SEQUENTIALLY INDEPENDENT EFFECTS

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ABSTRACT. A quantum effect is a yes-no measurement that may be unsharp. An effect is represented by an operator E on a Hilbert space that satisfies $0 \leq E \leq I$. We define effects $E_1, E_2, \ldots, E_n$ to be sequentially independent if the result of any sequential measurement of $E_1, E_2, \ldots, E_n$ does not depend on the order in which they are measured. We show that two effects are sequentially independent if and only if they are compatible. That is, their corresponding operators commute. We also show that three effects are sequentially independent if and only if all permutations of the product of their corresponding operators coincide. It is noted that this last condition does not imply that the three effects are mutually compatible.

1. INTRODUCTION

Sequential measurements are very important in quantum mechanics. In general, two measurements M_1, M_2 cannot be performed simultaneously so they are usually executed sequentially. Thus, they are either executed in the order (M_1, M_2) or (M_2, M_1). We may think of (M_1, M_2) as a sequential measurement in which M_1 is measured first and M_2 second. As is frequently done, both experimentally and theoretically, we shall simplify the situation by considering yes-no measurements. These are measurements that only have two possible results, usually taken to be 0 and 1. A paradigm situation is an optical bench in which a beam of particles prepared in a certain state is injected at the left and then subjected to a sequence of filters $F_1, \ldots, F_n$. Particles that pass through all the filters enter a detection device at the right of F_n. In general, the order of placement of the filters makes a difference. For example, let F_1, F_2, F_3 be polarizing filters in planes perpendicular to the particle beam, where F_1 polarizes vertically, F_2 horizontally and F_3 at a 45° angle. If we place the filters in order F_1, F_2, F_3, then no particles are detected, but in the order F_1, F_3, F_2 particles are detected. The difference is due to quantum interference and this phenomenon occurs for many other quantum measurements such as spin measurements in which the filters are given by Stern-Gerlach apparata.

A yes-no measurement may be thought of as an event that occurs when the measurement has value 1 and does not occur when the measurement has value 0. In classical probability theory, events are represented by sets and the sequential

measurement (A, B) occurs with probability $P(A, B) = P(A)P(B \mid A)$. But $P(B \mid A) = P(A \cap B)/P(A)$ so that $P(A, B) = P(A \cap B)$. (We are assuming that $P(A) \neq 0$ which we shall always do in the sequel.) In this case, $P(A, B) = P(B, A)$ so there is no interference between A and B and this is one of the main differences between classical and quantum probability theory. Another way of stating this is that the sequential measurement (A, B) is represented by the event $A \cap B$ and $(A, B) = (B, A)$ because $A \cap B = B \cap A$.

In quantum theory, sharp quantum events are represented by (self-adjoint) projection operators and the conditional probability is given by the von Neumann-Lüders formula [6]

$$P(B \mid A) = \frac{\mathrm{tr}(BAWA)}{\mathrm{tr}(AW)}$$

where W is a density operator representing a quantum state. Since $P(A) = \mathrm{tr}(AW)$, we have

$$P(A, B) = P(A)P(B \mid A) = \mathrm{tr}(BAWA) = \mathrm{tr}(ABAW).$$

In general, $P(A, B) \neq P(B, A)$ so A and B interfere. Stated differently, we can represent the sequential measurements (A, B) and (B, A) by the operators ABA and BAB, respectively, and $(A, B) \neq (B, A)$ in general.

We say that two events A and B are **sequentially independent** if $(A, B) = (B, A)$. Of course, $(A, B) = (B, A)$ if and only if $P(A, B) = P(B, A)$ for every probability measure P or for every state W in the quantum case. We have seen that classical events are always sequentially independent and would like to know when this happens for sharp quantum events. This is easily accomplished by letting ψ be a vector satisfying $A\psi = \psi$. We then have

$$\|B\psi\|^2 = \langle B\psi, \psi \rangle = \langle ABA\psi, \psi \rangle = \langle BAB\psi, \psi \rangle = \langle AB\psi, B\psi \rangle.$$

If $B\psi \neq 0$, then

$$\langle A \frac{B\psi}{\|B\psi\|}, \frac{B\psi}{\|B\psi\|} \rangle = 1.$$

It follows that $AB\psi = B\psi$. Hence, $BA = ABA$ and taking adjoints gives $AB = ABA = BA$. Thus, $(A, B) = (B, A)$ if and only if A and B are compatible (or simultaneously measurable) and this reduces to the classical theory when only A and B are considered.

Notice that sequential independence is unrelated to stochastic independence which says that $P(B \mid A) = P(B)$. In the classical case, this means that $P(A \cap B) = P(A)P(B)$. In the quantum case, we would have

$$\mathrm{tr}(ABAW) = \mathrm{tr}(AW)\mathrm{tr}(BW).$$

This last equation practically never holds and is rarely considered. Notice also that stochastic independence is a symmetric relation in classical probability while it is not in quantum probability.

The main goal of this paper is the study of sequentially independent quantum effects. An effect corresponds to a quantum event that may be unsharp and is represented by a linear operator A on a Hilbert space H that satisfies $0 \leq A \leq I$. That is,

$$0 \leq \langle A\psi, \psi \rangle \leq \langle \psi, \psi \rangle$$

for all $\psi \in H$. Effects play an important role in quantum measurement theory and have been intensely studied [1, 2, 3]. We can already see one reason why effects are important from our previous discussion. Even if A and B are projections representing sharp events, the sequential measurement BAB is not a projection unless A and B are compatible. However, $0 \leq BAB \leq I$ so BAB represents an unsharp event in general.

2. Sequential independent effects

Let H be a complex Hilbert space and let $B(H)$ be the set of bounded linear operators on H. An operator $A \in B(H)$ is **positive** if $A \geq 0$; that is, $\langle A\psi, \psi \rangle \geq 0$ for all $\psi \in H$. It is easy to show that a positive operator is self-adjoint. Recall that $B \in B(H)$ is **normal** if $BB^* = B^*B$. In the sequel we shall need the following theorem due to Fuglede, Putnam and Rosenblum [5].

Theorem 1. *Let $M, N, A \in B(H)$ with M and N normal. If $MA = AN$, then $M^*A = AN^*$.*

Let $E, F \in B(H)$ be effects. Then $E^{1/2}FE^{1/2}$ is an effect because

$$0 \leq \langle E^{1/2}FE^{1/2}\psi, \psi \rangle = \langle FE^{1/2}\psi, E^{1/2}\psi \rangle \leq \langle E^{1/2}\psi, E^{1/2}\psi \rangle$$
$$= \langle E\psi, \psi \rangle \leq \langle \psi, \psi \rangle$$

for every $\psi \in H$. The von Neumann-Lüders formula is generalized to effects [4] by defining

$$P(F \mid E) = \frac{\text{tr}\left(FE^{1/2}WE^{1/2}\right)}{\text{tr}(EW)} = \frac{\text{tr}\left(E^{1/2}FE^{1/2}W\right)}{\text{tr}(EW)}.$$

We call $E^{1/2}FE^{1/2}$ the **sequential effect** for (E, F) and say that E and F are **sequentially independent** if

$$E^{1/2}FE^{1/2} = F^{1/2}EF^{1/2}.$$

We now prove a general result from which a characterization of sequential independence follows.

Theorem 2. *For $A, B \in B(H)$ the following statements are equivalent.*
  (a) *$AB^*BA^* = BA^*AB^*$ and $A^*BB^*A = B^*AA^*B$.*
  (b) *$BA^*A = AA^*B$ and $AB^*B = BB^*A$.*

Proof. To show that (b) implies (a), suppose that (b) holds. We then have

$$AB^*BA^* = BB^*AA^* = B(AA^*B)^* = B(BA^*A)^* = BA^*AB^*$$

and

$$A^*BB^*A = A^*AB^*B = (BA^*A)^*B = (AA^*B)^*B = B^*AA^*B.$$

To show that (a) implies (b), suppose that (a) holds and let $M = AB^*$, $N = B^*A$. It follows from (a) that M and N are normal and we have

$$MA = AB^*A = AN.$$

Applying Theorem 1 gives

$$BA^*A = M^*A = AN^* = AA^*B.$$

Now let $M = BA^*$, $N = A^*B$. It follows from (a) that M and N are normal and we have

$$MB = BA^*B = BN.$$

Again, applying Theorem 1 gives

$$AB^*B = M^*B = BN^* = BB^*A.$$

$\square$

Corollary 3. *Two effects, E and F, are sequentially independent if and only if $EF = FE$.*

Proof. It is well known that if $EF = FE$, then $E^{1/2}F^{1/2} = F^{1/2}E^{1/2}$ [5] and it follows that E and F are sequentially independent. Conversely, suppose that E and F are sequentially independent and let $A = E^{1/2}$, $B = F^{1/2}$. We then have that $AB^2A = BA^2B$ so by Theorem 2, $BA^2 = A^2B$. It follows that $B^2A^2 = A^2B^2$ and hence $EF = FE$. $\square$

We next consider three effects E_1, E_2, E_3. The natural definition for the probability of the sequential measurement (E_1, E_2, E_3) becomes

$$P(E_1, E_2, E_3) = P(E_1)P(E_2 \mid E_1)P\left(E_3 \mid (E_1, E_2)\right)$$
$$= \mathrm{tr}\left(E_1^{1/2}E_2^{1/2}E_3E_2^{1/2}E_1^{1/2}W\right).$$

We then call $E_1^{1/2}E_2^{1/2}E_3E_2^{1/2}E_1^{1/2}$ the **sequential effect** for (E_1, E_2, E_3). It is thus reasonable to say that E_1, E_2, E_3 are **sequentially independent** if

$$E_i^{1/2}E_j^{1/2}E_kE_j^{1/2}E_i^{1/2} = E_r^{1/2}E_s^{1/2}E_tE_s^{1/2}E_r^{1/2}$$

for every $i \neq j \neq k$, $r \neq s \neq t$. Moreover, we call E_1, E_2, E_3 **permutable** if

$$E_iE_jE_k = E_rE_sE_t$$

for every $i \neq j \neq k$, $r \neq s \neq t$.

Lemma 4. *Effects E, F, G are permutable if and only if $E^{1/2}, F^{1/2}, G^{1/2}$ are permutable.*

Proof. Suppose that E, F, G are permutable. In particular we have

$$EFG = FEG = GEF = GFE.$$

It follows that if ψ is in the range $R(G)$ of G, then

$$EF\psi = FE\psi \in R(G).$$

Since the closure $\overline{R}(G^{1/2}) = \overline{R}(G)$ we have $EF = FE$ on $\overline{R}(G^{1/2})$ and hence $E^{1/2}F^{1/2} = F^{1/2}E^{1/2}$ on $\overline{R}(G^{1/2})$. Therefore,

$$E^{1/2}F^{1/2}G^{1/2} = F^{1/2}E^{1/2}G^{1/2}$$

and taking adjoints gives

$$G^{1/2}F^{1/2}E^{1/2} = G^{1/2}E^{1/2}F^{1/2}.$$

The last two equations hold for all permutations of $E^{1/2}, F^{1/2}, G^{1/2}$. Hence,

$$F^{1/2}G^{1/2}E^{1/2} = F^{1/2}E^{1/2}G^{1/2} = E^{1/2}F^{1/2}G^{1/2} = E^{1/2}G^{1/2}F^{1/2}$$
$$= G^{1/2}E^{1/2}F^{1/2} = G^{1/2}F^{1/2}E^{1/2}$$

so $A^{1/2}, B^{1/2}, C^{1/2}$ are permutable. The converse is similar. $\square$

We now characterize sequential independence for three effects.

Theorem 5. *Effects E, F, G are sequentially independent if and only if E, F, G are permutable.*

Proof. Suppose that E, F, G are permutable. By Lemma 4, $E^{1/2}, F^{1/2}, G^{1/2}$ are permutable. We then have

$$E^{1/2}F^{1/2}GF^{1/2}E^{1/2} = E^{1/2}F^{1/2}G^{1/2}G^{1/2}F^{1/2}E^{1/2} = E^{1/2}G^{1/2}FG^{1/2}E^{1/2}.$$

Since the other permutations are similar, we conclude that E, F, G are sequentially independent.

Conversely, suppose that E, F, G are sequentially independent. Letting $A = E^{1/2}$, $B = F^{1/2}$, $C = G^{1/2}$ we have

$$ABC^2BA = CBA^2BC.$$

Letting $N = ABC$ and $M = BCA$ we have that N and M are normal and $NA = AM$. Applying Theorem 1 gives $N^*A = AM^*$ and hence $CBA^2 = A^2CB$. Since $A \geq 0$, it follows that $CBA = ACB$ [5]. We conclude that all cyclic permutations coincide. We now show that transpositions coincide; that is, $CAB = CBA$. To accomplish this, we have

$$A^2BC^2B = ABCACB = ABC^2BA = BCA^2CB = BC^2BA^2.$$

Again, since $A \geq 0$ we have that $ABC^2B = BC^2BA$. Hence, $C^2AB^2 = B^2AC^2$. Since $B(BAC) = (BAC)B$, B and BAC are commuting normal operators. It follows that B^2AC is normal and similarly CAB^2 is normal. Since $(B^2AC)C = C(CAB^2)$, by Theorem 1 we have that $CAB^2C = CB^2AC$. Hence, $B^2CAC = CACB^2$. Since $B \geq 0$, we conclude that $BCAC = CACB$. Therefore,

$$C^2AB = CABC = BCAC = CACB = C^2BA.$$

Letting $M = AB - BA$ we have that $C^2M = 0$ and thus $M^*C^2M = 0$. Hence, for every $\psi \in H$ we have

$$\|CM\psi\|^2 = \langle CM\psi, CM\psi \rangle = \langle M^*C^2M\psi, \psi \rangle = 0$$

We conclude that $CM = 0$ so that $CAB = CBA$. By symmetry, all transpositions coincide so A, B, C are permutable. Applying Lemma 4, we conclude that E, F, G are permutable. $\square$

Comparing Corollary 3 and Theorem 5 we see that E, F are sequentially independent if and only if E and F are compatible while E, F, G are sequentially independent if and only if E, F, G are only permutable. It is interesting to note that three effects that are permutable need not be mutually compatible even if

they are sharp (that is, projections). For example, let E, F, G be the following projections on $\mathbb{C}^3$:

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad F = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad G = \frac{1}{2} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

Thus, three effects that are sequentially independent need not be compatible in pairs. However, by employing the methods in Lemma 4 we do have the following result.

Corollary 6. *Effects E_1, E_2, E_3 are sequentially independent if and only if $E_i E_j = E_j E_i$ on $R(E_k)$ for all $i \neq j \neq k$.*

This work can be continued for four or more effects and a result similar to Theorem 5 holds.

References

[1] P. Busch, P. J. Lahti and P. Mittlestaedt, *The Quantum Theory of Measurements*, Springer-Verlag, Berlin, 1991. MR **93m:**81014

[2] P. Busch, M. Grabowski and P. J. Lahti, *Operational Quantum Physics*, Springer-Verlag, Berlin, 1995. MR **96j:**81022

[3] E. B. Davies, *Quantum Theory of Open Systems*, Academic Press, London, 1976. MR **58:**8853

[4] S. Gudder, A histories approach to quantum mechanics, *J. Math. Phys.* **39** (1998), 5772–5788. MR **99h:**81011

[5] W. Rudin, *Functional Analysis*, McGraw-Hill, New York, 1991. MR **92k:**46001

[6] J. von Neumann, *Mathematical Foundations of Quantum Mechanics*, Princeton University Press, Princeton, New Jersey, 1955. MR **16:**654a

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