

HILBERT TRANSFORM OF $\log|f|$

JAVAD MASHREGHI

(Communicated by Juha Heinonen)

ABSTRACT. There are two general ways to evaluate the Hilbert transform of a function of real variable $u(x)$. We can extend u to a harmonic function in the upper half plane by the Poisson integral formula. Non-tangential limit of its harmonic conjugate exists almost everywhere and is defined to be the Hilbert transform of $u(x)$. There is also a singular integral formula for the Hilbert transform of $u(x)$. It is fairly difficult to directly evaluate the Hilbert transform of $u(x)$. In this paper we give an explicit formula for the Hilbert transform of $\log|f|$, where f is a function in the Cartwright class.

1. INTRODUCTION

Suppose $u(t)$ is a real valued function such that $\int_{-\infty}^{\infty} \frac{|u(t)|}{1+t^2} dt < \infty$. Then the integral

$$\begin{aligned} U(z) + i\tilde{U}(z) &:= \frac{i}{\pi} \int_{-\infty}^{\infty} \left(\frac{1}{z-t} + \frac{t}{1+t^2} \right) u(t) dt \\ &= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\Im z}{|z-t|^2} u(t) dt + i \frac{1}{\pi} \int_{-\infty}^{\infty} \left(\frac{\Re z - t}{|z-t|^2} + \frac{t}{1+t^2} \right) u(t) dt \end{aligned}$$

converges absolutely for $\Im z > 0$ and is an analytic function of z . U is the unique harmonic extension of u to the upper half plane and it is given by a Poisson integral formula. \tilde{U} is the unique harmonic conjugate of U such that $\tilde{U}(i) = 0$. It is well known that U and \tilde{U} have non-tangential limits at almost every $t \in \mathbb{R}$. The non-tangential limit of U is u . The non-tangential limit of \tilde{U} is denoted by \tilde{u} and is called the Hilbert transform of u . \tilde{u} can also be found by the following singular integral [3]:

$$\tilde{u}(t) = \lim_{\varepsilon \rightarrow 0} \int_{|x-t|>\varepsilon} \left(\frac{1}{t-x} + \frac{x}{1+x^2} \right) u(x) dx.$$

It is fairly difficult to evaluate this integral for $u(x) = \log|\sin x|$ or for $u(x) = \log|p(x)|$ where $p(x)$ is a polynomial. In this paper we give an explicit formula for $\log|f|$ where f belongs to a large class of functions which contains $\sin x$ and $p(x)$.

Received by the editors August 2, 2000.

2000 *Mathematics Subject Classification*. Primary 30D20; Secondary 42A50.

Key words and phrases. Entire functions of exponential type, Blaschke products, Hilbert transform.

This work was supported by Institut des sciences mathématiques (Montreal) and a J. W. McConnell McGill Major Fellowship.

In section 2 we discuss the argument of a meromorphic Blaschke product on the real line. The argument is a C^∞ function and it can be given explicitly by a formula. Then in section 3 the Cartwright class is defined, and an important representation theorem in the upper half plane is derived. Section 4 is about the behavior of these functions on the real line. Finally in section 5 we give an explicit formula for $\widehat{\log|f|}$.

2. BLASCHKE PRODUCTS

Let $\{z_n\}$ be a sequence of complex numbers in the upper half plane such that $\lim_{n \rightarrow \infty} |z_n| = \infty$ and $\sum_n \frac{\Im z_n}{|z_n|^2} < \infty$. Let $b_n(z) = \frac{1 - \frac{z}{z_n}}{1 - \frac{\bar{z}}{\bar{z}_n}}$ and put $B(z) = \prod_n b_n(z)$. $B(z)$ is a meromorphic function with zeros at $\{z_n\}$ and poles at $\{\bar{z}_n\}$. At each point $t \in \mathbb{R}$, B is analytic and $|B(t)| = 1$. Therefore, there is a real C^∞ function φ such that $B(t) = e^{i\varphi(t)}$ for every $t \in \mathbb{R}$ [2]. We can find an explicit formula for $\varphi(t)$:

$$b_n(t) = \frac{1 - \frac{t}{z_n}}{1 - \frac{t}{\bar{z}_n}} = \frac{(1 - \frac{\Re z_n}{|z_n|^2}t) + i(\frac{\Im z_n}{|z_n|^2}t)}{(1 - \frac{\Re z_n}{|z_n|^2}t) - i(\frac{\Im z_n}{|z_n|^2}t)} = \frac{e^{i\varphi_{z_n}(t)}}{e^{-i\varphi_{z_n}(t)}} = e^{2i\varphi_{z_n}(t)}.$$

The first candidate for $\varphi_{z_n}(t)$ is $\arctan\left(\frac{\frac{\Im z_n}{|z_n|^2}t}{1 - \frac{\Re z_n}{|z_n|^2}t}\right)$. If $\Re z_n = 0$, then $\varphi_{z_n}(t) = \arctan\left(\frac{t}{\Im z_n}\right)$ is a well defined C^∞ function on \mathbb{R} . But if $\Re z_n \neq 0$, $\arctan\left(\frac{\frac{\Im z_n}{|z_n|^2}t}{1 - \frac{\Re z_n}{|z_n|^2}t}\right)$ is not continuous at $t = \frac{|z_n|^2}{\Re z_n}$. We should use the following branches of \arctan .

If $\Re z_n > 0$, then

$$\varphi_{z_n}(t) = \begin{cases} \arctan\left(\frac{\frac{\Im z_n}{|z_n|^2}t}{1 - \frac{\Re z_n}{|z_n|^2}t}\right) + \pi, & t > \frac{|z_n|^2}{\Re z_n}, \\ \frac{\pi}{2}, & t = \frac{|z_n|^2}{\Re z_n}, \\ \arctan\left(\frac{\frac{\Im z_n}{|z_n|^2}t}{1 - \frac{\Re z_n}{|z_n|^2}t}\right), & t < \frac{|z_n|^2}{\Re z_n}, \end{cases}$$

and if $\Re z_n < 0$

$$\varphi_{z_n}(t) = \begin{cases} \arctan\left(\frac{\frac{\Im z_n}{|z_n|^2}t}{1 - \frac{\Re z_n}{|z_n|^2}t}\right), & t > \frac{|z_n|^2}{\Re z_n}, \\ \frac{-\pi}{2}, & t = \frac{|z_n|^2}{\Re z_n}, \\ \arctan\left(\frac{\frac{\Im z_n}{|z_n|^2}t}{1 - \frac{\Re z_n}{|z_n|^2}t}\right) - \pi, & t < \frac{|z_n|^2}{\Re z_n}. \end{cases}$$

Since $b_n(0) = 1$, we defined φ_{z_n} such that $\varphi_{z_n}(0) = 0$. Therefore

$$(2.1) \quad B(t) = \prod_n b_n(t) = \prod_n e^{2i\varphi_{z_n}(t)} = e^{2i\sum_n \varphi_{z_n}(t)}.$$

Hence $\varphi(t) = 2\sum_n \varphi_{z_n}(t)$. To prove φ is continuous suppose $t \in [a, b]$ where a and b are arbitrary real numbers. Since $\frac{|z_n|^2}{|\Re z_n|} \rightarrow \infty$, there exists N such that for each $n \geq N$ and for each $t \in [a, b]$, $\varphi_{z_n}(t) = \arctan\left(\frac{\frac{\Im z_n}{|z_n|^2}t}{1 - \frac{\Re z_n}{|z_n|^2}t}\right)$. Thus $|\varphi_{z_n}(t)| \leq c \frac{\Im z_n}{|z_n|^2}$ for each $n \geq N$. Therefore by the Weierstrass M-test φ is continuous. We can even prove that φ is a C^∞ function, but continuity is enough in what follows.

3. CARTWRIGHT CLASS

An entire function $f(z)$ is said to be of exponential type if there are constants A and B such that $|f(z)| \leq B e^{A|z|}$ everywhere. $Cart$ denotes the space of entire functions of exponential type which satisfy the boundedness condition

$$\int_{-\infty}^{\infty} \frac{\log^+ |f(x)|}{1+x^2} dx < \infty.$$

Let $\{z_n\}$ denote the sequence of the upper half plane zeros of f . Since $\sum_n \frac{\Im z_n}{|z_n|^2} < \infty$ and $\lim_{n \rightarrow \infty} |z_n| = \infty$, the Blaschke product formed with this sequence $B_+(z) = \prod_n \frac{1-\frac{z}{z_n}}{1-\frac{\bar{z}}{\bar{z}_n}}$ is a well defined meromorphic function. Let

$$h_+[f] = \limsup_{y \rightarrow +\infty} \frac{\log |f(iy)|}{y},$$

$$h_-[f] = \limsup_{y \rightarrow +\infty} \frac{\log |f(-iy)|}{y}.$$

Therefore $O(z) = \frac{e^{-ih_+z} f(z)}{B_+(z)}$ has removable singularities at zeros of f . It is at least an entire function free of zeros in the upper half plane. But since $f \in Cart$, then $O(z) \in Cart$ and for $\Im z > 0$, $O(z) = c \exp\left(\frac{i}{\pi} \int_{-\infty}^{\infty} \left(\frac{1}{z-t} + \frac{t}{1+t^2}\right) \log |f(t)| dt\right)$. c is a constant of absolute value one [1]. Therefore for $\Im z > 0$

(3.1)

$$f(z) = c e^{-ih_+z} B_+(z) \exp\left(\frac{i}{\pi} \int_{-\infty}^{\infty} \left(\frac{1}{z-t} + \frac{t}{1+t^2}\right) \log |f(t)| dt\right).$$

B_+ is analytic at each point of \mathbb{R} and by (2.1) for every $t \in \mathbb{R}$

(3.2)

$$B_+(t) = e^{2i \sum_n \varphi_{z_n}(t)}.$$

4. CARTWRIGHT FUNCTIONS ON THE REAL LINE

Let $f \in Cart$. Since $f \in Cart$, by definition, $\log^+ |f| \in L^1(\frac{dt}{1+t^2})$. If $f \neq 0$, by a deep theorem in function theory [1, 3], $\log^- |f| \in L^1(\frac{dt}{1+t^2})$. Therefore for almost every $t \in \mathbb{R}$ the Hilbert transform $\widetilde{\log |f|}(t)$ exists. Let α and β be two consecutive real zeros of f . Consider an arbitrary closed interval $[a, b] \subset (\alpha, \beta)$. On $[a, b]$, $\log |f(t)|$ is Lipschitz. Thus $\widetilde{\log |f|}(t)$ is at least continuous on $[a, b]$ [4]. Therefore, $\widetilde{\log |f|}(t)$ is continuous on (α, β) . Hence it exists at every $t \in \mathbb{R}$ except at real zeros of f .

Theorem 4.1. *Let $f \in Cart$ and let $\{x_n\}$ denote the sequence of real zeros of f . Then for every $t \in \mathbb{R} \setminus \{x_n\}$*

$$\frac{f(t)}{|f(t)|} = c B_+(t) e^{-ih_+t} e^{i \widetilde{\log |f|}(t)}$$

where c is a constant of modulus one.

Proof. By (3.1), in the upper half plane $\Im z > 0$

$$\begin{aligned} f(z) &= c e^{-ih_+ z} B_+(z) \exp \left(\frac{i}{\pi} \int_{-\infty}^{\infty} \left(\frac{1}{z-t} + \frac{t}{1+t^2} \right) \log |f(t)| dt \right) \\ &= c e^{-ih_+ z} B_+(z) \exp \left(\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\Im z}{|z-t|^2} \log |f(t)| dt \right. \\ &\quad \left. + i \frac{1}{\pi} \int_{-\infty}^{\infty} \left(\frac{\Re z - t}{|z-t|^2} + \frac{t}{1+t^2} \right) \log |f(t)| dt \right). \end{aligned}$$

Take the non-tangential limit of both sides as $z \rightarrow t$. Thus, for every $t \in R \setminus \{x_n\}$

$$f(t) = c e^{-ih_+ t} B_+(t) \exp(\log |f(t)| + i \widetilde{\log |f(t)|}) = c e^{-ih_+ t} B_+(t) |f(t)| e^{i \widetilde{\log |f(t)|}}.$$

□

Let $f^*(z) := \bar{f}(\bar{z})$. Thus

$$(4.1) \quad h_+[f^*] = \limsup_{y \rightarrow +\infty} \frac{\log |f^*(iy)|}{y} = \limsup_{y \rightarrow +\infty} \frac{\log |f(-iy)|}{y} = h_-[f],$$

$$(4.2) \quad f^*(t) = \bar{f}(\bar{t}) = \bar{f}(t), \quad \log |f^*(t)| = \log |f(t)|.$$

Since $f \in \text{Cart}$ by (4.2), $f^* \in \text{Cart}$. The upper half plane zeros of f^* are conjugates of the lower half plane zeros of f , say (\bar{w}_n) . Put $B_-(z) = \prod_n \frac{1 - \frac{z}{\bar{w}_n}}{1 - \frac{z}{w_n}}$. B_- is analytic at each point of real line and by equation (2.1) for every $t \in \mathbb{R}$

$$(4.3) \quad B_-(t) = e^{2i \sum_n \varphi_{\bar{w}_n}(t)}.$$

Corollary 4.2. *Let $f \in \text{Cart}$ and let $\{x_n\}$ denote the sequence of real zeros of f . Then for every $t \in R \setminus \{x_n\}$*

$$\frac{\bar{f}(t)}{|f(t)|} = c B_-(t) e^{-ih_- t} e^{i \widetilde{\log |f(t)|}}$$

where c is a constant of modulus one.

Proof. Apply Theorem 4.1 to $f^*(z)$. □

5. HILBERT TRANSFORM OF $\log |f|$

Let $\{x_n\}$ be a sequence of real numbers such that $\lim_{|n| \rightarrow \infty} |x_n| = \infty$ and $x_n < x_m$ if $n < m$. Let $\{k_n\}$ be a sequence of integers. The distribution function $\nu_{\{x_n\}}(t)$ is constant between x_{n-1} and x_n and at each point x_n jumps up k_n units. The value of $\nu_{\{x_n\}}(t)$ at x_n is not important. Therefore $\nu_{\{x_n\}}(t)$ is defined up to an additive constant. For example, if we let x_n repeat k_n times, we can define $\nu_{\{x_n\}}(t)$ by the following formula:

$$\nu_{\{x_n\}}(t) = \begin{cases} \#x_n \in [0, t], & t \geq 0, \\ (-1) \#x_n \in (t, 0], & t < 0. \end{cases}$$

Lemma 5.1. *Let $f \in \text{Cart}$ and let $\{x_n\}$ denote the sequence of real zeros of f and $\{k_n\}$ be the sequence of orders of f at each zero. Then $\widetilde{\log |f|} + \pi \nu_{\{x_n\}}$ is a continuous function.*

Proof. Let x_{n-1} and x_n be two consecutive real zeros of f . On $[a, b] \subset (x_{n-1}, x_n)$, $\log|f|$ is Lipschitz. Therefore, $\widetilde{\log|f|}$ is at least continuous on $[a, b]$. On the other hand $\nu_{\{x_n\}}$ is constant on $[a, b]$. Hence $\widetilde{\log|f| + \pi\nu_{\{x_n\}}}$ is continuous on $[a, b]$ and thus on (x_{n-1}, x_n) . We can decompose f as $f(z) = (z - x_n)^{k_n} g(z)$, where g is analytic and nonzero on a neighborhood of x_n . Thus $\log|f(t)| = k_n \log|t - x_n| + \log|g(t)|$. According to the above discussion, since g is not zero in an interval around x_n , $\widetilde{\log|g|}$ is continuous in this interval. The Hilbert transform of $k_n \log|t - x_n|$ is a step function jumping down by $k_n \pi$ at x_n . Therefore, $\widetilde{\log|f|}$ is continuous on $(x_n - \delta, x_n) \cup (x_n, x_n + \delta)$ for a small δ , and jumps down by $k_n \pi$ at x_n . On the other hand $\pi\nu_{\{x_n\}}$ is also continuous on this neighborhood and jumps up by $k_n \pi$ at x_n . Hence $\widetilde{\log|f| + \pi\nu_{\{x_n\}}}$ has a removable discontinuity at x_n . \square

Lemma 5.2. *Let $f \in \text{Cart}$. Let $\{x_n\}$, $\{z_n\}$ and $\{w_n\}$ be respectively the sequence of real, upper and lower half plane zeros of f . Then for every $t \in R \setminus \{x_n\}$*

$$\widetilde{\log|f|}(t) \equiv \theta + \left(\frac{h_+ + h_-}{2}\right)t - \sum_n \varphi_{z_n}(t) - \sum_n \varphi_{\bar{w}_n}(t) \pmod{\pi},$$

where θ is a constant.

Proof. By Theorem 4.1 and Corollary 4.2, for every $t \in R \setminus \{x_n\}$ we have

$$c_1 B_+(t) e^{-ih_+ t} e^{i\widetilde{\log|f|}(t)} = c_2 \bar{B}_-(t) e^{ih_- t} e^{-i\widetilde{\log|f|}(t)}.$$

Hence by (3.2) and (4.3)

$$\begin{aligned} \exp(2i\widetilde{\log|f|}(t)) &= c e^{i(h_+ + h_-)t} \frac{\bar{B}_-(t)}{B_+(t)} = c e^{i(h_+ + h_-)t} \frac{e^{-2i\sum_n \varphi_{\bar{w}_n}(t)}}{e^{2i\sum_n \varphi_{z_n}(t)}} \\ &= c e^{i(h_+ + h_-)t} e^{-2i\sum_n \varphi_{\bar{w}_n}(t) - 2i\sum_n \varphi_{z_n}(t)} \\ &= e^{2i\left(\theta + \left(\frac{h_+ + h_-}{2}\right)t - \sum_n \varphi_{\bar{w}_n}(t) - \sum_n \varphi_{z_n}(t)\right)}. \end{aligned}$$

$c = \frac{c_2}{c_1} = e^{2i\theta}$ is a constant of modulus one. Thus for every $t \in R \setminus \{x_n\}$

$$\widetilde{\log|f|}(t) \equiv \theta + \left(\frac{h_+ + h_-}{2}\right)t - \sum_n \varphi_{\bar{w}_n}(t) - \sum_n \varphi_{z_n}(t) \pmod{\pi}.$$

\square

Main Theorem 5.3. *Let $f \in \text{Cart}$. Let $\{x_n\}$, $\{z_n\}$ and $\{w_n\}$ be respectively the sequence of real, upper and lower half plane zeros of f . Then for every $t \in R \setminus \{x_n\}$*

$$\widetilde{\log|f|}(t) = -\pi\nu_{\{x_n\}}(t) + \left(\frac{h_+ + h_-}{2}\right)t - \sum_n \varphi_{z_n}(t) - \sum_n \varphi_{\bar{w}_n}(t) + \theta$$

where θ is a constant.

Proof. $\nu_{\{x_n\}}(t)$ is a step function which jumps up at $\{x_n\}$ by an integer. Thus for every $t \in R \setminus \{x_n\}$, $\pi\nu_{\{x_n\}}(t) \equiv c_1 \pmod{\pi}$, where c_1 is a constant. Hence by Lemma 5.2, for every $t \in R \setminus \{x_n\}$

$$\begin{aligned} \widetilde{\log|f|}(t) + \pi\nu_{\{x_n\}}(t) &\equiv \widetilde{\log|f|}(t) + c_1 \\ &\equiv c_1 + c_2 + \left(\frac{h_+ + h_-}{2}\right)t - \sum_n \varphi_{\bar{w}_n}(t) - \sum_n \varphi_{z_n}(t) \pmod{\pi}. \end{aligned}$$

The right side is a continuous function of t . By Lemma 5.1 the left side is also continuous. Hence there is a constant c_3 such that for every $t \in R \setminus \{x_n\}$

$$\widetilde{\log|f|}(t) + \pi\nu_{\{x_n\}}(t) = c_1 + c_2 + c_3 + \left(\frac{h_+ + h_-}{2}\right)t - \sum_n \varphi_{\bar{w}_n}(t) - \sum_n \varphi_{z_n}(t).$$

□

Corollary 5.4. *Let $f \in \text{Cart}$. Let $\{x_n\}$ be the sequence of real zeros of f . Suppose f has no other zeros. Then for every $t \in R \setminus \{x_n\}$*

$$\widetilde{\log|f|}(t) = \left(\frac{h_+ + h_-}{2}\right)t - \pi\nu_{\{x_n\}}(t) + \theta,$$

where θ is a constant.

Let us consider the function $f(z) = \sin z \in \text{Cart}$. $h_+[f] = h_-[f] = 1$ and f has only simple zeros at $\{n\pi\}_{n \in \mathbb{Z}}$. Hence $\nu_{\{x_n\}}(t) = [\frac{t}{\pi}]$. Therefore, for every $t \in R \setminus \{n\pi\}_{n \in \mathbb{Z}}$

$$\widetilde{\log|\sin x|}(t) = t - \pi\left[\frac{t}{\pi}\right] + \theta.$$

The following result is due to Cartwright and Levinson.

Corollary 5.5. *Let $f \in \text{Cart}$. Suppose f has only real zeros $\{x_n\}$. Then*

$$\lim_{|t| \rightarrow \infty} \frac{\nu_{\{x_n\}}(t)}{t} = \frac{h_+ + h_-}{2\pi}.$$

Proof. By Corollary 5.4 for every $t \in R \setminus \{x_n\}$

$$\frac{\nu_{\{x_n\}}(t)}{t} = \frac{h_+ + h_-}{2\pi} - \frac{\widetilde{\log|f|}(t)}{\pi t} + \frac{\theta}{\pi t}.$$

$\frac{\widetilde{\log|f|}(t)}{t} \rightarrow 0$ as $|t| \rightarrow \infty$ [1].

□

REFERENCES

- [1] Paul Koosis, *The Logarithmic Integral I*, Cambridge Studies in Advanced Mathematics 12, 1988. MR **90a**:30097
- [2] John B. Conway, *Functions of one complex variable*, Second Edition, Springer-Verlag, 1978. MR **80c**:30003
- [3] A. Zygmund, *Trigonometric Series*, Cambridge, 1968. MR **58**:29731
- [4] R. Boas, *Entire Functions*, Academic Press, 1954. MR **16**:914f

DEPARTMENT OF MATHEMATICS AND STATISTICS, MCGILL UNIVERSITY, MONTREAL, CANADA H3A 2K6

E-mail address: mashregh@math.mcgill.ca