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INTERPRETATION OF THE DEFORMATION SPACE OF A DETERMINANTAL BARLOW SURFACE VIA SMOOTHINGS

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ABSTRACT. In this present paper, we provide an interpretation of the deformation space of a determinantal Barlow surface via smoothings.

1. INTRODUCTION

Let (y_1, y_2, y_3, y_4) be a coordinate of $\mathbb{P}^3$, and consider a $D_5 = \langle b, a\rangle$ action on $\mathbb{P}^3$ via

$$b : (y_1, y_2, y_3, y_4) \to (\epsilon y_1, \epsilon^2 y_2, \epsilon^3 y_3, \epsilon^4 y_4),$$
$$a : (y_1, y_2, y_3, y_4) \to (y_4, y_3, y_2, y_1)$$

where ϵ is the primitive 5-th root of unity. In [C1], Catanese studied a four-dimensional family of surfaces that are double coverings of $\mathbb{Z}_5$-quotients of the $\mathbb{Z}_5$-invariant symmetric determinantal quintics. We will refer to these $\mathbb{Z}_5$-quotients of the $\mathbb{Z}_5$-invariant symmetric determinantal quintics as *determinantal Godeaux surfaces*. Inside of this four-dimensional family, there is a two-dimensional D_5-invariant symmetric determinantal Godeaux surface. After providing a twisted $\mathbb{Z}_2$-action on the double cover, we have a two-dimensional subfamily of the moduli space of Barlow's example [B]. We will refer to these surfaces as *determinantal Barlow surfaces*. Let Σ be a D_5-invariant symmetric determinantal quintic surface in $\mathbb{P}^3$ and let $\sigma : Y \to \Sigma$ be a double cover. Consider a commutative diagram

$$\begin{array}{ccc} Z = Y/\langle b\rangle & \longrightarrow & B = Y/\langle b, a\sigma\rangle \\ \downarrow & & \\ X = Y/\langle b, \sigma\rangle = \Sigma/\langle b\rangle. & & \end{array} \tag{1.1}$$

In diagram (1.1), X is a D_5-invariant determinantal Godeaux surface, and B is a determinantal Barlow surface. By the construction of a D_5-invariant determinantal Godeaux surface, there is an extra involution. The fixed divisor of this involution is a -3-curve D_X ($\mathbb{P}^1$ with $N_{\mathbb{P}^1|X} = \mathcal{O}_{\mathbb{P}^1}(-3)$) without passing through four nodes (the double cover $Z \to X$ is branched over these four nodes). So the preimage of D_X in Z is two disjoint -3-curves, D_1, D_2. Since a-action fixes each point on D_1

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and D_2, $\mathbb{Z}_2$-action of $Z \to B$ interchanges D_1 and D_2. So these two -3-curves go to a -3-curve D_B in B without passing through four nodes (the double cover $Z \to B$ is branched over these four nodes) [B]. By parameterizing $\mathbb{Z}_5$-invariant symmetric determinantal quintics with 20 nodes, we can easily see that the general element of those set do not have the preimage of D_X. This argument shows that the natural map of $H^1(T_X) \to H^1(N_{D_X|X})$ is surjective. Therefore we have two directions of the first order deformation space induced by the cohomology of $H^1(N_{D_X|X})$. In this paper, we will prove that the surjectivity of the map of $H^1(T_X) \to H^1(N_{D_X|X})$ produces the surjectivity of the map of $H^1(T_B) \to H^1(N_{D_B|B})$ by the lifting of the first order deformation space to the commutative diagram (1.1) and the vanishing of $H^2(T_Z) = 0$. This vanishing was proved by Catanese-Le Brun [CL] and by the author [L] independently. Then we provide an interpretation of the deformation space of a determinantal Barlow surface via smoothings:

Theorem. *Consider a determinantal Barlow surface. Then the extension of the two-dimensional family of the determinantal Barlow surface, in the recovering six directions, can be interpreted as follows:*

(1) *one independent smoothing direction for each of the four nodes of B,*
(2) *two independent directions corresponding to the global smoothings induced by the surjectivity of the map $H^1(T_B) \to H^1(N_{D_B|B})$.*

Throughout we work over the complex number field $\mathbb{C}$. The notation here follows the standard textbook [H].

2. Proof of the Theorem

Let V be a smooth projective surface and D a smooth curve in V. There is a natural map $n : T_V \to N_{D|V}$ induced by the composition of the maps $T_V \to T_V|_D \to N_{D|V}$. Define $T_{V,D}$ by the kernel of the map of n:

$$0 \to T_{V,D} \to T_V \xrightarrow{n} N_{D|V} \to 0. \tag{2.1}$$

The short exact sequence (2.1) induces the long exact sequence

$$\begin{aligned} 0 &\to H^0(T_{V,D}) \to H^0(T_V) \to H^0(N_{D|V}) \to H^1(T_{V,D}) \\ &\to H^1(T_V) \to H^1(N_{D|V}) \to H^2(T_{V,D}) \to H^2(T_V). \end{aligned} \tag{2.2}$$

In the long exact sequence (2.2), each cohomology group relates the first order infinitesimal deformation or obstruction:

(2.2.1) $H^0(N_{D|V})$ classes the first order infinitesimal deformation of D in V and the obstruction lies in $H^1(N_{D|V})$,

(2.2.2) $H^1(T_{V,D})$ classes the first order infinitesimal deformation of the pair (V,D) and the obstruction lies in $H^2(T_{V,D})$,

(2.2.3) $H^1(T_V)$ classes the first order infinitesimal deformation of V and the obstruction lies in $H^2(T_V)$.

We also have the commutative diagram of the obstruction maps in (2.2):

$$\begin{array}{ccccccc} H^0(T_V) & \longrightarrow & H^0(N_{D|V}) & \longrightarrow & H^1(T_{V,D}) & \longrightarrow & H^1(T_V) \\ & & \downarrow{\scriptstyle ob_1} & & \downarrow{\scriptstyle ob_2} & & \downarrow{\scriptstyle ob_3} \\ & \xrightarrow{n} & H^1(N_{D|V}) & \longrightarrow & H^2(T_{V,D}) & \longrightarrow & H^2(T_V). \end{array} \tag{2.3}$$

The next lemma is easily obtained by the commutative diagram (2.3).

Lemma 1. (1) *Assume that the natural map* $n : H^1(T_V) \to H^1(N_{D|V})$ *is surjective. Then* $ob_3 = 0$ *implies that* $ob_2 = 0$.

(2) *If* $H^1(N_{D|V}) = 0$, *then* $ob_2 = 0$ *if and only if* $ob_3 = 0$.

The definition of $T_{V,D}$ induces a short exact sequence:

$$0 \to T_V(I_D) \to T_{V,D} \xrightarrow{t} T_D \to 0. \tag{2.4}$$

It is interesting to find the cases when the natural maps $n : H^1(T_V) \to H^1(N_{D|V})$ or $t : H^1(T_{V,D}) \to H^1(T_D)$ are zero or surjective.

Definition 2. A simple normal crossing surface X (a projective surface with normal crossing, and each irreducible component is smooth) is smoothable if there is an analytic disc Δ in $\mathbb{C}$ and a projective flat family of varieties $\pi : \mathcal{X} \to \Delta$ whose central fiber is X, and a general fiber $X_t = \pi^{-1}(t)$ is smooth for $t \neq 0$. Then $\pi : \mathcal{X} \to \Delta$ is called a smoothing of X.

Denote by T^i_X and $\mathbb{T}^i_X$ the local and global deformation objects of Lichtenbaum-Schlessinger, respectively. In the case of reduced local complete intersection, we have

$$T^i_X = \mathcal{E}xt^i_X(\Omega_X, \mathcal{O}_X), \ \ \mathbb{T}^i_X = \mathrm{Ext}^i_X(\Omega_X, \mathcal{O}_X).$$

Since X is normal crossing, X is locally embedded in a smooth variety $\mathcal{X}$ and we have

$$0 \to I_X/I_X^2 \to \Omega_{\mathcal{X}}|_X \to \Omega_X \to 0$$

where I_X/I_X^2 is locally free $\mathcal{O}_X$-module. Therefore $T^2_X = 0$.

There exists a spectral sequence $E_2^{p,q} = H^p(X, \mathcal{E}xt^q_X(\Omega_X, \mathcal{O}_X))$ and E_∞ goes to $\mathcal{E}xt^{p+q}_X(\Omega_X, \mathcal{O}_X)$. Also, there is a local-global exact sequence ([Go], §7.3)

$$\begin{aligned} 0 \to & H^1(X, T^0_X) \to \mathrm{Ext}^1_X(\Omega_X, \mathcal{O}_X) \to H^0(X, \mathcal{E}xt^1_X(\Omega_X, \mathcal{O}_X)) \\ \to & H^2(X, T^0_X) \to \mathrm{Ext}^2_X(\Omega_X, \mathcal{O}_X) \to H^1(X, \mathcal{E}xt^1_X(\Omega_X, \mathcal{O}_X)), \end{aligned}$$

so the following long exact sequence holds:

$$0 \to H^1(T^0_X) \to \mathbb{T}^1_X \to H^0(T^1_X) \to H^2(T^0_X) \to \mathbb{T}^2_X \to H^1(T^1_X). \tag{2.5}$$

The space $H^1(T^0_X)$ classifies all "locally trivial" deformations of X, i.e. for which the singularities remain locally a product, and its obstruction lies in $H^2(T^0_X)$. So if there is a smoothing of X, then $\mathbb{T}^1_X \to H^0(T^1_X)$ is not zero. Let $D = Sing\, X$. If there is a smoothing $\mathcal{X}$ of X, then $T^1_X \cong \mathcal{O}_D(E)$ for some effective divisor E. In addition, if $\mathcal{X}$ is assumed to be smooth, then $T^1_X \cong \mathcal{O}_D$. In general, T^1_X is not $\mathcal{O}_D$ [F, §2]. For example if $X = V \cup_D W$, then $T^1_X \cong N_{D|V} \otimes N_{D|W}$.

Definition 3. Let X be a simple normal crossing variety. Assume $T^1_X \cong \mathcal{O}_D$ for $D = Sing\, X$. Then X is called d-semistable.

We consider the simplest case: $X = V \cup_D W$ with $T^1_X \cong \mathcal{O}_D$. Then there are relations between T^0_X, and $T_{V,D}$, $T_{W,D}$ by definition of T^0_X.

$$0 \to T^0_X \to T_{V,D} \oplus T_{W,D} \xrightarrow{\frac{1}{2}(t_1+t_2)_*} T_D \to 0, \tag{2.6}$$

$$0 \to T_V(-D) \oplus T_W(-D) \to T^0_X \to T_D \to 0. \tag{2.7}$$

Then the map $H^0(N_{D|V} \otimes N_{D|W}) \to H^2(T_V) \oplus H^2(T_W)$, that is a composition of the map $H^0(N_{D|V} \otimes N_{D|W}) \to H^2(T_X^0)$ and the maps induced by the exact sequences (2.6), (2.1), can be obtained via the following natural maps [DF].

Let ω_W be the obstruction class of the extension of the first-order neighborhood of D in W to the flat model, which is in $H^1(N^*_{D|W} \otimes T_D)$. The splitting of the exact sequence of $0 \to T_D \to T_W|_D \to N_{D|W} \to 0$ is the same as $\omega_W = 0$. Consider the natural map

$$\begin{aligned} H^0(N_{D|V} \otimes N_{D|W}) \otimes \omega_W &\to H^1(N_{D|V} \otimes T_D) \\ &\to H^1(N_{D|V} \otimes T_V|_D) \\ &\to H^2(T_V), \end{aligned}$$

induced by the exact sequence $0 \to T_V \to T_V(D) \to T_V(D)|_D \to 0$. The other direction is exactly the same as this. Then the next lemma is easily obtained by the observation.

Lemma 4. *Assume that the exact sequence $0 \to T_D \to T_W|_D \to N_{D|W} \to 0$ splits and that $H^2(T_W)$ vanishes. Then the map $H^0(T_X^1) \to H^2(T_V) \oplus H^2(T_W)$ is zero. In particular, if W is a ruled surface and D is a section in W, then the map $H^0(T_X^1) \to H^2(T_V) \oplus H^2(T_W)$ is zero.*

The next example in [PP] shows that the vanishing of the above map is not enough for the obstruction map to be zero even for a d-semistable case.

Example 5. Let V be a smooth projective surface and D a smooth curve in V. Choose an element $\eta \in H^1(N_{D|V})$. Since $H^1(N_{D|V}) = \operatorname{Ext}^1(\mathcal{O}_D, N_{D|V})$, η corresponds to a vector bundle $\mathcal{E}$ of rank two over D by the extension

$$0 \to N_{D|V} \to \mathcal{E} \to \mathcal{O}_D \to 0.$$

Let $W = \mathbb{P}_D(\mathcal{E})$ and let $X = V \cup_D W$. Then X satisfies d-semistability by its construction. In [PP, §2], Persson and Pinkham prove that the image of the map $H^0(T_X^1) \to H^2(T_{V,D})$, induced by the composition of the maps $H^0(T_X^1) \to H^2(T_X^0) \to H^2(T_{V,D}) + H^2(T_{W,D}) \to H^2(T_{V,D})$ where the last map is the projection, is same as the image of η in $H^2(T_{V,D})$ induced by the exact sequence (2.1). Therefore if η is not in the image of the map $n : H^1(T_V) \to H^1(N_{D|V})$, then there is no smoothing of X.

Lemma 6. *In Example 5, if we assume that the map $H^1(T_W) \to H^1(N_{D|W})$ is surjective, then the obstructiom map $H^0(T_X^1) \to H^2(T_X^0)$ is zero if and only if η is the image of n for $\eta \in H^1(N_{D|V})$.*

Proof. Since W is a ruled surface, we have the vanishing $H^2(T_W) = 0$. Then the assumption implies that $H^2(T_{W,D}) = 0$ by the exact sequence (2.1). Since W is a ruled surface, the map $H^1(T_W) \to H^1(T_D)$, induced by the splitting of the exact sequence (2.1), is surjective. This splitting also applies to the surjectivity of the map $t : H^1(T_{W,D}) \to H^1(T_D)$ induced by the exact sequence (2.4). Therefore $H^2(T_X^0) = H^2(T_{V,D})$ by the exact sequence (2.6). □

Lemma 7. *In Example 5, if we assume that $H^2(T_V) = 0$, then X has a smoothing if and only if η is the image of n.*

Proof. If X has a smoothing, then the map $H^0(T^1_X) \to H^2(T^0_X)$ is zero. So η is in the image of n by the argument in Example 5. Assume that η is in the image of n. So there is an element θ in $H^1(T_V)$ such that $n(\theta) = \eta$. Since $H^2(T_V) = 0$, we have the θ-direction deformation of V i.e. there is a flat family $\pi : \mathcal{V} \to \Delta$ with $\pi^{-1}(0) = V$ and θ corresponds to the extension $0 \to T_V \to T_{\mathcal{V}}|_V \to \mathcal{O}_V \to 0$. Blow up D in $\mathcal{V}$; then we have a smoothing of X. □

Example 8. Let V_1, W_1 be smooth projective surfaces. Assume that $\sigma : V_1 \to W_1$ is a double covering branched over a finite disjoint union of irreducible smooth curves. Assume that there is a smooth curve D in W_1 outside of branch loci. Write $\sigma^{-1}(D) = D_1 + D_2,\ D \cong D_i$ for $i = 1, 2$. Then the analytic neighborhood of D in W_1 is isomorphic to that of D_1 in V_1, in particular $N_{D|V_1} \cong N_{D|W_1}$. For convenience we also write $D_1 = D$.

Choose a nonzero element η in $H^1(N_{D|V_1}) = H^1(N_{D|W_1})$. Construct $W_2 = \mathbb{P}_D(\mathcal{E})$ where $\mathcal{E}$ corresponds to the extension sheaf of η (Example 5). Let $X = V_1 \cup_D W_2$ and $Y = W_1 \cup_D W_2$. Then one may guess the following statement: X has a smoothing if and only if Y has a smoothing. But this is not true in general by the next example.

Example 9. Choose a smooth curve C with $g(C) \geq 2$ and a finite morphism $f : C \to \mathbb{P}^1$. This induces a finite morphism $\hat{f}$ and a commutative diagram

$$\begin{array}{ccc} C \times D & \xrightarrow{\hat{f}} & \mathbb{P}^1 \times D \\ \downarrow & & \downarrow \\ C & \xrightarrow{f} & \mathbb{P}^1 \end{array}$$

where D is a smooth curve with $g(D) \geq 2$. Then $\hat{f}$ is branched over finite fibers $D_1, \dots, D_n$. Choose a fiber D_0 such that D_0 is outside of $\{D_1, \dots, D_n\}$. Let $V_1 = C \times D$, $W_1 = \mathbb{P}^1 \times D$. Since we choose D_0 outside of the branch loci,

$$N_{D_0|V_1} \cong N_{D_0|W_1} \cong \mathcal{O}_{D_0}.$$

Choose a nonzero element $\eta \in H^1(\mathcal{O}_{D_0})$. Construct $W_2 = \mathbb{P}_{D_0}(\mathcal{E})$. Then $X = V_1 \cup_{D_0} W_2$ has no smoothing by Lemma 10 and Example 5, but $Y = W_1 \cup_{D_0} W_2$ has a smoothing by Lemma 11.

Lemma 10. *Let C, D be smooth curves of genus $g \geq 2$, and let $X = C \times D$. Then the map $n : H^1(T_X) \to H^1(N_{C|X})$ is zero.*

Proof. It holds that $H^1(T_X) = H^1(T_C) \oplus H^1(T_D)$. The map n restricted on $H^1(T_C)$ to $H^1(N_{C|X})$ is derived from the splitting of the long exact sequence of $0 \to T_C \to T_X|_C \to N_{C|X} \to 0$. Also, the map n restricted on $H^1(T_D)$ to $H^1(N_{C|X})$ is zero since D lies on the fiber of $X \to C$. □

Lemma 11. *In Example 9, $Y = W_1 \cup_{D_0} W_2$ has a smoothing. And the natural map $n : H^1(T_{W_1}) \to H^1(N_{D_0|W_1})$ is surjective.*

Proof. Choose any $\eta \in H^1(N_{D_0|W_1}) = H^1(\mathcal{O}_{D_0})$. Consider a trivial family $\mathcal{W}$ of $W_2 = \mathbb{P}_{D_0}(\mathcal{E})$ where $\mathcal{E}$ corresponds to an extension sheaf of η. Blow up the central fiber; then we have $Y = W_1 \cup_{D_0} W_2$ in the central fiber. So Y has a smoothing. By Lemma 7, η is in the image of n. □

According to the above example, the same analytic neighborhood of the double curve is not enough to determine whether a smoothing exists or not. Let us go back to the question in Example 8. Let $\sigma : V_1 \to W_1$ be a double covering branched over C where C consists of a finite disjoint union of smooth curves. There is a relation between the cohomology of the tangent sheaf of V_1 and the cohomology of the tangent sheaf of W_1 [C2]:

$$0 \to \sigma_*(\sigma^*\Omega_{W_1}(K_{V_1})) \to \sigma_*(\Omega_{V_1}(K_{V_1})) \to \mathcal{O}_C(K_{W_1}) \to 0 \tag{2.8}$$

and it holds the long exact sequence by Serre duality

$$0 \to H^0(T_{V_1}) \to H^0(T_{W_1}) \oplus H^0(T_{W_1}(-L)) \to H^0(\mathcal{O}_C(C)) \to H^1(T_{V_1}) \to \dots$$

where $2L \sim C$. In particular, $H^2(T_{V_1}) = 0$ implies that $H^2(T_{W_1}) = 0$.

Proposition 12. *Consider Example 8 under the assumption $H^2(T_{V_1}) = 0$. Let C be a branched divisor. Assume that $D = \mathbb{P}^1$ or L itself is effective for $2L \sim C$. Then we have the following:*

1. *(1) If X has a smoothing, then Y has a smoothing.*
2. *(2) If Y has a smoothing and the preimage of η in $H^1(T_{W_1})$ goes to zero in $H^1(N_{C|W_1})$, then X has a smoothing.*

Proof. Since we have the vanishing $H^2(T_{V_1}) = H^2(T_{W_1}) = 0$, the smoothability of X and Y is determined by the image of the following natural maps by Lemma 7:

$$n_1 : H^1(T_{V_1}) \to H^1(N_{D|V_1}) \quad n_2 : H^1(T_{W_1}) \to H^1(N_{D|W_1}).$$

Since $D = \mathbb{P}^1$ or L is effective and since D is outside of the branched divisor C, we have $\sigma_*\sigma^* N_{D|W_1}(-L) = N_{D|W_1}$.

From the commutative diagram (in the diagram, the vertical map is induced by the dual sequence of the exact sequence (2.8))

$$\begin{array}{ccc} \sigma_* T_{V_1} & \longrightarrow & \sigma_* N_{D|V_1} \\ \downarrow & & \| \\ \sigma_*\sigma^* T_{W_1}(-L) & \longrightarrow & \sigma_*\sigma^* N_{D|W_1}(-L) = N_{D|W_1} \\ \downarrow & & \\ N_{C|W_1} & & \end{array} \tag{2.9}$$

the following commutative diagram in the long exact sequence of cohomology holds:

$$\begin{array}{ccc} H^1(T_{V_1}) & \longrightarrow & H^1(N_{D|V_1}) \\ \downarrow & & \| \\ H^1(T_{W_1}) \oplus H^1(T_{W_1}(-L)) & \longrightarrow & H^1(N_{D|W_1}) \\ \downarrow & & \\ H^1(N_{C|W_1}) & & \end{array} \tag{2.10}$$

By the splitting of the eigenspaces of $\mathbb{Z}_2$-action, the maps of $H^1(T_{W_1}(-L)) \to H^1(N_{D|W_1})$ and of $H^1(T_{W_1}(-L)) \to H^1(N_{C|W_1})$ are zero. Then the proof is obtained by the commutative diagram (2.10) and Lemma 7. □

By Proposition 12, we finish the proof of the Theorem.

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