

## INVARIANT IDEALS OF ABELIAN GROUP ALGEBRAS UNDER THE MULTIPLICATIVE ACTION OF A FIELD. I

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ABSTRACT. Let  $D$  be a division ring and let  $V = D^n$  be a finite-dimensional  $D$ -vector space, viewed multiplicatively. If  $G = D^\bullet$  is the multiplicative group of  $D$ , then  $G$  acts on  $V$  and hence on any group algebra  $K[V]$ . Our goal is to completely describe the semiprime  $G$ -stable ideals of  $K[V]$ . As it turns out, this result follows fairly easily from the corresponding results for the field of rational numbers (due to Brookes and Evans) and for infinite locally-finite fields. Part I of this work is concerned with the latter situation, while Part II deals with arbitrary division rings.

### INTRODUCTION I

In a long series of papers (see [Z]), the second author studied the ideal structure of various complex group algebras  $\mathbb{C}[\mathfrak{H}]$ , with  $\mathfrak{H}$  an infinite locally-finite simple group. It now appears that the next family of groups to be considered will have the form  $\mathfrak{H} = V \rtimes \mathfrak{G}$ , where  $V$  is an elementary abelian group and  $\mathfrak{G}$  is an infinite locally-finite “almost” simple group (see [PZ1]). For example,  $\mathfrak{G}$  might be the group  $\mathrm{GL}_n(F)$  where  $F$  is an infinite locally-finite field, and  $V$  could be a suitable finite-dimensional  $F$ -vector space viewed multiplicatively. Note that a field is *locally finite* or *absolute* if every finite subset generates a finite subfield. In other words,  $F$  is locally finite precisely when it is a subfield of the algebraic closure of a finite field. Now  $\mathfrak{G}$  acts on  $V$ , so it acts on  $\mathbb{C}[V]$ . Thus, a necessary first ingredient in the ideal structure of  $\mathbb{C}[\mathfrak{H}]$  is a description of the  $\mathfrak{G}$ -stable ideals of  $\mathbb{C}[V]$ . Indeed, since  $\mathfrak{G}$  might contain an isomorphic copy of  $F^\bullet$ , the multiplicative group of  $F$  acting naturally on the  $F$ -vector space  $V$ , it is appropriate to consider the  $F^\bullet$ -stable ideals of  $\mathbb{C}[V]$ . For this problem, there is really no need to restrict our attention to the complex field. Thus, we let  $K$  be any field, although we usually require that its characteristic be different from that of  $F$ .

Let  $V$  be a multiplicative abelian group and let  $K[V]$  denote its group algebra over the field  $K$ . If  $A$  is a subgroup of  $V$ , then there exists a natural epimorphism  $K[V] \rightarrow K[V/A]$  and we let  $\omega(A; V) = \omega_K(A; V)$ , the *augmentation ideal* of  $A$  in  $V$ , denote its kernel. Thus,  $\omega(A; V)$  is the  $K$ -linear span of all elements of the form  $(1 - a)v$  with  $a \in A$  and  $v \in V$ . If  $G$  is a group which acts as automorphisms on

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$V$ , then  $G$  also acts on  $K[V]$ , and it is clear that  $A$  is a  $G$ -stable subgroup of  $V$  if and only if  $\omega(A; V)$  is a  $G$ -stable ideal of  $K[V]$ . The main result of this paper is

**Theorem A.** *Let  $F$  be an infinite locally-finite field and let  $V = F^n$  be a finite-dimensional  $F$ -vector space, viewed multiplicatively. If  $G = F^\bullet$ , then  $G$  acts on  $V$  and hence on the group algebra  $K[V]$ . Suppose, in addition, that  $\text{char } K \neq \text{char } F$ . Then every  $G$ -stable ideal of  $K[V]$  can be written uniquely as a finite irredundant intersection  $\bigcap_{i=1}^k \omega(A_i; V)$  of augmentation ideals, where each  $A_i$  is an  $F$ -subspace of  $V$ . As a consequence, the set of these  $G$ -stable ideals is Noetherian.*

Note that, if  $V$  is a torsion abelian group having no elements of order equal to the characteristic of  $K$ , then  $K[V]$  is a commutative von Neumann regular algebra (see [P, Theorem 1.1.5]). It follows that if  $I, J \triangleleft K[V]$ , then  $I \cap J = IJ$ . In particular, finite products and finite intersections of ideals coincide here. Furthermore, every ideal of  $K[V]$  is semiprime. In view of this latter comment, it is clear that Theorem A is the locally-finite analog (with some enhancements) of [BE, Proposition 6] which studies finite-dimensional rational vector spaces.

### 1. GENERALITIES

Let  $G$  be a group of operators on the abelian group  $V$ . We start with a simple observation.

**Lemma 1.1.** *Let  $W$  be a subgroup of  $V$ . Then:*

- (i)  $K[W] \cap \omega(A; V) = \omega(W \cap A; W)$ .
- (ii) If  $I = \bigcap_{i \in \mathcal{I}} \omega(A_i; V)$ , then  $K[W] \cap I = \bigcap_{i \in \mathcal{I}} \omega(W \cap A_i; W)$ .

*Proof.* Part (i) follows since the restriction to  $K[W]$  of the map  $K[V] \rightarrow K[V/A]$  is clearly  $K[W] \rightarrow K[W/(W \cap A)]$ . Part (ii) is immediate from (i). □

Now we say that  $\{(V_i, G_i) \mid i = 1, 2, \dots\}$  is a *local system* for  $(V, G)$  if:

- (1)  $V_1 \subseteq V_2 \subseteq \dots \subseteq V_n \subseteq \dots$  and  $\bigcup_1^\infty V_i = V$ .
- (2)  $G_1 \subseteq G_2 \subseteq \dots \subseteq G_n \subseteq \dots$  and  $\bigcup_1^\infty G_i = G$ .
- (3)  $G_i$  stabilizes  $V_i$  and both of these subgroups are finite.

The next lemma shows that the property of being an intersection of augmentation ideals can be lifted from a local system to the entire group.

**Lemma 1.2.** *Let  $\{(V_i, G_i) \mid i = 1, 2, \dots\}$  be a local system for  $(V, G)$ , and let  $I$  be a  $G$ -stable ideal of  $K[V]$ . Suppose that, for each  $i$ , the ideal  $K[V_i] \cap I$  of  $K[V_i]$  is an intersection of augmentation ideals of  $G_i$ -stable subgroups of  $V_i$ . Then  $I$  is an intersection of augmentation ideals of  $G$ -stable subgroups of  $V$ .*

*Proof.* For each  $i$ , let  $\mathbb{A}_i$  be the set of all families  $\mathfrak{A}$  of  $G_i$ -stable subgroups of  $V_i$  such that  $I_i = K[V_i] \cap I = \bigcap_{A \in \mathfrak{A}} \omega(A; V_i)$ . By assumption, each  $\mathbb{A}_i$  is nonempty. Furthermore, since  $V_i$  is a finite group, it is clear that each  $\mathfrak{A}$ , as above, is finite, and so also is each  $\mathbb{A}_i$ . Since

$$I_i = K[V_i] \cap I = K[V_i] \cap (K[V_{i+1}] \cap I) = K[V_i] \cap I_{i+1},$$

Lemma 1.1(ii) implies that there is an intersection map  $\text{Int}_i: \mathbb{A}_{i+1} \rightarrow \mathbb{A}_i$  given by

$$\{A_1, A_2, \dots, A_t\} \mapsto \{V_i \cap A_1, V_i \cap A_2, \dots, V_i \cap A_t\}.$$

Thus  $\mathbb{T} = \{\mathbb{A}_i, \text{Int}_i \mid i = 1, 2, \dots\}$  can be viewed as a tree with  $\mathbb{A}_i$  being the set of elements at level  $i$  and with  $\text{Int}_i$  describing the branches from level  $i + 1$  to level  $i$ .

Since each  $\mathbb{A}_i$  is finite and nonempty, there exists an infinite branch in this tree. In other words, we can choose fixed  $\mathfrak{A}_i \in \mathbb{A}_i$  such that  $\text{Int}_i: \mathfrak{A}_{i+1} \mapsto \mathfrak{A}_i$  for all  $i$ . Say  $\mathfrak{A}_i = \{A_{i,1}, A_{i,2}, \dots, A_{i,t_i}\}$ .

Since  $\text{Int}_i: \mathfrak{A}_{i+1} \mapsto \mathfrak{A}_i$ , there is a second intersection map  $\text{int}_i: \mathfrak{A}_{i+1} \rightarrow \mathfrak{A}_i$  given by  $\text{int}_i: A_{i+1,j} \mapsto V_i \cap A_{i+1,j}$ . Thus  $\mathcal{T} = \{\mathfrak{A}_i, \text{int}_i \mid i = 1, 2, \dots\}$  can be viewed as a tree with  $\mathfrak{A}_i$  being the set of elements at level  $i$  and with  $\text{int}_i$  describing the branches from level  $i + 1$  to level  $i$ . Again, each  $\mathfrak{A}_i$  is finite and nonempty. Furthermore, since  $\text{Int}_i(\mathfrak{A}_{i+1}) = \mathfrak{A}_i$ , this tree has the additional property that each node at level  $i$  is the image of a node at level  $i + 1$ . Equivalently, each  $A_{i,j}$  is equal to  $V_i \cap A_{i+1,j'}$  for some subscript  $j'$ .

Now let  $\mathfrak{B}$  be any full branch of the tree  $\mathcal{T}$ . Then  $\mathfrak{B}$  has nodes

$$A_{1,j_1} \subseteq A_{2,j_2} \subseteq \dots \subseteq A_{n,j_n} \subseteq \dots$$

for suitable subscripts  $j_1, j_2, \dots, j_n, \dots$ , and we let  $B_{\mathfrak{B}} = \bigcup_i A_{i,j_i}$ . We claim that each  $B_{\mathfrak{B}}$  is a  $G$ -stable subgroup of  $V$  and that  $I$  is equal to the ideal  $J = \bigcap_{\mathfrak{B}} \omega(B_{\mathfrak{B}}; V)$ , where the intersection is over all such branches  $\mathfrak{B}$ . This will certainly yield the result.

To start with, it is clear that each  $B_{\mathfrak{B}}$  is a subgroup of  $V$ . Furthermore, for any  $k \geq i$ , we know that  $G_k \supseteq G_i$  and that  $A_{k,j_k}$  is  $G_i$ -stable. Thus  $B_{\mathfrak{B}}$  is  $G_i$ -stable for all  $i$ , and hence it is  $G$ -stable. Next, by definition of the map  $\text{int}_i$ , if  $B_{\mathfrak{B}}$  is defined as above, then  $V_i \cap A_{k,j_k} = A_{i,j_i}$  for all  $k \geq i$  and hence  $V_i \cap B_{\mathfrak{B}} = A_{i,j_i}$ . Thus, by Lemma 1.1(ii) and the fact that each  $A_{i,j}$  is a member of some full branch  $\mathfrak{B}$ , we have

$$J_i = K[V_i] \cap J = \bigcap_{\mathfrak{B}} \omega(V_i \cap B_{\mathfrak{B}}; V_i) = \bigcap_{j=1}^{t_i} \omega(A_{i,j}; V_i) = K[V_i] \cap I = I_i,$$

where the latter uses the fact that  $\{A_{i,1}, A_{i,2}, \dots, A_{i,t_i}\} = \mathfrak{A}_i \in \mathbb{A}_i$ . Since  $V$  is the ascending union of the subgroups  $V_i$ , it now follows from  $J_i = I_i$  that  $J = I$ , and the lemma is proved.  $\square$

As is readily apparent, the preceding argument only shows that  $I$  is an infinite intersection of augmentation ideals. Even so, this turns out to be a fairly powerful conclusion. For example, if  $I$  is properly smaller than  $\omega(V; V)$ , then it follows from the above that  $I \subseteq \omega(A; V)$  for some proper  $G$ -stable subgroup  $A$  of  $V$ . To proceed further, it is necessary to recall some standard notation.

First, a  $G$ -stable ideal  $I$  of  $K[V]$  is said to be  $G$ -prime if the inclusion  $J_1 J_2 \subseteq I$  implies that one of the  $G$ -stable ideals  $J_1$  or  $J_2$  is contained in  $I$ . Second,  $K[V]$  is a  $G$ -prime algebra if  $I = 0$  is a  $G$ -prime ideal. In other words,  $K[V]$  is  $G$ -prime if and only if the product of any two nonzero  $G$ -stable ideals is again nonzero. Finally,  $B/A$  is a  $G$ -section of  $V$  if  $A \subset B$  are any two distinct  $G$ -stable subgroups of  $V$ .

**Lemma 1.3.** *Suppose that all  $G$ -sections of  $V$  are infinite. Then:*

- (i)  $K[V]$  is a  $G$ -prime algebra.
- (ii) If  $A$  is a  $G$ -stable subgroup of  $V$ , then  $\omega(A; V)$  is a  $G$ -prime ideal of  $K[V]$ .

*Proof.* (i) Form the group  $H = V \rtimes G$ , and suppose that  $I$  and  $J$  are  $G$ -stable ideals of  $K[V]$  with  $IJ = 0$ . Then  $I' = I \cdot K[H]$  and  $J' = J \cdot K[H]$  are two-sided ideals of  $K[H]$  with  $I'J' = 0$ . In particular, if  $\Delta(H)$  is the f.c. center of  $H$  and if  $\theta: K[H] \rightarrow K[\Delta(H)]$  denotes the natural projection, then [P, Theorem 4.2.9] implies that  $\theta(I')\theta(J') = 0$  and hence that  $\theta(I)\theta(J) = 0$ . But all  $G$ -sections of  $V$

are infinite, so it follows from [P, Lemma 4.1.8] that  $V \cap \Delta(H)$  must be torsion-free abelian. In particular,  $K[V \cap \Delta(H)]$  is a domain, so  $\theta(I)\theta(J) = 0$  implies that either  $\theta(I) = 0$  or  $\theta(J) = 0$ , and consequently that either  $I = 0$  or  $J = 0$ .

(ii) Since  $K[V]/\omega(A; V) = K[V/A]$  and  $V/A$  has the same  $G$ -structure as  $V$ , the result follows from part (i). □

As a consequence, we can now easily describe the uniqueness aspects of finite intersections of augmentation ideals. If  $I = \bigcap_1^n \omega(A_i; V)$  and if  $A_1 \subseteq A_2$ , then  $\omega(A_1; V) \subseteq \omega(A_2; V)$  and  $\omega(A_2; V)$  is not needed. Thus, we say that the intersection is *irredundant* if  $A_i \subseteq A_j$  implies that  $i = j$ .

**Lemma 1.4.** *Suppose that all  $G$ -sections of  $V$  are infinite. Let  $A_1, A_2, \dots, A_n$  and  $B_1, B_2, \dots, B_m$  be  $G$ -stable subgroups of  $V$  and assume that*

$$I = \bigcap_1^n \omega(A_i; V) \quad \text{and} \quad J = \bigcap_1^m \omega(B_j; V)$$

*are irredundant intersections. Then  $I \supseteq J$  if and only if each  $A_i$  contains some  $B_j$ . In particular, if  $I = J$ , then  $n = m$  and, by reordering the  $B_j$ 's if necessary, we have  $A_i = B_i$  for all  $i$ .*

*Proof.* Suppose first that  $I \supseteq J$ . Then, for each subscript  $i$ , we have

$$\omega(A_i; V) \supseteq I \supseteq J \supseteq \prod_1^m \omega(B_j; V).$$

Thus, since  $\omega(A_i; V)$  is a  $G$ -prime ideal of  $K[V]$  by Lemma 1.3(ii), we conclude that  $\omega(A_i; V) \supseteq \omega(B_j; V)$  for some  $j$ . In particular,  $A_i \supseteq B_j$ . Conversely, suppose that each  $A_i$  contains some  $B_j$ . Then, for each  $i$ , we have  $\omega(A_i; V) \supseteq \omega(B_j; V) \supseteq J$  and hence  $I = \bigcap_i \omega(A_i; V) \supseteq J$ .

Finally, suppose that  $I = J$  and let  $i$  be given. Then  $I \supseteq J$ , so  $A_i \supseteq B_j$  for some  $j$ . On the other hand,  $J \supseteq I$ , so  $B_j \supseteq A_{i'}$  for some  $i'$ . Thus  $A_i \supseteq B_j \supseteq A_{i'}$  and the irredundancy of the  $I$ -intersection implies that  $i = i'$  and  $A_i = B_j$ . We have therefore shown that

$$\{A_1, A_2, \dots, A_n\} \subseteq \{B_1, B_2, \dots, B_m\}.$$

By symmetry, we get the reverse inclusion, and the result follows. □

We will also need the following variant of the above.

**Lemma 1.5.** *Assume that all  $G$ -sections on  $V$  are infinite. Let  $I$  be a  $G$ -stable ideal of  $K[V]$ , and let  $X \subseteq Y$  be  $G$ -stable subgroups of  $V$ . Suppose that  $I \cap K[X] = \bigcap_{i=1}^n \omega(A_i; X)$  and  $I \cap K[Y] = \bigcap_{j=1}^m \omega(B_j; Y)$  are finite irredundant intersections, where the  $A_i$  and  $B_j$  are  $G$ -stable subgroups of  $X$  and  $Y$ , respectively. Then each  $B_j$  contains some  $A_{j'}$ , and each  $A_i$  can be written as  $A_i = B_{i'} \cap X$  for some  $B_{i'}$ .*

*Proof.* By assumption and Lemma 1.1(ii), we have

$$\bigcap_{i=1}^n \omega(A_i; X) = I \cap K[X] = (I \cap K[Y]) \cap K[X] = \bigcap_{j=1}^m \omega(B_j \cap X; X).$$

Since  $\omega(B_j \cap X; X) \supseteq \bigcap_{i=1}^n \omega(A_i; X)$ , it follows from Lemma 1.3(ii) that  $B_j \supseteq B_j \cap X \supseteq A_{j'}$  for some  $j'$ . Conversely, if we eliminate unnecessary terms from  $\bigcap_{j=1}^m \omega(B_j \cap X; X)$ , then we have a finite irredundant intersection, so Lemma 1.4 implies that  $A_i = B_{i'} \cap X$  for some  $i'$ . □

Next, we introduce an assumption to guarantee that any arbitrary intersection of augmentation ideals can be reduced to a finite intersection.

**Lemma 1.6.** *Assume that  $V$  has a finite-length composition series as a  $G$ -module. Let  $\mathfrak{A}$  be a nonempty family of  $G$ -stable subgroups of  $V$  and let  $I = \bigcap_{A \in \mathfrak{A}} \omega(A; V)$ . Then  $I$  can be written as a finite intersection of augmentation ideals  $\omega(B; V)$  with each  $B$  a  $G$ -stable subgroup of  $V$ .*

*Proof.* For convenience we can assume that  $V \in \mathfrak{A}$  since  $\omega(V; V) \supseteq \omega(A; V)$  for all  $A$ . We proceed by induction on the  $G$ -composition length of  $V$ , the result being trivial if the length is 0. Now if  $I = 0$ , then  $I = \omega(1; V)$  and we are done. So assume that  $I \neq 0$  and choose  $0 \neq \gamma = \sum_0^n k_i x_i \in I$  with  $x_0 = 1$  and  $0 \neq k_0 \in K$ . Let  $X_i$  be the  $G$ -stable subgroup of  $V$  generated by  $x_i$  for  $i = 1, 2, \dots, n$ , and let  $\mathfrak{A}_i = \{A \in \mathfrak{A} \mid A \supseteq X_i\}$ . Note that each  $\mathfrak{A}_i$  is nonempty since  $V \in \mathfrak{A}_i$ . We claim that  $\mathfrak{A} = \bigcup_1^n \mathfrak{A}_i$ .

To this end, let  $A \in \mathfrak{A}$ . Then

$$\gamma = \sum_0^n k_i x_i \in I \subseteq \omega(A; V),$$

so  $x_0 = 1$  and  $k_0 \neq 0$  implies that  $x_i \in A$  for some  $i \neq 0$ . But  $A$  is  $G$ -stable, so  $X_i \subseteq A$  and hence  $A \in \mathfrak{A}_i$ , as required. Now set  $I_i = \bigcap_{A \in \mathfrak{A}_i} \omega(A; V)$ , so that  $I = I_1 \cap I_2 \cap \dots \cap I_n$ . It clearly suffices to show that each  $I_i$  is a finite intersection of suitable  $\omega(B; V)$ .

For this, note that  $A \supseteq X_i$  implies that  $\omega(A; V) \supseteq \omega(X_i; V)$  and that

$$\omega(A; V)/\omega(X_i; V) = \omega(A/X_i; V/X_i) \subseteq K[V/X_i].$$

Thus

$$I_i/\omega(X_i; V) = \bigcap_{A \in \mathfrak{A}_i} \omega(A/X_i; V/X_i).$$

Now,  $X_i \neq 1$ , so  $V/X_i$  has smaller  $G$ -composition length than  $V$ . Thus, by induction,  $I_i/\omega(X_i; V)$  can be written as a finite intersection  $\bigcap_j \omega(B_{i,j}/X_i; V/X_i)$  where each  $B_{i,j}$  is a  $G$ -stable subgroup of  $V$  containing  $X_i$ . Thus  $I_i = \bigcap_j \omega(B_{i,j}; V)$  and we have the finite intersection

$$I = \bigcap_i I_i = \bigcap_{i,j} \omega(B_{i,j}; V),$$

as required. □

A slight generalization of part of the preceding argument yields the following result which requires no assumption on the  $G$ -module structure of  $V$ .

**Lemma 1.7.** *Let  $\mathfrak{A}$  be a nonempty family of subgroups of  $V$ , let  $I = \bigcap_{A \in \mathfrak{A}} \omega(A; V)$ , and let  $\alpha \in I$ . Then we can write  $\mathfrak{A}$  as a finite union  $\mathfrak{A} = \mathfrak{A}_1 \cup \mathfrak{A}_2 \cup \dots \cup \mathfrak{A}_m$ , depending upon  $\alpha$ , so that if  $A_i = \bigcap_{A \in \mathfrak{A}_i} A$ , then  $\alpha \in \bigcap_1^m \omega(A_i; V)$ .*

*Proof.* Let  $S$  be the support of  $\alpha$ , namely the finite set of elements of  $V$  which appear in the representation of  $\alpha \in K[V]$ . Note that each  $A \in \mathfrak{A}$  partitions  $V$  into disjoint  $A$ -cosets, and hence each such  $A$  gives rise to a partition of  $S$ . But  $S$  is finite, so there are only finitely many possible partitions. Thus, we can write  $\mathfrak{A} = \mathfrak{A}_1 \cup \mathfrak{A}_2 \cup \dots \cup \mathfrak{A}_m$  as a finite union of nonempty sets, where all  $A$ 's in a fixed  $\mathfrak{A}_i$  yield the same partition  $\mathcal{P}_i$  of  $S$ . If  $A_i = \bigcap_{A \in \mathfrak{A}_i} A$ , it suffices to show that

$\alpha \in \omega(A_i; V)$ . To this end, fix  $i$  and write  $\alpha = \alpha_1 + \alpha_2 + \cdots + \alpha_k$  where the  $\alpha_j$ 's have support corresponding to the various subsets of  $S$  in the partition  $\mathcal{P}_i$ . Now for any  $A \in \mathfrak{A}_i$ , we have  $\alpha = \sum_j \alpha_j \in \omega(A; V)$ . Thus, since each  $\alpha_j$  has support in precisely one coset of  $A$  and since these cosets are disjoint, we conclude that  $\alpha_j \in \omega(A; V)$ . In particular, the sum of the coefficients of each  $\alpha_j$  is 0, and since the support of  $\alpha_j$  is contained in precisely one coset of  $A_i = \bigcap_{A \in \mathfrak{A}_i} A$ , we conclude that  $\alpha_j \in \omega(A_i; V)$ . It follows that  $\alpha \in \omega(A_i; V)$ , as required.  $\square$

Next, we prove the expected Noetherian result.

**Lemma 1.8.** *Assume that  $V$  has a finite-length composition series as a  $G$ -module and that all composition factors are infinite. If  $\mathcal{I}$  is the set of ideals of  $K[V]$  which can be written as intersections of augmentation ideals of  $G$ -stable subgroups of  $V$ , then  $\mathcal{I}$  satisfies the ascending chain condition.*

*Proof.* It is clear from the assumption that every  $G$ -section of  $V$  is infinite. Furthermore, Lemma 1.6 implies that every ideal in  $\mathcal{I}$  can be written as a finite intersection of augmentation ideals of  $G$ -stable subgroups of  $V$ . Indeed, we can then assume that these intersections are irredundant and hence satisfy the uniqueness condition of Lemma 1.4. In other words, each  $I \in \mathcal{I}$  corresponds uniquely to a finite irredundant set  $\{A_1, A_2, \dots\}$  of  $G$ -stable subgroups of  $V$ . Moreover, if  $J$  corresponds to  $\{B_1, B_2, \dots\}$ , then  $I \supseteq J$  if and only if each  $A_i$  contains some  $B_j$ .

Let  $\mathbb{S}$  denote the collection of all finite irredundant sets  $\{A_1, A_2, \dots\}$  of  $G$ -stable subgroups of  $V$ , and define  $\{B_1, B_2, \dots\} \preceq \{A_1, A_2, \dots\}$  if and only if each  $A_i$  contains some  $B_j$ . In view of the comments of the preceding paragraph, it suffices to show that  $\mathbb{S}, \preceq$  satisfies the ascending chain condition. To this end, for every  $G$ -stable subgroup  $A$  of  $V$ , define the *depth* of  $A$ ,  $d(A)$ , to equal the  $G$ -composition length of  $V/A$ , and note that this parameter is at most equal to the  $G$ -composition length of  $V$ . Now let  $\mathcal{T}_1 \preceq \mathcal{T}_2 \preceq \mathcal{T}_3 \preceq \cdots$  be an ascending sequence of elements of  $\mathbb{S}$  and let  $d(\mathcal{T})$ , the depth of this sequence  $\mathcal{T}$ , equal the largest depth of all the  $G$ -stable subgroups which occur as members of the various  $\mathcal{T}_i$ s. We prove by induction on  $d(\mathcal{T})$  that the sequence terminates, the result being trivial if  $d(\mathcal{T}) = 0$  since  $V$  is the only  $G$ -stable subgroup of  $V$  having depth  $\leq 0$ .

Now suppose that  $d(\mathcal{T}) = n$  and that the result holds for all sequences of smaller depth. Write  $\mathcal{T}_i = \mathcal{T}'_i \cup \mathcal{T}''_i$ , where  $\mathcal{T}'_i$  contains the  $G$ -stable subgroups of depth smaller than  $n$  and  $\mathcal{T}''_i$  contains those of depth precisely  $n$ . Suppose  $r \leq s$  and let  $A \in \mathcal{T}''_s \subseteq \mathcal{T}_s$ . Then  $\mathcal{T}_r \preceq \mathcal{T}_s$  implies that  $A$  contains some  $B \in \mathcal{T}_r$ . But  $B \subseteq A$  implies that  $d(B) \geq d(A) = n$ , so the definition of  $n = d(\mathcal{T})$  implies that  $d(B) = n$  and  $A = B$ . In other words, if  $r \leq s$ , then  $\mathcal{T}''_r \supseteq \mathcal{T}''_s$  and we obtain the decreasing sequence of finite sets  $\mathcal{T}''_1 \supseteq \mathcal{T}''_2 \supseteq \mathcal{T}''_3 \supseteq \cdots$  which clearly terminates. By deleting the first few terms if necessary, we can now assume that all  $\mathcal{T}''_i$  are equal.

Again let  $r \leq s$  and now take  $A \in \mathcal{T}'_s \subseteq \mathcal{T}_s$ . Then  $\mathcal{T}_r \preceq \mathcal{T}_s$  implies that  $A$  contains some  $B \in \mathcal{T}_r = \mathcal{T}'_r \cup \mathcal{T}''_r$ . If  $B \in \mathcal{T}''_r$ , then  $B \in \mathcal{T}''_s$  and  $A \supseteq B$  violates the irredundancy of the set  $\mathcal{T}_s$ . Thus  $B \in \mathcal{T}'_r$  and we conclude that  $\mathcal{T}'_r \preceq \mathcal{T}'_s$ . Since each  $\mathcal{T}'_i$  is obviously irredundant, we now have a new ascending sequence  $\mathcal{T}'_1 \preceq \mathcal{T}'_2 \preceq \mathcal{T}'_3 \preceq \cdots$  in  $\mathbb{S}$ , and this one has depth smaller than  $n$ . By induction, this new sequence terminates, and since all  $\mathcal{T}''_i$  are equal, the original sequence  $\mathcal{T}$  also terminates.  $\square$

Finally, we show that the property of being an intersection of augmentation ideals can be lifted from finitely generated submodules to the entire group.

**Lemma 1.9.** *Let  $G$  act on  $V$  in such a way that all  $G$ -sections are infinite, and let  $I$  be a  $G$ -stable ideal of the group algebra  $K[V]$ . Assume that, for every finitely generated  $G$ -submodule  $W$  of  $V$ , the  $G$ -stable ideal  $I \cap K[W]$  is a finite intersection of augmentation ideals of  $G$ -stable subgroups of  $W$ . Then  $I$  is an intersection of augmentation ideals of  $G$ -stable subgroups of  $V$ .*

*Proof.* Let  $\mathbb{S}$  denote the set of all finitely generated  $G$ -submodules of  $V$ . If  $X \in \mathbb{S}$ , then, by assumption and Lemma 1.4,  $I \cap K[X] = \bigcap_{i=1}^n \omega(A_i; X)$  is uniquely a finite irredundant intersection of augmentation ideals with each  $A_i$  a  $G$ -submodule of  $X$ . For convenience, write  $\mathbb{A}_X = \{A_1, A_2, \dots, A_n\}$ .

For each  $X \in \mathbb{S}$  and  $A \in \mathbb{A}_X$ , let  $\mathbb{B}_{X,A}$  denote the set of  $G$ -submodules  $B$  of  $V$  satisfying:

- (i)  $B \cap X \subseteq A$ , and
- (ii)  $B \cap Y$  contains a member of  $\mathbb{A}_Y$  for each  $Y \in \mathbb{S}$  with  $Y \supseteq X$ .

We first prove that each  $\mathbb{B}_{X,A}$  is nonempty. To this end, let  $\mathbb{S}_X$  denote the subset of  $\mathbb{S}$  consisting of all  $Y$  with  $Y \supseteq X$ , and consider all “choice” functions  $f$  defined on subsets  $\mathcal{D}$  of  $\mathbb{S}_X$  satisfying

$$f: \mathcal{D} \rightarrow \bigcup_{Y \in \mathbb{S}_X} \mathbb{A}_Y = \mathbb{T}_X \quad \text{and} \quad f(Y) \in \mathbb{A}_Y.$$

Let us say that such a function  $f$  is “good” if for all finite subsets  $\{Y_1, Y_2, \dots, Y_m\}$  of  $\mathcal{D}$ , we have

$$X \cap \langle f(Y_1), f(Y_2), \dots, f(Y_m) \rangle \subseteq A.$$

It is clear that if  $f: \mathcal{D} \rightarrow \mathbb{T}_X$  is good, then so is the restriction of  $f$  to any subset of  $\mathcal{D}$ . Next, suppose that  $Y_1, Y_2, \dots, Y_n \in \mathbb{S}_X$  and let  $Z = \langle Y_1, Y_2, \dots, Y_n \rangle$ . By assumption and Lemma 1.5, there exist some  $E \in \mathbb{A}_Z$  with  $E \cap X = A$ . Furthermore, by Lemma 1.5 again,  $E \cap Y_i \supseteq C_i$  for some  $C_i \in \mathbb{A}_{Y_i}$ . Thus, if we define

$$f: \{Y_1, Y_2, \dots, Y_m\} \rightarrow \mathbb{T}_X$$

by  $f(Y_i) = C_i$ , then

$$X \cap \langle f(Y_1), f(Y_2), \dots, f(Y_m) \rangle = X \cap \langle C_1, C_2, \dots, C_m \rangle \subseteq X \cap E = A$$

and  $f$  is a good function.

Since each  $\mathbb{A}_Y$  is finite, the Compactness Theorem (see [P, Theorem 6.3.1] for a slightly weaker version of this result) implies that there exists a good function  $g: \mathbb{S}_X \rightarrow \mathbb{T}_X$ , and we set  $B = \langle g(Y) \mid Y \in \mathbb{S}_X \rangle$ . Then  $B$  is certainly a  $G$ -stable subgroup of  $V$  and, for all  $Y \in \mathbb{S}_X$ , we have  $B \cap Y \supseteq g(Y)$ , so  $B \cap Y$  contains a member of  $\mathbb{A}_Y$ . In addition, if  $v \in X \cap B$ , then there exist  $Y_1, Y_2, \dots, Y_m \in \mathbb{S}_X$  with  $v \in X \cap \langle g(Y_1), g(Y_2), \dots, g(Y_m) \rangle \subseteq A$ . Thus  $X \cap B \subseteq A$ , so  $B \in \mathbb{B}_{X,A}$  and  $\mathbb{B}_{X,A}$  is indeed a nonempty set.

It is now a simple matter to prove that  $I$  is equal to  $J = \bigcap \omega(B; V)$ , where the latter intersection is over all  $X \in \mathbb{S}$ ,  $A \in \mathbb{A}_X$ , and  $B \in \mathbb{B}_{X,A}$ . For this, first note that if  $B \in \mathbb{B}_{X,A}$  and if  $Y \in \mathbb{S}_X$ , then  $B \cap Y \supseteq C$  for some  $C \in \mathbb{A}_Y$  by condition (ii). Thus

$$\omega(B; V) \cap K[Y] = \omega(B \cap Y; Y) \supseteq \omega(C; Y) \supseteq I \cap K[Y]$$

and, since this holds for all such  $Y$ , it follows that  $\omega(B; V) \supseteq I$ . Consequently,  $J = \bigcap \omega(B; V) \supseteq I$ . Conversely, let  $X \in \mathbb{S}$  and let  $\mathbb{A}_X = \{A_1, A_2, \dots, A_n\}$ . Then,

for each  $i$ ,  $\mathbb{B}_{X,A_i} \neq \emptyset$ , so we can choose some  $B_i$  in this set, and condition (i) yields

$$\begin{aligned} K[X] \cap J &\subseteq K[X] \cap \bigcap_{i=1}^n \omega(B_i; V) \\ &= \bigcap_{i=1}^n \omega(B_i \cap X; X) \subseteq \bigcap_{i=1}^n \omega(A_i; X) = K[X] \cap I. \end{aligned}$$

Since this holds for all  $X \in \mathbb{S}$ , we have the reverse inclusion, and therefore  $I = J$  is a suitable intersection. □

## 2. LOCALLY FINITE FIELDS

We continue to assume that  $K$  is an arbitrary field. In addition, we take  $F$  to be a locally-finite field and we let  $V$  be an  $F$ -vector space, viewed multiplicatively. Then  $V$  is an elementary abelian  $p$ -group, where  $p = \text{char } F$ , and  $G = F^\bullet$  acts on  $V$  in a natural manner. The following extension of [BE, Lemma 9] handles the case where both  $F$  and  $V$  are finite.

**Lemma 2.1.** *Let  $F$  be a finite field and let  $V$  be a finite-dimensional  $F$ -vector space, viewed multiplicatively. Assume that  $\text{char } F \neq \text{char } K$ . Then  $G = F^\bullet$  acts on  $V$  and every  $G$ -stable ideal of  $K[V]$  contained in  $\omega(V; V)$  is a finite intersection of augmentation ideals  $\omega(A; V)$  with  $A$  a  $G$ -stable subgroup of  $V$ .*

*Proof.* Suppose first that  $K$  is algebraically closed or at least that it contains a primitive  $p$ th root of unity for  $p = \text{char } F$ . Let  $\Lambda = \Lambda(V)$  be the set of nonprincipal irreducible representations of  $K[V]$  and let  $\{e_\lambda \mid \lambda \in \Lambda\}$  be the corresponding set of primitive idempotents in  $K[V]$ . Then we know that the latter are orthogonal idempotents and that  $\omega(V; V) = \bigoplus_{\lambda \in \Lambda} K e_\lambda$ . Furthermore, if  $I$  is an ideal of  $K[V]$  contained in  $\omega(V; V)$ , then  $I = \bigoplus_{\lambda \in \Lambda} I e_\lambda$  and, for each  $\lambda$ ,  $I e_\lambda$  is either 0 or  $K e_\lambda$ . Thus  $I$  is uniquely determined by its *support*, namely  $\text{supp } I = \{\lambda \in \Lambda \mid e_\lambda \in I\}$ . Indeed,  $I = \bigoplus_{\lambda \in \text{supp } I} K e_\lambda$ , and  $I$  is  $G$ -stable if and only if  $\text{supp } I$  is  $G$ -stable.

Let  $\mathcal{O}_1, \mathcal{O}_2, \dots, \mathcal{O}_m$  be the orbits of the natural action of  $G$  on  $\Lambda$ , and let  $I_i$  be the ideal of  $K[V]$  contained in  $\omega(V; V)$  and satisfying  $\text{supp } I_i = \Lambda \setminus \mathcal{O}_i$ . Then, the  $I_i$ 's are clearly the maximal  $G$ -stable ideals of  $K[V]$  properly contained in  $\omega(V; V)$ . Furthermore, if  $I$  is a  $G$ -stable ideal of  $K[V]$  with  $I \subseteq \omega(V; V)$  and  $\text{supp } I = \Lambda \setminus \bigcup_{i \in \mathcal{I}} \mathcal{O}_i$ , then clearly  $I = \bigcap_{i \in \mathcal{I}} I_i$ . Thus, it suffices to show that each  $I_i$  is a suitable augmentation ideal.

To this end, let  $|F| = q$  and  $|V| = q^n$ . Since the split extension  $V \rtimes G$  is a Frobenius group, it follows that  $G$  acts in a semiregular fashion, that is with full orbit sizes, on  $\Lambda$ . Thus, since  $|\Lambda| = q^n - 1$ , we see that the number of orbits  $\mathcal{O}_i$  is precisely equal to  $m = (q^n - 1)/(q - 1)$  and this is the same as the number of  $F$ -subspaces of  $V$  of codimension 1. In particular, if  $n = 1$ , then  $m = 1$  and we see that  $\omega(V; V)$  is the unique nonzero  $G$ -stable ideal of  $K[V]$  contained in  $\omega(V; V)$ . For the general case, let  $A_1, A_2, \dots, A_m$  be the  $m$  subspaces of  $V$  of codimension 1. Then each  $\omega(A_i; V)$  is  $G$ -stable and

$$\omega(V; V)/\omega(A_i; V) = \omega(V/A_i; V/A_i) \triangleleft K[V/A_i].$$

But  $\dim_F V/A_i = 1$ , so the above  $n = 1$  observation implies that  $\omega(V/A_i; V/A_i)$  is a minimal nonzero  $G$ -stable ideal of  $K[V/A_i]$ . Consequently, each  $\omega(A_i; V)$  is a  $G$ -stable ideal of  $K[V]$  maximal subject to being properly contained in  $\omega(V; V)$ . In other words, we have constructed  $m$  distinct  $G$ -stable ideals  $\omega(A_i; V)$ , all maximal

subject to being properly contained in  $\omega(V; V)$ . Since this obviously accounts for all the ideals  $I_1, I_2, \dots, I_m$ , the algebraically closed case is proved.

Finally, let  $K$  be arbitrary, subject to  $\text{char } K \neq \text{char } F$ , and let  $K'$  be its algebraic closure. Note that if  $J$  is any ideal of  $K[V]$ , then  $J' = J \cdot K' = J \cdot K'[V]$  is an ideal of  $K'[V]$ . Furthermore, since  $K$  is a  $K'$ -module direct summand of  $K'$ , it follows that  $J = J' \cap K[V]$ . In particular, since  $\omega_K(A; V)' = \omega_{K'}(A; V)$ , we see that  $\omega_{K'}(A; V) \cap K[V] = \omega_K(A; V)$ . Now  $I$  is a  $G$ -stable ideal of  $K[V]$  contained in  $\omega_K(V; V)$ , so  $I'$  is a  $G$ -stable ideal of  $K'[V]$  contained in  $\omega_{K'}(V; V)$ . Thus, by the algebraically closed result,  $I' = \bigcap_1^t \omega_{K'}(B_i; V)$  for suitable  $G$ -stable subgroups  $B_i$  of  $V$ , and consequently

$$I = I' \cap K[V] = \bigcap_1^t (\omega_{K'}(B_i; V) \cap K[V]) = \bigcap_1^t \omega_K(B_i; V),$$

as required. □

It is now a simple matter, using the results of the preceding section, to lift this lemma to the infinite situation. Note that, if  $V$  is an  $F$ -vector space and if  $G = F^\bullet$ , then the  $G$ -stable subgroups of  $V$  are precisely the  $F$ -subspaces of  $V$ . In particular, when  $F$  is infinite, each  $G$ -section of  $V$  is a nontrivial  $F$ -vector space and hence is also infinite.

**Lemma 2.2.** *Let  $F$  be an infinite-locally finite field and suppose that  $V$  is an  $F$ -vector space, viewed multiplicatively. Assume that  $\text{char } K \neq \text{char } F$ . Then  $G = F^\bullet$  acts on  $V$  and every  $G$ -stable ideal of  $K[V]$  is an intersection of augmentation ideals  $\omega(A; V)$  with  $A$  a  $G$ -stable subgroup of  $V$ .*

*Proof.* Assume first that  $V = F^n$  is finite dimensional, and let  $F_1 \subseteq F_2 \subseteq \dots$  be an ascending union of finite subfields of  $F$  with  $\bigcup_1^\infty F_i = F$ . If  $G_i = F_i^\bullet$  and  $V_i = F_i^n$ , then it is easy to see that  $\{(V_i, G_i) \mid i = 1, 2, \dots\}$  is a local system for  $(V, G)$ . Now if  $I$  is any  $G$ -stable ideal of  $K[V]$  contained in  $\omega(V; V)$ , then Lemma 1.1(i) implies that, for each  $i$ ,  $K[V_i] \cap I$  is a  $G_i$ -stable ideal of  $K[V_i]$  contained in  $\omega(V_i; V_i)$ . Hence, by Lemma 2.1, each  $K[V_i] \cap I$  is a finite intersection of augmentation ideals  $\omega(A; V_i)$  with  $A$  a  $G_i$ -stable subgroup of  $V_i$ , and it follows from Lemma 1.2 that  $I$  is an intersection of augmentation ideals of  $G$ -stable subgroups of  $V$ .

Next, let  $I$  be an arbitrary  $G$ -stable ideal of  $K[V]$ . If  $I \supseteq \omega(V; V)$ , then either  $I = \omega(V; V)$  or  $I = K[V]$ , and the latter is an empty intersection of augmentation ideals. On the other hand, if  $I \not\supseteq \omega(V; V)$ , then  $J = I \cap \omega(V; V)$  is a  $G$ -stable ideal of  $K[V]$  properly contained in  $\omega(V; V)$ . By the above,  $J$  is contained in some augmentation ideal  $\omega(B; V)$  with  $B$  a  $G$ -stable subgroup of  $V$  properly smaller than  $V$ . In particular,  $\omega(B; V) \not\supseteq \omega(V; V)$ . But  $\omega(B; V) \supseteq J = I \cap \omega(V; V)$  and  $\omega(B; V)$  is a  $G$ -prime ideal by Lemma 1.3(ii). Hence  $I \subseteq \omega(B; V) \subseteq \omega(V; V)$  and the result of the preceding paragraph applies to  $I$ .

Finally, if  $V$  is an arbitrary  $F$ -vector space, then the finitely generated  $G$ -submodules of  $V$  are precisely the finite-dimensional  $F$ -subspaces of  $V$ . With this, the preceding remarks and Lemma 1.9 yield the result. □

We can now offer the

*Proof of Theorem A.* If  $V$  is a finite-dimensional  $F$ -vector space, then it is clear that  $V$  has  $G$ -composition length equal to  $\dim_F V < \infty$ . Now, by Lemma 2.2, every  $G$ -stable ideal of  $K[V]$  is an intersection of augmentation ideals  $\omega(A; V)$  with

A a  $G$ -stable subgroup of  $V$ . Next, Lemma 1.6 implies that every such ideal is a finite intersection of suitable augmentation ideals. We can then assume that these intersections are irredundant, and conclude from Lemma 1.4 that the corresponding  $G$ -stable subgroups are unique. Finally, by Lemma 1.8, the set of all such ideals satisfies the ascending chain condition.  $\square$

Let  $\mathfrak{G}$  be a group acting on a set  $\Lambda$ . Then we recall that an element  $\lambda \in \Lambda$  is said to be  $\mathfrak{G}$ -*orbital* if it has finitely many  $\mathfrak{G}$ -conjugates or equivalently if the stabilizer in  $\mathfrak{G}$  of  $\lambda$  has finite index in the group.

**Lemma 2.3.** *Let  $F$  be an infinite locally-finite field, let  $V$  be a finite-dimensional  $F$ -vector space, viewed multiplicatively, and let  $\mathfrak{G}$  be a group which acts on  $V$  and contains  $F^\bullet$ , in its natural action, as a normal subgroup. If  $\text{char } K \neq \text{char } F$ , then every  $\mathfrak{G}$ -stable ideal of  $K[V]$  is uniquely an irredundant intersection  $\bigcap_{A \in \mathfrak{A}} \omega(A; V)$ , where  $\mathfrak{A}$  is a finite  $\mathfrak{G}$ -stable set of  $F$ -subspaces of  $V$ . In particular, each  $A \in \mathfrak{A}$  is an  $F^\bullet$ -stable,  $\mathfrak{G}$ -orbital subgroup of  $V$ .*

*Proof.* If  $I$  is  $\mathfrak{G}$ -stable, then it is  $F^\bullet$ -stable. Hence by Theorem A,  $I$  can be written uniquely as the irredundant intersection  $I = \bigcap_{i=1}^m \omega(A_i; V)$  where each  $A_i$  is  $F^\bullet$ -stable. In particular, if  $g \in \mathfrak{G}$ , then since  $I$  is  $\mathfrak{G}$ -stable, we have

$$\bigcap_{i=1}^m \omega(A_i; V) = I = I^g = \bigcap_{i=1}^m \omega(A_i^g; V).$$

But  $F^\bullet$  is normal in  $\mathfrak{G}$ , so each  $A_i^g$  is certainly an  $F^\bullet$ -stable subgroup of  $V$ . Hence, by uniqueness,  $\{A_1^g, A_2^g, \dots, A_m^g\} = \{A_1, A_2, \dots, A_m\}$  and consequently  $\mathfrak{G}$  permutes the finite set  $\mathfrak{A} = \{A_1, A_2, \dots, A_m\}$ .  $\square$

With this, we can consider a question posed in [HZ]. Namely, let  $F$  be a locally-finite field and suppose that  $F$  is generated by two infinite subfields  $F_1$  and  $F_2$ . If  $V$  is an  $F$ -vector space and if  $I$  is an ideal of  $K[V]$  stable under both  $F_1^\bullet$  and  $F_2^\bullet$ , must  $I$  also be stable under  $F^\bullet$ ? We obtain an affirmative answer, at least when  $F_1 \cap F_2$  is infinite.

**Lemma 2.4.** *Let  $F$  be a locally finite field generated by the two subfields  $F_1$  and  $F_2$  with  $F_1 \cap F_2$  infinite. Let  $V$  be an  $F$ -vector space, viewed multiplicatively, and let  $\text{char } K \neq \text{char } F$ . If  $I$  is an ideal of  $K[V]$  stable under both  $F_1^\bullet$  and  $F_2^\bullet$ , then  $I$  is stable under  $F^\bullet$ .*

*Proof.* Let  $I$  be as above and choose  $\alpha \in I$  and  $f \in F^\bullet$ . The goal is to show that  $\alpha^f \in I$ . Now  $f \in \langle F_1, F_2 \rangle$ , so there exists a subfield  $\tilde{F}_2$  of  $F_2$  with  $f \in \langle F_1, \tilde{F}_2 \rangle = \tilde{F}$  and with degree  $(\tilde{F}_2 : F_1 \cap F_2) < \infty$ . Furthermore, there exists a finite-dimensional  $\tilde{F}$ -subspace  $\tilde{V}$  of  $V$  with  $\alpha \in I \cap K[\tilde{V}] = \tilde{I}$ . Since  $(\tilde{F} : F_1) \leq (\tilde{F}_2 : F_1 \cap F_2) < \infty$  and  $\dim_{\tilde{F}} \tilde{V} < \infty$ , we see that  $\tilde{V}$  is also finite dimensional as an  $F_1$ -space. Now  $\tilde{I}$  is certainly both  $F_1^\bullet$ - and  $\tilde{F}_2^\bullet$ -stable and  $F_1$  is an infinite field, so Lemma 2.3, applied to the group  $F_1^\bullet \tilde{F}_2^\bullet$ , implies that  $\tilde{I} = \bigcap_{A \in \mathfrak{A}} \omega(A; \tilde{V})$  where each  $A \in \mathfrak{A}$  is an  $F_1^\bullet$ -stable,  $\tilde{F}_2^\bullet$ -orbital subgroup of  $\tilde{V}$ . By assumption,  $F_1 \cap F_2$  is infinite, so  $\tilde{F}_2$  is infinite, and therefore any subgroup of finite index in  $\tilde{F}_2^\bullet$  must additively generate the field  $\tilde{F}_2$ . In particular, since each  $A \in \mathfrak{A}$  is  $\tilde{F}_2^\bullet$ -orbital, it follows that each such  $A$  is  $\tilde{F}_2^\bullet$ -stable. Furthermore, since each  $A$  is  $F_1^\bullet$ -stable and since  $\tilde{F} = \langle F_1, \tilde{F}_2 \rangle$ , we see that each  $A \in \mathfrak{A}$  is  $\tilde{F}^\bullet$ -stable. Thus,  $\tilde{I}$  is an intersection of the  $\tilde{F}^\bullet$ -stable

augmentation ideals  $\omega(A; \tilde{V})$ , so  $\tilde{I}$  is also  $\tilde{F}^\bullet$ -stable. Since  $\alpha \in \tilde{I}$  and  $f \in \tilde{F}^\bullet$ , we conclude that  $\alpha^f \in \tilde{I} \subseteq I$ , as required.  $\square$

It follows from the above and a trivial induction that if  $F = \langle F_1, F_2, \dots \rangle$  is locally finite and if  $F_{n+1} \cap \langle F_1, F_2, \dots, F_n \rangle$  is infinite for all  $n \geq 1$ , then an ideal of  $K[V]$  is  $F^\bullet$ -stable if and only if it is  $F_n^\bullet$ -stable for all  $n$ .

Finally, as we observed in the introduction, Theorem A is essentially the locally-finite analog of [BE, Proposition 6], a result on the field of rational numbers. Both of these facts will be used in Part II of this work to handle arbitrary division rings.

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