

A NOTE ON IDEMPOTENTS IN FINITE AW*-FACTORS

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ABSTRACT. We prove that the value of the quasi-trace on an idempotent element in an AW*-factor of type II₁ is the same as the dimension of its left (or right) support.

It is a long-standing open problem (due to Kaplansky) to prove that *an AW*-factor of type II₁ is in fact a von Neumann algebra*. A remarkable answer, in the affirmative, was found by Haagerup ([Ha]), who proved that if an AW*-factor A is generated by an exact C*-algebra, then A is indeed a von Neumann algebra.

The main object, that was investigated in connection with Kaplansky's problem, is the *quasi-trace*, whose construction we briefly recall below.

One starts with an AW*-factor of type II₁, say A . Denote by $\mathcal{P}(A)$ the collection of projections in A , that is

$$\mathcal{P}(A) = \{p \in A : p = p^* = p^2\}.$$

A key fact is then the existence of a (unique) *dimension function* $D : \mathcal{P}(A) \rightarrow [0, 1]$ with the following properties:

- $D(p) = D(q) \iff p \sim q$;
- if $p \perp q$, then $D(p + q) = D(p) + D(q)$;
- $D(1) = 1$.

The symbol “ \sim ” denotes the Murray-von Neumann equivalence relation ($p \sim q \iff \exists x \in A$ with $p = x^*x$ and $q = xx^*$), while “ \perp ” denotes the orthogonality relation ($p \perp q \iff pq = 0$; this implies that $p + q$ is again a projection).

Once the dimension function is defined, it is extended to self-adjoint elements with finite spectrum. More explicitly, if $a \in A$ is self-adjoint with finite spectrum, then there are (real) numbers $\alpha_1, \dots, \alpha_n$ and pairwise orthogonal projections p_1, \dots, p_n , such that $a = \sum_{k=1}^n \alpha_k p_k$. We then define $d(a) = \sum_{k=1}^n \alpha_k D(p_k)$.

For an arbitrary self-adjoint element $a \in A$, one can approximate uniformly a with a sequence $(a_n)_{n \geq 1} \in \{a\}''$ of elements with finite spectrum. (Here $\{a\}''$ stands for the AW*-subalgebra generated by a and 1.) It turns out that the limit $q(a) = \lim_{n \rightarrow \infty} d(a_n)$ is independent of the particular choice of $(a_n)_{n \geq 1}$.

Finally, for an arbitrary element $x \in A$, one defines $Q(x) = q(\operatorname{Re} x) + iq(\operatorname{Im} x)$, where $\operatorname{Re} x = \frac{1}{2}(x + x^*)$ and $\operatorname{Im} x = \frac{1}{2i}(x - x^*)$.

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The map $Q : A \rightarrow \mathbb{C}$, defined this way, is the unique one with the properties:

- (i) Q is *linear*, when restricted to *abelian* C^* -subalgebras of A ;
- (ii) $Q(x^*x) = Q(xx^*) \geq 0$, for all $x \in A$;
- (iii) $Q(x) = Q(\operatorname{Re} x) + iQ(\operatorname{Im} x)$, for all $x \in A$;
- (iv) $Q(1) = 1$.

It is obvious that $Q|_{\mathcal{P}(A)} = D$. The map Q is called *the quasi-trace of A* .

It is well known that an AW^* -factor of type II_1 is a von Neumann algebra if and only if its quasi-trace is *linear*. Haagerup's solution for Kaplansky's problem goes through the proof of the linearity of the quasi-trace.

On the one hand, one can easily see that the linearity of the quasi-trace is equivalent to its scalar homogeneity (compare with (i) above):

$$(H) \quad Q(\alpha x) = \alpha Q(x), \text{ for all } x \in A, \alpha \in \mathbb{C}.$$

Notice that (H) holds when either $\alpha \in \mathbb{R}$ or when x is *normal*. On the other hand, it is again easy to note that the linearity of the quasi-trace is equivalent to the similarity invariance property

$$(S) \quad Q(sxs^{-1}) = Q(x), \text{ for all } x \in A, s \in GL(A).$$

(Here $GL(A)$ denotes the group of invertible elements in A .) Notice that (S) is true if s is *unitary*.

The purpose of this note is to prove that *both (H) and (S) hold if $x \in A$ is an idempotent (i.e. $x^2 = x$)*.

1. *Notations.* If A is an AW^* -algebra, for an element $x \in A$, we denote by $\mathbf{L}(x)$ (resp. $\mathbf{R}(x)$) the *left* (resp. *right support*) of x . Recall that both $\mathbf{L}(x)$ and $\mathbf{R}(x)$ are projections, and moreover we have

$$\mathbf{L}(x) \sim \mathbf{R}(x), \text{ for all } x \in A.$$

It is known (see [Ka]) that, for every $x \in A$, there exists a unique partial isometry v such that

- (i) $x = v(x^*x)^{1/2}$;
- (ii) $vv^* = \mathbf{L}(x)$;
- (iii) $v^*v = \mathbf{R}(x)$.

(This property is referred to as the *Polar Decomposition*.)

2. *Remark.* If A is a finite AW^* -algebra, then the group $GL(A)$, of invertible elements, is dense in A in the norm topology.

Indeed, on the one hand, since $\mathbf{L}(x) \sim \mathbf{R}(x)$, by the finiteness assumption we also have $1 - \mathbf{L}(x) \sim 1 - \mathbf{R}(x)$. In particular, there exists a partial isometry $w \in A$ such that $1 - \mathbf{L}(x) = ww^*$ and $1 - \mathbf{R}(x) = w^*w$. On the other hand, if we take $x = v(x^*x)^{1/2}$ to be the polar decomposition described above, then we obviously have $w^*v = 0$, so the element $u = v + w$ is *unitary*, and we still have $x = u(x^*x)^{1/2}$. Then, for every $\varepsilon > 0$ the positive element $(x^*x)^{1/2} + \varepsilon 1$ is invertible, and so is $u\{(x^*x)^{1/2} + \varepsilon 1\}$. The result then follows from the obvious equality $\|x - u\{(x^*x)^{1/2} + \varepsilon 1\}\| = \varepsilon$.

3. *Notation.* For a C^* -algebra A and an integer $n \geq 2$, we denote by $M_n(A)$ the C^* -algebra of $n \times n$ matrices with coefficients in A .

The key technical result in this paper is the following.

4. Lemma. *Suppose A is a unital C^* -algebra and $x \in GL(A)$. Then there exist a unitary element $U \in M_2(A)$ and elements $y, z \in A$, such that*

$$(1) \quad U^* \begin{bmatrix} 2 & x \\ 0 & 0 \end{bmatrix} U = \begin{bmatrix} 1 & y \\ z & 1 \end{bmatrix}.$$

Proof. Consider the function defined by

$$f(t) = \frac{(1-t)^2}{t}, \quad 0 < t < 1.$$

It is obvious that $f : (0, 1) \rightarrow (0, \infty)$ is a homeomorphism. Since x is invertible, the spectrum of xx^* is contained in $(0, \infty)$, so by functional calculus there exists an invertible positive element $w \in A$, with $\|w\| < 1$, such that $f(w) = xx^*$, which means

$$(2) \quad xx^* = (1-w)^2w^{-1}.$$

Define the elements

$$\begin{aligned} a &= (1+w)^{-1/2}w^{1/2}, & b &= (1+w)^{-1/2}, \\ c &= x^{-1}(1+w)^{-1/2}w^{-1/2}(1-w), & d &= -x^{-1}(1+w)^{-1/2}(1-w). \end{aligned}$$

First, we have

$$(3) \quad aa^* + bb^* = (1+w)^{-1/2}[w+1](1+w)^{-1/2} = 1,$$

and using (2) we also have

$$(4) \quad \begin{aligned} cc^* + dd^* &= x^{-1}(1+w)^{-1/2}(1-w)[w^{-1}+1](1-w)(1+w)^{-1/2}(x^*)^{-1} \\ &= x^{-1}(1-w)^2w^{-1}(x^*)^{-1} = x^{-1}(xx^*)(x^*)^{-1} = 1. \end{aligned}$$

Secondly, since by taking inverses, (2) yields

$$(5) \quad (x^*)^{-1}x^{-1} = w(1-w)^{-2},$$

so we also get

$$\begin{aligned} (6) \quad a^*a + c^*c &= w^{1/2}(1+w)^{-1/2}[1+(1-w)(x^*)^{-1}x^{-1}(1-w)](1+w)^{-1/2}w^{1/2} \\ &= w^{1/2}(1+w)^{-1/2}[1+w^{-1}](1+w)^{-1/2}w^{1/2} = 1, \\ b^*b + d^*d &= (1+w)^{-1/2}[1+(1-w)(x^*)^{-1}x^{-1}(1-w)](1+w)^{-1/2} \\ (7) \quad &= (1+w)^{-1/2}1+w^{-1/2}1. \end{aligned}$$

Finally, we notice that

$$(8) \quad ac^* + bd^* = (1+w)^{-1/2}[w^{1/2}w^{-1/2} - 1](1-w)(1+w)^{-1/2}(x^*)^{-1} = 0,$$

and using (5) we also have

$$(9) \quad \begin{aligned} a^*b + c^*d &= (1+w)^{-1/2}[w^{1/2} - w^{-1/2}(1-w)(x^*)^{-1}x(1-w)](1+w)^{-1/2} \\ &= (1+w)^{-1/2}[w^{1/2} - w^{-1/2}w](1+w)^{-1/2} = 0. \end{aligned}$$

If we define the matrix $U = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then (3), (4), and (8) give $UU^* = I$, while (6), (7), and (9) give $U^*U = I$, so U is indeed unitary. (Here I denotes the unit in $M_2(A)$.)

Let us now observe that

$$(10) \quad \begin{aligned} 2a + xc &= 2(1 + w)^{-1/2}w^{1/2} + (1 + w)^{1/2}(1 - w)w^{-1/2} \\ &= (1 + w)^{-1/2}(w^{1/2} + w^{-1/2}) = a + bw^{-1/2}, \end{aligned}$$

$$(11) \quad \begin{aligned} 2b + xd &= 2(1 + w)^{-1/2} - (1 + w)^{-1/2}(1 - w) \\ &= (1 + w)^{-1/2}(w + 1) = aw^{1/2} + b, \end{aligned}$$

$$(12) \quad c + dw^{-1/2} = x^{-1}(1 + w)^{-1/2}[(1 - w)w^{-1/2} - (1 - w)w^{-1/2}] = 0,$$

$$(13) \quad cw^{1/2} + d = (c + dw^{-1/2})w^{1/2} = 0.$$

These equalities prove exactly that

$$\begin{bmatrix} 2 & x \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & w^{1/2} \\ w^{-1/2} & 1 \end{bmatrix} \cdot \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

□

The next result is a particular case of the main result.

5. Lemma. *Let A be an AW^* -factor of type II_1 , and let $e \in A$ be an idempotent with $D(\mathbf{L}(e)) = \frac{1}{2}$. Then, for any $\lambda \in \mathbb{C}$, we have*

$$(14) \quad Q(\lambda e) = \lambda/2.$$

Proof. Denote, for simplicity, the projection $\mathbf{L}(x)$ by p . The assumption is that $p \sim 1 - p$. Then we have a $*$ -isomorphism $\Phi : A \rightarrow M_2(pAp)$, such that $\Phi(p) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. It is obvious that, since $e = pe$, there exists an element $x \in pAp$ such that $\Phi(e) = \begin{bmatrix} 1 & 0 \\ x & 0 \end{bmatrix}$. By Remark 2, we can find a sequence $(x_n)_{n \geq 1}$ of invertible elements in pAp (the unit in pAp is p) with $\lim_{n \rightarrow \infty} \|x_n - x\| = 0$. Define the sequence

$$E_n = \begin{bmatrix} 1 & x_n \\ 0 & 0 \end{bmatrix} \in M_2(pAp), \quad n \geq 1.$$

By Lemma 4, one can find two sequences $(y_n)_{n \geq 1}$ and $(z_n)_{n \geq 1}$ in pAp , and a sequence of unitaries $(U_n)_{n \geq 1} \subset M_2(pAp)$, such that

$$(15) \quad E_n = U_n \begin{bmatrix} \frac{1}{2} & y_n \\ z_n & \frac{1}{2} \end{bmatrix} U_n^*, \quad \text{for all } n \geq 1.$$

Define $e_n = \Phi^{-1}(E_n)$, $u_n = \Phi^{-1}(U_n)$, and $a_n = u_n^* e_n u_n - \frac{1}{2}1$, $n \geq 1$, so that we have

$$(16) \quad \Phi(a_n) = \begin{bmatrix} 0 & y_n \\ z_n & 0 \end{bmatrix}, \quad \text{for all } n \geq 1.$$

Now fix a complex number λ . On the one hand, we have

$$\lambda e_n = u_n \left(\frac{\lambda}{2} + a_n \right) u_n^* = \frac{\lambda}{2}1 + u_n a_n u_n^*, \quad \text{for all } n \geq 1.$$

This gives, for every $n \geq 1$, the equalities

$$(17) \quad \operatorname{Re}(\lambda e_n) = \left(\operatorname{Re} \frac{\lambda}{2} \right) 1 + \operatorname{Re}(\lambda u_n a_n u_n^*),$$

$$(18) \quad \operatorname{Im}(\lambda e_n) = \left(\operatorname{Im} \frac{\lambda}{2} \right) 1 + \operatorname{Im}(\lambda u_n a_n u_n^*).$$

Notice however that, using the *unitary invariance* (property (S) for s unitary), together with (17) and (18), gives, for every $n \geq 1$, the equalities

$$\begin{aligned}
 Q(\lambda e_n) &= Q((\operatorname{Re} \frac{\lambda}{2})1 + \operatorname{Re}(\lambda u_n a_n u_n^*)) + iQ((\operatorname{Im} \frac{\lambda}{2})1 + \operatorname{Im}(\lambda u_n a_n u_n^*)) \\
 (19) \quad &= \operatorname{Re} \frac{\lambda}{2} + Q(\operatorname{Re}(\lambda u_n a_n u_n^*)) + i\operatorname{Im} \frac{\lambda}{2} + iQ(\operatorname{Im}(\lambda u_n a_n u_n^*)) \\
 &= \frac{\lambda}{2} + Q(\lambda u_n a_n u_n^*) = \frac{\lambda}{2} + Q(u_n(\lambda a_n)u_n^*) = \frac{\lambda}{2} + Q(\lambda a_n).
 \end{aligned}$$

On the other hand, if we define $v = 1 - 2p$ (which is obviously a unitary in A), then for all $n \geq 1$ we have

$$\Phi(v(\lambda a_n)v^*) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} 0 & \lambda y_n \\ \lambda z_n & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & -\lambda y_n \\ -\lambda z_n & 0 \end{bmatrix} = \Phi(-\lambda a_n).$$

This gives

$$(20) \quad v(\lambda a_n)v^* = -\lambda a_n, \text{ for all } n \geq 1.$$

Using the unitary invariance, combined with *real scalar homogeneity* (property (H) with $\alpha \in \mathbb{R}$), equality (20) gives

$$Q(\lambda a_n) = Q(v(\lambda a_n)v^*) = Q(-\lambda a_n) = -Q(\lambda a_n),$$

which forces

$$(21) \quad Q(\lambda a_n) = 0, \text{ for all } n \geq 1.$$

Combining (21) with (19) gives

$$(22) \quad Q(\lambda e_n) = \lambda/2, \text{ for all } n \geq 1.$$

It is obvious that, by construction, we have $\lim_{n \rightarrow \infty} \|E_n - \Phi(e)\| = 0$, which means that $\lim_{n \rightarrow \infty} \|e_n - e\| = 0$. Using the norm continuity of the quasi-trace (see [BH]), combined with (22), gives the desired result. \square

We are now ready to prove the main result.

6. Theorem. *Let A be an AW*-factor of type II_1 , and let $e \in A$ be an idempotent element. Then, for any $\lambda \in \mathbb{C}$, one has the equality*

$$(23) \quad Q(\lambda e) = \lambda D(\mathbf{L}(e)).$$

Proof. The proof will be carried out in two steps.

PARTICULAR CASE: Assume $D(\mathbf{L}(e)) \leq \frac{1}{2}$.

Denote, for simplicity $\mathbf{L}(e)$ by p , and $\mathbf{R}(e)$ by q . By the *Parallelogram Law* (see [Ka]) we have

$$p \vee q - p \sim q - p \wedge q.$$

Since $p \sim q$, we get

$$D(p \vee q - p) = D(q - p \wedge q) = D(q) - D(p \wedge q) \leq D(q) = D(p).$$

Using the intermediate value property for D , we can find a projection $r \leq 1 - p \vee q$ such that $D(r) + D(p \vee q - p) = D(p)$. Put $q_0 = r + p \vee q - p$. We have $p \perp q_0$ and $p + q_0 \geq p \vee q$. Let us work in the AW*-algebra $A_0 = (p + q_0)A(p + q_0)$. Obviously A_0 is again an AW*-factor of type II_1 , so it carries its quasi-trace Q_0 . By the uniqueness of the quasi-trace, it is obvious that

$$(24) \quad Q_0(x) = \frac{Q(x)}{D(p + q_0)}, \text{ for all } x \in A_0.$$

Notice that $e \in (p \vee q)A(p \vee q)$, so in particular e belongs to A_0 . In A_0 , we have $\mathbf{L}(e) = p$, and $p \sim q_0 = 1_{A_0} - p$, which means that $D_0(p) = \frac{1}{2}$. (Here D_0 denotes the dimension function in A_0). By Lemma 5, we get

$$Q_0(\lambda e) = \lambda/2,$$

which combined with (24) yields

$$\frac{\lambda}{2} = \frac{Q(\lambda e)}{D(p + q_0)} = \frac{Q(\lambda e)}{2D(p)},$$

which obviously proves (23).

GENERAL CASE. One knows (see [Be]) that $M_2(A)$ is also an AW*-factor of type II_1 . Moreover, if we denote by $Q^{(2)}$ the quasi-trace of $M_2(A)$, we have (as above)

$$(25) \quad Q(x) = 2Q^{(2)}\left(\begin{bmatrix} x & 0 \\ 0 & 0 \end{bmatrix}\right), \text{ for all } x \in A.$$

Define the idempotent $E = \begin{bmatrix} e & 0 \\ 0 & 0 \end{bmatrix} \in M_2(A)$, and the projection $P = \begin{bmatrix} p & 0 \\ 0 & 0 \end{bmatrix} \in M_2(A)$. Using (25) we have $Q(e) = 2Q^{(2)}(E)$. If we denote by $D^{(2)}$ the dimension function of $M_2(A)$, then by (25) we also have $D(p) = 2D^{(2)}(P)$, which gives $D^{(2)}(P) \leq \frac{1}{2}$. Using the obvious equality $\mathbf{L}(E) = P$, by the particular case above applied to $M_2(A)$, we get

$$Q(\lambda e) = 2Q^{(2)}(\lambda E) = 2\lambda D^{(2)}(P) = \lambda D(p).$$

□

7. Corollary. *Let A be an AW*-factor of type II_1 , and let $e \in A$ be an idempotent. Then, for every $s \in GL(A)$, we have*

$$(26) \quad Q(ses^{-1}) = Q(e).$$

Proof. Put $p = \mathbf{L}(e)$. If we define $t = 1 - e(1 - p)$, then t is invertible, and $e = tpt^{-1}$. This computation shows that it is enough to prove (26) in the case when $e = e^*$. Let $q = \mathbf{L}(ses^{-1})$. Arguing as above, there exists some $x \in GL(A)$ such that $ses^{-1} = xqx^{-1}$. So if we put $y = x^{-1}s$, we have $yey^{-1} = q$, with both e and q self-adjoint idempotents. Since this obviously forces $e \sim q$, we get $D(e) = D(q) = Q(ses^{-1})$. □

8. Corollary. *Let A be an AW*-factor of type II_1 and let $e_1, e_2 \in A$ be idempotents, such that $e_1e_2 = e_2e_1 = 0$. (This implies that $e_1 + e_2$ is again an idempotent.) Then*

$$Q(e_1 + e_2) = Q(e_1) + Q(e_2).$$

Proof. Let $p = \mathbf{L}(e_1 + e_2)$. As in the proof of the preceding corollary, there exists an element $s \in GL(A)$, such that $p = s(e_1 + e_2)s^{-1}$. Put $f_k = se_k s^{-1}$, $k = 1, 2$, so that $f_1 + f_2 = p$. But now we also have $pf_k = f_k p = f_k$, $k = 1, 2$, which means in particular that $f_1, f_2 \in pAp$. So, if we work in the AW*-factor (again of type II_1) $A_0 = pAp$, we will have $f_1 + f_2 = 1_{A_0}$. On the one hand, using the notations from the proof of Theorem 6, we have

$$Q_0(f_1) + Q_0(f_2) = 1.$$

On the other hand, we have

$$Q_0(f_k) = \frac{Q(f_k)}{D(p)}, \quad k = 1, 2,$$

so we get

$$D(p) = Q(f_1) + Q(f_2).$$

Finally, using the preceding corollary, we get

$$Q(e_1 + e_2) = D(p) = Q(f_1) + Q(f_2) = Q(se_1s^{-1}) + Q(se_2s^{-1}) = Q(e_1) + Q(e_2).$$

□

9. Comment. A. Theorem 6 can be analyzed from a different point of view, as follows. In principle, one can extend the dimension function D to the collection of all idempotents by $\tilde{D}(e) = D(\mathbf{L}(e))$. One can easily prove that this extended dimension function will have the same properties as the usual dimension function (when Murray-von Neumann equivalence is extended to idempotents). The point of Theorem 6 is then the fact that $\tilde{D} = Q$.

B. It is interesting to note that the linearity of the quasi-trace is equivalent to the following condition:

(*) For any family of idempotents $e_1, \dots, e_n \in A$, such that $e_j e_k = 0$ for all $j, k \in \{1, \dots, n\}$ with $j \neq k$, and any family of numbers $\alpha_1, \dots, \alpha_n \in \mathbb{R}$, one has:

$$Q(\alpha_1 e_1 + \dots + \alpha_n e_n) = \alpha_1 Q(e_1) + \dots + \alpha_n Q(e_n).$$

This will be discussed in a future paper.

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