{5}$$

Since $e_1(A) = 1$ implies $N(1/2, A) \ge 2$ we get

$$K(\varepsilon_0) \ \ge\ \frac{H(1/2, A)}{f(1/2)} \ \ge\ 2^{-p'}\ (\log 6)^{\alpha}\ \log 2 \ \ge\ 2^{-2p'} \log 3,$$

which allows us to estimate

$$\log 3 + 2\ K(\varepsilon_0) f(\varepsilon) \ \le\ K(\varepsilon_0)\ \Big(2^{2p'} + 2f(\varepsilon)\Big) \ \le\ 2^{3p'} K(\varepsilon_0)\ f(\varepsilon)$$

for all $\varepsilon \in [\varepsilon_0, 1/2]$. Thus, (5) can be continued to

$$\big|D_k'\big| \ \le\ \exp\big(2^{3p'}\ K(\varepsilon_0)\ f(\gamma 2^{-k})\big), \quad k = 1, \dots, n.$$

Let us now define $C_k := \operatorname{co} D_k' = \operatorname{aco} D_k$ and $E_n := \sum_{k=1}^n C_k$. For $k \ge 2$ and $t_k \in N_k \subseteq A$ there is always a $t_{k-1} \in N_{k-1}$ such that $t_k - t_{k-1} \in D_k$. From this one easily deduces $N_n \subseteq E_n$, so E_n is an ε_0–net of A. Since E_n is absolutely convex, it is even an ε_0-net of $\operatorname{aco} A$. Hence we have

$$H\big(2\varepsilon_0, \operatorname{aco} A\big) \;\le\; H(\varepsilon_0, E_n) \tag{6}$$

by the triangle inequality. We will prove in the following that, for suitable numbers $m_1, \dots, m_n$ which we specify later, the set

$$X := \Big\{\sum_{k=1}^{n} \frac{1}{m_k} \sum_{i=1}^{m_k} d_{k,i} : d_{k,i} \in D'_k\Big\} \tag{7}$$

forms an ε_0–net of E_n. For this we will use an argument which originally goes back to Maurey (cf. [Pis]). We denote by $x_1^k, \dots, x_{d_k}^k$ the elements of $D'_k \backslash \{0\}$. Now we fix $z \in E_n$ and write $z = \sum_{k=1}^n z_k$ with $z_k \in C_k$. Then every z_k can be represented by

$$z_k = \sum_{i=1}^{d_k} a_{k,i} x_i^k, \quad \text{where} \quad a_{k,i} \ge 0 \quad \text{and} \quad \sum_{i=1}^{d_k} a_{k,i} \le 1.$$

Let us define Z_k to be a random vector with values in D'_k, namely

$$\mathbb{P}(Z_k = x_i^k) = a_{k,i} \quad \text{for } i = 1, \dots, d_k, \quad \text{and} \quad \mathbb{P}(Z_k = 0) = 1 - \sum_{i=1}^{d_k} a_{k,i}.$$

Trivially, we obtain $\mathbb{E} Z_k = z_k$. Moreover, we take independent random vectors $Z_{1,1}, \dots, Z_{1,m_1}, \dots, Z_{n,1}, \dots, Z_{n,m_n}$ such that $Z_{k,i}$ is distributed as Z_k for all $k = 1, \dots, n$ and $i = 1, \dots, m_k$. With $Y_{k,i} := \frac{1}{m_k} Z_{k,i}$ and inequality (3) it then yields

$$\begin{aligned} \mathbb{E}\Big\|\sum_{k=1}^{n} z_k - \sum_{k=1}^{n} \frac{1}{m_k} \sum_{i=1}^{m_k} Z_{k,i}\Big\| \;&=\; \mathbb{E}\Big\|\sum_{k=1}^{n}\sum_{i=1}^{m_k} (\mathbb{E} Y_{k,i} - Y_{k,i})\Big\| \\ &\le\; 4\,\tau_p(E)\,\Big(\sum_{k=1}^{n}\sum_{i=1}^{m_k} \mathbb{E}\|Y_{k,i}\|^p\Big)^{1/p} \\ &\le\; 4\,\tau_p(E)\,\left(\sum_{k=1}^{n} \frac{1}{m_k^{p-1}}\; 2^{-p(k-1)}\right)^{1/p} . \end{aligned} \tag{8}$$

Now we want to specify the integers m_k. For this, we first set $\delta := \frac{1}{2}(p' - \alpha)$ and then let $c_{\alpha,p}^{(1)} := \big(4\tau_p(E)\big)^{p'} \big(\delta\,(p-1)\big)^{-p'/p}$. Since $\varepsilon_0 \le 2^{-(n-1)}$ we know that

$$\frac{k^{p'-1-\delta}\; n^{\delta}}{\varepsilon_0^{p'} 2^{p'(k-1)}} \ge\; 1, \quad k = 1, \dots, n,$$

and hence there is an integer m_k with

$$c_{\alpha,p}^{(1)}\; \frac{k^{p'-1-\delta}\; n^{\delta}}{\varepsilon_0^{p'} 2^{p'(k-1)}} \;\le\; m_k \;\le\; 2\; c_{\alpha,p}^{(1)}\; \frac{k^{p'-1-\delta}\; n^{\delta}}{\varepsilon_0^{p'} 2^{p'(k-1)}}.$$

For these m_k's we apply estimate (8) and obtain

$$\mathbb{E}\Big\|z - \sum_{k=1}^{n} \frac{1}{m_k} \sum_{i=1}^{m_k} Z_{k,i}\Big\| \;\le\; \big(\delta\,(p-1)\big)^{1/p}\; \varepsilon_0\; n^{-\delta/p'} \left(\sum_{k=1}^{n} k^{-1+\delta(p-1)}\right)^{1/p} \;\le\; \varepsilon_0.$$

Since $\sum_{k=1}^n \frac{1}{m_k} \sum_{i=1}^{m_k} Z_{k,i}$ only takes values in the set X defined in (7), there is an $x \in X$ such that $\|z - x\| \le \varepsilon_0$. Hence X is an ε_0–net of E_n and moreover, we have

$$\begin{aligned} \log|X| \;\le\; \log\Big(\prod_{k=1}^n |D_k'|^{m_k}\Big) &\le 2^{3p'} K(\varepsilon_0) \sum_{k=1}^n m_k \; f(\gamma 2^{-k}) \\ &\le c_{\alpha,p}^{(2)} \; K(\varepsilon_0) \; \varepsilon_0^{-p'} \; n^\delta \; \sum_{k=1}^n k^{p'-1-\delta-\alpha} \\ &\le c_{\alpha,p}^{(3)} \; K(\varepsilon_0) \; \varepsilon_0^{-p'} \; n^{p'-\alpha} \\ &\le c_{\alpha,p}^{(4)} \; K(\varepsilon_0) \; \varepsilon_0^{-p'} \; (\log(3/\varepsilon_0))^{p'-\alpha} \end{aligned}$$

for suitable constants $c_{\alpha,p}^{(2)}, c_{\alpha,p}^{(3)}$ and $c_{\alpha,p}^{(4)}$ depending only on α, p and $\tau_p(E)$. With inequality (6) this yields (4) and therefore the assertion. □

Although it seems self–evident that Proposition 2.1 is easily translated into Theorem 1.3, one needs to be careful with arising constants.

Proof of Theorem 1.3. Let us first assume that $A \subset E$ is symmetric and without loss of generality we also suppose $e_1(A) = 1$. We use $a_n := \frac{n}{(\log(n+1))^{p'}} + 1$ for short and define

$$C_n \;:=\; 2 \sup_{k \le a_n} k^{1/p'} \; (\log(k+1))^\beta \; e_k(A).$$

With the help of standard arguments, it suffices to show that

$$n^{1/p'} \big(\log(n+1)\big)^{\beta-1} e_{cn}(\operatorname{aco} A) \;\le\; 8 \; C_n \tag{9}$$

for all $n \ge (4p')^{4p'}$ and some $c \in \mathbb{N}$ only depending on β, p and $\tau_p(E)$. Therefore, we fix an arbitrary integer $n \ge (4p')^{4p'}$. Since we have $e_k(A) < C_n \; k^{-1/p'} \; (\log(k+1))^{-\beta}$ for all $1 \le k \le a_n$, we obtain

$$N(C_n \; k^{-1/p'} \; (\log(k+1))^{-\beta}, A) \;\le\; 2^{k-1}, \quad 1 \le k \le a_n.$$

We let

$$\varepsilon_0 \;:=\; C_n \, \lfloor a_n \rfloor^{-1/p'} \; (\log(\lfloor a_n \rfloor + 1))^{-\beta}$$

and first assume $\varepsilon_0 < 1$. Then for all $\varepsilon \in [\varepsilon_0, 1]$ there is an integer k_ε with $2 \le k_\varepsilon \le a_n$ such that

$$C_n \; k_\varepsilon^{-1/p'} (\log(k_\varepsilon + 1))^{-\beta} \;\le\; \varepsilon \;\le\; C_n \; (k_\varepsilon - 1)^{-1/p'} (\log k_\varepsilon)^{-\beta}.$$

Hence we get

$$\begin{aligned} H(\varepsilon, A) &\le H(C_n \; k_\varepsilon^{-1/p'} \; (\log(k_\varepsilon + 1))^{-\beta}, A) \\ &\le k_\varepsilon - 1 \\ &\le C_n^{p'} \; \varepsilon^{-p'} \; (\log k_\varepsilon)^{-\beta p'} \\ &\le 5^{p'} \; C_n^{p'} \; \varepsilon^{-p'} \; (\log(3/\varepsilon))^{-\beta p'} \end{aligned}$$

for all $\varepsilon \in [\varepsilon_0, 1]$. Now Proposition 2.1 provides a constant $c_{\beta,p} > 1$ such that

$$H(2\varepsilon_0, \operatorname{aco} A) \;\le\; c_{\beta,p} \; C_n^{p'} \; \varepsilon_0^{-p'} \; \left(\log \frac{3}{2\varepsilon_0}\right)^{p'(1-\beta)} \;\le\; (cn - 1) \; \log 2 \tag{10}$$

with $c := \lfloor 4^{2+2p'} c_{\beta,p} \rfloor + 2$. On the other hand, if $\varepsilon_0 \geq 1$, we have $H(2\varepsilon_0, \operatorname{aco} A) = 0$, hence estimate (10) holds in this case, too. This leads to

$$e_{cn}(\operatorname{aco} A) \;\leq\; 2\varepsilon_0 \;\leq\; 8\, C_n\, n^{-1/p'}\, (\log(n+1))^{1-\beta}\ ,$$

where the last estimate uses $n \geq (4p')^{4p'}$. Thus we have shown the assertion for symmetric A.

For arbitrary precompact $A \subseteq E$, the set $A' := A \cup (-A)$ is precompact and symmetric. Moreover, we have $\operatorname{aco} A' = \operatorname{aco} A$, $e_1(A') \leq 2\, \|A\|$ and $e_{2k}(A') \leq e_k(A)$ for all $k \geq 1$. Therefore, we obtain

$$\begin{aligned}
\sup_{k \leq a_n} k^{1/p'}\, (\log(k+1))^{\beta}\, e_k(A') \;&\leq\; 2 \max\Big\{ \|A\|, \sup_{2 \leq k \leq a_n} k^{1/p'} (\log(k+1))^{\beta} e_k(A') \Big\} \\
&\leq\; 18\, c_A \sup_{k \leq a_n} k^{1/p'}\, (\log(k+1))^{\beta}\, e_k(A),
\end{aligned}$$

which completes the proof. □

Before we turn to the construction of A in Theorem 1.5, let us prove the following lemma which was essentially obtained in [Ga, proof of Theorem 1]:

Lemma 2.2. *Let $1 \leq p < \infty$ and $E_1, \ldots, E_N$ be Banach spaces. We equip the product space $E_1 \times \cdots \times E_N$ with the p-product norm*

$$\|(x_1, \ldots, x_N)\|_p := \Big(\sum_{i=1}^{N} \|x_i\|^p \Big)^{1/p}.$$

Then for all subsets $A_i \subset E_i$ and every $n \geq 6$ we have

$$N^{1/p} \min_{i \leq N} e_{n+1}(A_i) \;\leq\; 4\; e_{\lfloor \frac{nN}{3} \rfloor}\Big(A_1 \times \cdots \times A_N \Big).$$

Proof. Let $\varrho < \min_{i \leq N} e_{n+1}(A_i)$ be arbitrary. Then there exist sets $S_i \subseteq A_i$ of cardinality $L := 2^n$ such that for all $x_i, y_i \in S_i$ with $x_i \neq y_i$ we have $\|x_i - y_i\| \geq \varrho$. We set

$$S := S_1 \times \cdots \times S_N$$

and $M := \lfloor N/2 \rfloor$. For $x, y \in S$ with $x = (x_i)$ and $y = (y_i)$ we define the Hamming distance

$$h(x,y) := |\{ i : \; x_i \neq y_i \}| \quad \text{and set} \quad B_h(y, M) := \big\{ x \in S : h(x,y) \leq M \big\}.$$

Then for every $y \in S$ we obtain

$$\big| B_h(y, M) \big| \;\leq\; \binom{N}{M} L^M \;\leq\; 2^N L^{N/2} \;\leq\; L^{N/6}\, L^{N/2} \;=\; L^{\frac{2}{3}N}.$$

Thus, for any $k < L^{N/3}$ and $x^1, \ldots, x^k \in S$ we have

$$\Big| \bigcup_{j=1}^{k} B_h(x^j, M) \Big| \;\leq\; \sum_{j=1}^{k} \big| B_h(x^j, M) \big| \;\leq\; k\, L^{\frac{2}{3}N} \;<\; \big| S \big|.$$

Therefore, one can find elements $x^1, \ldots, x^m \in S$ with $m \geq L^{N/3} > 2^{\lfloor nN/3 \rfloor - 1}$ such that for $i \neq j$ we have $h(x^i, x^j) \geq M+1 \geq N/2$. Thus we get $\|x^i - x^j\|_p \geq (N/2)^{1/p} \varrho$ for $i \neq j$ and hence $N^{1/p} \varrho \leq 4\; e_{\lfloor nN/3 \rfloor}(A_1 \times \cdots \times A_N)$. But this finally yields the assertion. □

For the proof of Theorem 1.5, we will combine ideas and techniques from [Ga] and [St2].

Proof of Theorem 1.5. Before we start the proof, we would like to point out that for p, β and f given as in Theorem 1.5, there is a constant $c > 0$ which we keep fixed throughout the proof such that

$$\sum_{k=1}^{\infty} 2^{-k}\ k^{\max\{0,-\beta\}}\ f(2^{p'k})\ \leq\ c$$

as well as $f(xy) \leq cf(x)f(y)$ and $f(x\log x) \leq cf(\sqrt{x})$ for all $x, y \geq 1$.

For the construction of the subset $A \subset E$ we set $\alpha_j := \exp(2^{p'j})$ for $j \geq 0$ and $\beta_k := \lfloor \alpha_{2k} - \alpha_k \rfloor$ for $k \geq 1$. Due to Theorem 1.7, there are isomorphisms

$$T_k : \ell_p^{\beta_k} \to E_k \subseteq E$$

with $\|T_k\| \leq 1$ and $\|T_k^{-1}\| \leq 2$. Let $\mathbf{e}_1, \ldots, \mathbf{e}_{\beta_k}$ denote the standard basis of $\ell_p^{\beta_k}$. Then for $k \in \mathbb{N}$ and $j = k+1, \ldots, 2k$ we define

$$C_j^k := \Big\{ \pm\ 2^{-j} j^{-\beta} f(2^{p'j})\ \mathbf{e}_i\ :\ \alpha_{j-1} - \alpha_k < i \leq \alpha_j - \alpha_k \Big\}$$

and $D_j^k := T_k(C_j^k)$. Moreover, we let $A_k := \sum_{j=k+1}^{2k} D_j^k$. For further calculations we mention that we have

$$\Big|\sum_{j=k+1}^{N} D_j^k\Big|\ \leq\ \prod_{j=k+1}^{N} |D_j^k|\ \leq\ \prod_{j=k+1}^{N} 2\ \alpha_j\ \leq\ \alpha_{N+2} \tag{11}$$

for $k < N \leq 2k$. This implies $|A_k| \leq \alpha_{2k+2}$ in particular. We finally set A to be

$$A := \bigcup_{k=1}^{\infty} A_k,$$

which completes the desired construction.

Now we begin with verifying estimate (1). We fix $n > 1 + \log_2 \alpha_{11}$. Then there is an $N \geq 8$ with $\alpha_{N+3}\ \leq\ 2^{n-1}\ \leq\ \alpha_{N+4}$. We divide A into

$$B_1 := \bigcup_{k=1}^{\lfloor \frac{N-1}{2} \rfloor} A_k, \qquad B_2 := \bigcup_{k=\lfloor \frac{N+1}{2} \rfloor}^{N-2} A_k, \qquad B_3 := \bigcup_{k=N-1}^{\infty} A_k$$

and set $\varepsilon := 2c^2 2^{-N} N^{-\beta} f(2^{p'N})$. Of course, B_1 is an ε–net for itself, and (11) yields

$$\big|B_1\big|\ \leq\ \sum_{k=1}^{\lfloor \frac{N-1}{2} \rfloor} \big|A_k\big|\ \leq\ \sum_{k=1}^{\lfloor \frac{N-1}{2} \rfloor} \alpha_{2k+2}\ \leq\ N\ \alpha_{N+2}.$$

To see that $\{0\}$ is an ε-net of B_3 we estimate

$$\begin{aligned} \|A_k\| &\le \sum_{j=k+1}^{2k} \|D_j^k\| \\ &\le \sum_{j=N}^{\infty} 2^{-j}\; j^{-\beta} f(2^{p'j}) \\ &\le c\; 2^{-N+1}\; N^{-\beta}\; f(2^{p'N}) \Big(\sum_{j=1}^{\infty} 2^{-j}\; j^{\max\{0,-\beta\}} f(2^{p'j})\Big) \\ &\le \varepsilon \end{aligned} \tag{12}$$

for $k \ge N-1$. Finally, for considering B_2 we first investigate A_k for $\lfloor \frac{N+1}{2} \rfloor \le k \le N-2$. Given an $x = \sum_{j=k+1}^{2k} x_j^k \in A_k$ with $x_j^k \in D_j^k$, we define $y := \sum_{j=k+1}^{N} x_j^k \in \sum_{j=k+1}^{N} D_j^k$. Then

$$\|x-y\| \;=\; \Big\| \sum_{j=N+1}^{2k} x_j^k \Big\| \;\le\; \sum_{j=N+1}^{2k} \|D_j^k\| \;\le\; \varepsilon$$

holds similar to estimate (12). Thus, $\sum_{j=k+1}^{N} D_j^k$ is an ε-net of A_k and hence

$$R := \bigcup_{k=\lfloor \frac{N+1}{2} \rfloor}^{N-2} \; \sum_{j=k+1}^{N} D_j^k$$

is an ε-net of B_2. Moreover, (11) ensures that

$$|R| \;\le\; \sum_{k=\lfloor \frac{N+1}{2} \rfloor}^{N-2} \alpha_{N+2} \;<\; N\; \alpha_{N+2}.$$

Summing up our results for B_1, B_2 and B_3, we have shown that $B_1 \cup R \cup \{0\}$ is an ε-net of A of at most $2N\alpha_{N+2}$ points. Since $2N\alpha_{N+2} \le \alpha_{N+3} \le 2^{n-1}$ we conclude

$$e_n(A) \;\le\; \varepsilon \;=\; 2\; c^2\; 2^{-N}\; N^{-\beta}\; f(2^{p'N}) \;\le\; c_{p,\beta,f}\; n^{-1/p'}\; \big(\log(n+1)\big)^{-\beta}\; f(n),$$

where $c_{p,\beta,f} > 0$ is independent of n.

Now we verify estimate (2). Given $n \ge 8 \cdot 2^{16p'}$, we fix an $N \ge 9$ with

$$(N-1) \cdot 2^{2p'(N-1)} \;\le\; 3n \;\le\; N \cdot 2^{2p'N}.$$

We set $m := 2^{\lfloor 2p'N \rfloor + 1}$ and observe that $n \le mN/3$. Thus, with $T_N^{-1}(\operatorname{aco} A_N) = \sum_{j=N+1}^{2N} \operatorname{aco} C_j^N$ and Gao's Lemma we obtain

$$\begin{aligned} e_n(\operatorname{aco} A) \;&\ge\; e_{\lfloor \frac{mN}{3} \rfloor}(\operatorname{aco} A_N) \\ &\ge\; \frac{1}{2}\; e_{\lfloor \frac{mN}{3} \rfloor}\big(\operatorname{aco} C_{N+1}^N \times \cdots \times \operatorname{aco} C_{2N}^N\big) \\ &\ge\; \frac{1}{8}\; N^{1/p} \min_{N \le j \le 2N} e_{m+1}(\operatorname{aco} C_j^N). \end{aligned} \tag{13}$$

To continue this estimate we first note that for j with $N \leq j \leq 2N$ we have

$$2^{p'j} \leq m+1 \leq \exp(2^{p'(j-2)-1}), \tag{14}$$

where the right inequality follows from $N \geq 9$. Moreover, for $d_j := \dim\big(\text{span } C_j^N\big)$ we know $\alpha_{j-2} \leq d_j \leq \alpha_j$ and with (14) this implies $2^{p'(j-2)} \leq \log d_j \leq m+1 \leq d_j^{1/2}$. Now applying Theorem 1.6 for j with $N \leq j \leq 2N$ we get

$$\begin{aligned} e_{m+1}(\operatorname{aco} C_j^N) &= 2^{-j}\, j^{-\beta}\, f(2^{p'j})\, e_{m+1}\big(\mathrm{id} : \ell_1^{d_j} \to \ell_p^{d_j}\big) \\ &\geq c_p\, 2^{-j}\, j^{-\beta}\, f(2^{p'j}) \left(\frac{\log d_j^{1/2}}{m+1}\right)^{1/p'} \\ &\geq \frac{1}{8}\, c_p\, N^{-\beta}\, f(2^{p'N})\, m^{-1/p'}. \end{aligned}$$

Therefore, we can continue (13) by

$$\begin{aligned} e_n(\operatorname{aco} A) &\geq \frac{1}{64}\, c_p\, N^{1/p-\beta}\, f(2^{p'N})\, m^{-1/p'} \\ &= \frac{1}{64}\, c_p\, (mN)^{-1/p'}\, N^{1-\beta}\, f(2^{p'N}) \\ &\geq c^{(1)}_{p,\beta,f}\, (mN)^{-1/p'}\, (\log(mN))^{1-\beta}\, f(mN) \\ &\geq c^{(2)}_{p,\beta,f}\, n^{-1/p'}\, (\log n)^{1-\beta}\, f(n), \end{aligned}$$

where $c^{(1)}_{p,\beta,f}$ and $c^{(2)}_{p,\beta,f}$ are suitable constants depending only on p, β and f. $\square$

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References

[BP] K. Ball and A. Pajor, *The entropy of convex bodies with "few" extreme points*, London Math. Soc. Lecture Note Ser. **158** (1990), 25-32. MR **93b:**46024

[CKP] B. Carl, I. Kyrezi and A. Pajor, *Metric entropy of convex hulls in Banach spaces*, J. London Math. Soc. **60** (1999), 871-896. MR **2001c:**46019

[Du] R.M. Dudley, *Universal Donsker classes and metric entropy*, Ann. Probab. **15** (1987), 1306-1326. MR **88g:**60081

[Ho] J. Hoffmann-Jørgensen, *Sums of Independent Banach Space Valued Random Variables*, Studia Math. **52** (1974), 159-186. MR **50:**8626

[Ga] F. Gao, *Metric entropy of convex hulls*, preprint.

[LL] W.V. Li and W. Linde, *Metric Entropy of Convex Hulls in Hilbert spaces*, Studia Math. **139** (2000), 29-45. CMP 2000:14

[MP] B. Maurey and G. Pisier, *Séries de variables aléatoires vectorielles indépendantes et propriétés géométriques des espaces de Banach*, Studia Math. **58** (1976), 45-90. MR **56:**1388

[MS] V.D. Milman and G. Schechtman, *Asymptotic Theory of Finite Dimensional Normed Spaces*, Lecture Notes in Mathematics **1200**, Springer, 1986. MR **87m:**46038

[Pis] G. Pisier, *Remarques sur un résultat non publié de B. Maurey*, "Séminaire d'Analyse Fonctionnelle 1980-1981, Ecole Polytechnique, Centre de Mathématiques, Palaiseau". MR **83h:**46026

[Sch] C. Schütt, *Entropy numbers of diagonal operators between symmetric Banach spaces*, J. Approx. Theory **40** (1984), 121-128. MR **85e:**47029

[St1] I. Steinwart, *Entropy of $C(K)$-valued operators*, J. Approx. Theory, **103** (2000), 302-328. MR **2001a:**47022

[St2] I. Steinwart, *Entropy of $C(K)-$valued operators and some applications*, Dissertation, Jena 2000. `http://www.minet.uni-jena.de/Math-Net/reports/shadows/99-51report.html`

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