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# LOWER ESTIMATE FOR THE INTEGRAL MEANS SPECTRUM FOR $p = -1$

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Abstract. In this paper we show that there exists a function $f$ bounded and univalent in the unit disk, such that $\int |f'(re^{i\theta})|^{-1} d\theta \geq C(1-r)^{-0.127}$, $0 \leq r < 1$.

The aim of the paper is to obtain a new lower estimate for the integral means spectrum

$$\beta(p) = \overline{\lim}_{r \to 1} \frac{\log \int |f'(re^{i\theta})|^p d\theta}{\log \frac{1}{1-r}}$$

of bounded univalent functions in $D = \{|z| < 1\}$ for $p = -1$. Rohde [Roh89], [Pom91] proved that there exists a bounded univalent function such that $\beta(-1) \geq 0.109$. Using Carleson-Jones ideas [CJ92], Kraetzer [Kra96] obtained numerical evidence that for every $p \in [-2, 2]$ there exists a bounded univalent function for which $\beta(p) \geq p^2/4$.

In this paper we analytically show that there exists a bounded univalent function $f$ for which $\beta(-1) \geq 0.127$.

Define the function

$$f(z) = z \exp \int_0^z \frac{e^{at} - 1}{t} dt, \quad a = 1.7646.$$

We shall prove that $f$ is univalent in $D = \{|z| < 1\}$ function. Since $f$ is a real function it is enough to show that $L = \{f(e^{i\theta}), 0 < \theta < \pi\}$ is a simple curve and that $L \cap \mathbf{R}$ is empty. It is useful to mention that

$$\frac{d}{d\theta} \log f(e^{i\theta}) = i \exp(ae^{i\theta}).$$

It follows from

$$\frac{d|f|}{d\theta} = -|f| e^{a\cos\theta} \sin(a \sin\theta) < 0, \quad 0 < \theta < \pi,$$

that $L$ is a simple curve.

Consider

$$\frac{d}{d\theta} \arg f(e^{i\theta}) = e^{a\cos\theta} \cos(a \sin\theta).$$

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In our case, the equation $\frac{d}{d\theta}\arg f = 0$ is equivalent to the equation $a\sin\theta = \pi/2$ which has two roots $\theta_1 < \theta_2$ on $(0,\pi)$. Now, it is clear that $Imf(e^{i\theta_1}) > 0$ implies $L \cap \mathbf{R} = \emptyset$. But this follows from a straightforward calculation.

Therefore, $f$ is univalent and bounded in the unit disk $D$. Hence the functions $f_n(z) = f(z^n)^{1/n}$ are also bounded and univalent in $D$. Note that

$$f_n'(z) = \exp\left(az^n + \frac{1}{n}\int_0^{z^n} \frac{e^{at}-1}{t}dt\right).$$

Put

$$\Phi(z) = M^{1+1/q}\lim_{n\to\infty} g_n(z), \quad M = \exp\int_0^1 \frac{e^{at}-1}{t}dt$$

where $g_0(z) = z, g_n(z) = g_{n-1}(M^{-1/q^{n-1}} f_{q^{n-1}}(z))$, $n = 1, 2, ....$

Applying standard methods of geometric function theory it is easy to establish that the function $\Phi$ is well defined, bounded, and univalent in $D$. The idea of using compositions of univalent functions was first used by Pommerenke [Pom91]. At the present time it is a most effective method for constructing patalogic mappings.

We have

$$\log\Phi'(z) = \sum_{k=0}^{\infty}\log f'_{q^k}(\phi_k(z)) = \sum_{k=0}^{\infty}\left(a\phi_k^{q^k}(z) + \frac{1}{q^k}\int_0^{\phi_k^{q^k}(z)}\frac{e^{at}-1}{t}dt\right),$$

where $\phi_k(z) = M^{\frac{-1}{q^k(q-1)}}z + ...$ and $|\phi_k| < 1, z \in D$. Therefore

$$\left|\log\Phi'(z) - \sum_{k=0}^{\infty} a\phi_k^{q^k}(z)\right| \le \text{const}, z \in D.$$

Since the Taylor coefficients of $\phi_k$ are positive then

$$\left|\phi_k(z) - M^{\frac{-1}{q^k(q-1)}}z\right| \le |z|\left(1 - M^{\frac{-1}{q^k(q-1)}}\right)$$

and

$$|\phi_k^{q^k}(z) - M^{\frac{-1}{q-1}}z^{q^k}| \le |\phi_k(z) - M^{\frac{-1}{q^k(q-1)}}z|q^k|z|^{q^k-1} \le \frac{|z|^{q^k}\log M}{q-1}.$$

It is known [Pom91] that

$$\sum_{k=1}^{\infty} r^{q^k} \le \log\frac{1}{1-r}/\log q + \text{const}.$$

Thus,

$$\left|\log\Phi'(z) - \sum_{k=0}^{\infty} aM^{-1/(q-1)}z^{q^k}\right| \le \frac{a\log M}{(q-1)\log q}\log\frac{1}{1-r} + const, \quad r = |z|, \tag{1}$$

and we can prove the following

**Theorem 1.**

$$\int_0^{2\pi}|\Phi'(re^{i\theta})|^{-1}d\theta \ge \text{const}(1-r)^{-0.127}.$$

*Proof.* Define $\log f_*'(z) = \sum_{k=1}^{\infty} aM^{-1/(q-1)} z^{q^k}$. Rohde [Roh89], [Pom91] proved that

$$\int_0^{2\pi} |f_*'(re^{i\theta})|^{-1} d\theta \geq \mathrm{Const}(1-r)^{-\alpha}$$

where $\alpha = \log I_0(aM^{-1/(q-1)})/\log q$ and

$$I_0(x) = \sum_{\nu=0}^{\infty} \frac{x^{2\nu}}{2^{2\nu}\nu!} \text{ is a modified Bessel function.}$$

Now, it follows from (1) that

$$\int_0^{2\pi} |\Phi'(re^{i\theta})|^{-1} d\theta \geq \mathrm{const}(1-r)^{-\gamma}$$

where

$$\gamma = \frac{\log I_0(aM^{-1/(q-1)})}{\log q} - \frac{a \log M}{(q-1)\log q}.$$

With the choice $q = 69$ we obtain our estimate. □

Let us remark that the author [Kay01] used the Koebe function as a starting function for lower estimates when $p$ is positive.

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