PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 130, Number 5, Pages 1371–1376
S 0002-9939(01)06210-4
Article electronically published on October 12, 2001

ON THE FAILURE OF THE FACTORIZATION CONDITION FOR NON-DEGENERATE FOURIER INTEGRAL OPERATORS

MICHAEL RUZHANSKY

(Communicated by Christopher D. Sogge)

Abstract. In this paper we give examples of polynomial phase functions for which the factorization condition of Seeger, Sogge and Stein (Ann. Math. **134** (1991)) fails. The corresponding Fourier integral operators turn out to be still continuous in L^p. We also give examples of the failure of the factorization condition for translation invariant operators. In this setting the frequency space must be at least 5-dimensional, which shows that the examples are optimal. We briefly discuss the stationary phase method for the corresponding operators.

Let X and Y be open subsets of $\mathbb{R}^n$. A Fourier integral operator $T \in I^\mu(X, Y; \Lambda)$ is an operator which can be locally written in the form

$$Tu(x) = \int_Y \int_{\mathbb{R}}^n e^{i\Phi(x,y,\theta)} a(x, y, \theta) u(y) d\theta dy,$$

where $a \in S^\mu(X, Y, \mathbb{R}^n)$ is a symbol of order m, i.e. a smooth function with the property that

$$|\partial_x^\beta \partial_\xi^\alpha a(x, y, \theta)| \leq C(1 + |\xi|)^{\mu - |\alpha|},$$

locally uniformly in x, y for all multi-indices α and β. The canonical relation Λ is a conic Lagrangian submanifold of the cotangent bundle $T^*(X \times Y)\backslash 0$ with the symplectic form $\sigma_X \bigoplus -\sigma_Y$, where σ_X and σ_Y are the canonical symplectic forms in T^*X and T^*Y respectively. Let $\pi_{X\times Y}$ be the canonical projection from $T^*(X \times Y)$ to $X \times Y$. The canonical relation Λ can be locally parametrized as the set of points

$$\Lambda_\Phi = \{(x, y, d_x\Phi, d_y\Phi) : d_\theta\Phi(x, y, \theta) = 0\}.$$

We will assume that Λ is a local canonical graph, which means that $\partial_y\partial_\theta\Phi$ is a non-degenerate matrix. The regularity properties of Fourier integral operators are related to the geometric properties of Λ. Let Λ satisfy the smooth factorization condition. This means that for every $\lambda = (x_0, y_0, \xi_0, \eta_0) \in \Lambda$ there is a conic neighborhood Λ_0 of λ_0 in Λ and a smooth map $\pi_{\lambda_0} : \Lambda_0 \to \Lambda$ homogeneous of degree 0 such that $\operatorname{rank} d\pi_{\lambda_0} \equiv n + k$ and $\pi_{X\times Y}|_{\Lambda_0} = \pi_{X\times Y} \circ \pi_{\Lambda_0}$, for some k. Under this condition it was shown in [7] that operators $T \in I^\mu(X, Y; \Lambda)$ are bounded from $L^p_{comp}(Y)$ to $L^p_{loc}(Y)$, provided that $1 < p < \infty$ and $\mu \leq -k|1/p - 1/2|$. It was also shown in [4] that the order $-k|1/p-1/2|$ is optimal; that is, for $\mu > -k|1/p-1/2|$ elliptic operators $T \in I^\mu(X, Y; \Lambda)$ are not bounded in L^p. For general properties of

Received by the editors June 22, 1999 and, in revised form, October 30, 2000.
1991 *Mathematics Subject Classification.* Primary 35A20, 35S30, 58G15, 32D20.

operators with ranks k and their relation to the singularity theory of affine fibrations we refer to [5] for the real valued phase functions, and to [6] for the complex valued phase functions, respectively. For the backgrounds on the L^p theory of Fourier integral operators we refer to [8], [9], and a survey [5] for smaller ranks k.

In this paper we will consider a case for which the factorization condition fails. We give an example of an operator for which the factorization condition fails but the L^p result holds. Such a family of phase functions was suggested in [3]. Let X, Y be open subsets of $\mathbb{R}^3$ and define Φ by

$$\Phi(x,y,\xi) = \langle x-y,\xi\rangle - \frac{1}{\xi_3}(y_1\xi_1 + y_2\xi_2)^2 \tag{1}$$

in the cone $|(\xi_1,\xi_2)| \leq C|\xi_3|$ for some $C > 0$. The factorization condition clearly fails for this phase function. The maximal rank of $d\pi_{X\times Y}|_\Lambda$ equals 4, so $k = 1$ and it follows from [4] that the best order for the L^p continuity can be $-|1/p - 1/2|$.

Theorem 1. *Let $T \in I^\mu(\mathbb{R}^3, \mathbb{R}^3; \Lambda_\Phi)$ with Λ_Φ defined by* (1). *Then T is bounded from $L^p_{comp}(Y)$ to $L^p_{loc}(Y)$, provided that $1 < p < \infty$ and $\mu \leq -|1/p - 1/2|$.*

This statement can be generalized to higher dimensions, but this is not the purpose of the paper. Our point is to present operators for which the factorization condition fails but L^p estimates are still valid.

We will give a brief proof of this result based on the technique and notations of [7]. The operator T is defined by

$$Tu(x) = \int\int e^{i\Phi(x,y,\xi)} b(x,y,\xi)u(y)d\xi dy \tag{2}$$

and the support of its symbol $b(x,y,\xi)$ away from $\xi_3 = 0$. The set $\Sigma = \pi_{X\times Y}(\Lambda_\Phi) \subset \mathbb{R}^3 \times \mathbb{R}^3$ can be represented by the set of points $(\nabla_\xi \phi(y,\xi), y)$ parametrized by ξ, where

$$\phi(y,\xi) = \langle y,\xi\rangle + \frac{1}{\xi_3}(y_1\xi_1 + y_2\xi_2)^2.$$

In a neighborhood of $x = y$ the set Σ_y after a choice $\sigma = (y_1\xi_1 + y_2\xi_2)/\xi_3$ can be parametrized by

$$\Sigma_y = \{(y_1 + 2y_1\sigma, y_2 + 2y_2\sigma, y_3 - \sigma^2)\}.$$

By the analytic interpolation technique it is sufficient to check that operators in (2) are bounded from the Hardy space H^1 to L^1 when the order of b is $-1/2$ (see [9] or [7] for details). From the atomic decomposition of Hardy space, it is sufficient to check $||Ta_Q||_{L^1} \leq C$ for any atom a_Q with C independent of a_Q and a cube Q with a small sidelength. Recall that a_Q is supported in a cube Q and satisfies $|a_Q| \leq |Q|^{-1}$ almost everywhere as well as the cancellation property $\int a_Q(x)dx = 0$.

From the atomic decomposition we need to consider only the atoms a_Q with Q containing the points where the rank of $\phi''_{\xi\xi}$ drops, because otherwise T is conormal in Q. We also assume $|Q| \leq 1$. The singularities of Σ occur at the points $y = (0,0,y_3)$. For $y_1^2 + y_2^2 \neq 0$ we denote by N^y a tubular neighborhood of Σ_y with width $|Q|^{2/3}$. Then $|N^y| \leq c|Q|^{2/3}$. Now, for y with $y_1 = y_2 = 0$ the singular set Σ_y is a point, but we enlarge it by taking a limit of Σ_y with $y_1^2 + y_2^2 \to 0$, and the limit is a straight ray from y. For these y we define N^y in a similar way as a tubular neighborhood of this ray with width $|Q|^{2/3}$. Finally, we define

$$N_Q = \bigcup_{y \in Q} N^y.$$

The size of N_Q is $|N_Q| \leq C|Q|^{2/3}$. Now we want to estimate the L^1 norm of Ta_Q on N_Q. By Cauchy-Schwartz inequality we have

$$||Ta_Q||_{L^1(N_Q)} \leq C|Q|^{1/3}||Ta_Q||_{L^2(N_Q)}.$$

Since T is of order $-1/2$, the operator $T(I-\Delta)^{1/4}$ is bounded on L^2, and hence we get

$$||Ta_Q||_{L^2} \leq C||(I-\Delta)^{-1/4}a_Q||_{L^2} \leq C||a_Q||_{p_n},$$

the last inequality following from the Hardy-Littlewood-Sobolev inequality with $p_n = 3/2$. Now, using $||a_Q||_\infty \leq |Q|^{-1}$, we obtain $||a_Q||_{3/2} \leq |Q|^{-1}|Q|^{2/3}$, so that

$$||Ta_Q||_{L^1(N_Q)} \leq C|Q|^{1/3}|Q|^{-1}|Q|^{2/3} \leq C.$$

Next we want to estimate in $L^1(\mathbb{R}^3 \backslash N_Q)$.

1. First we decompose T into a finite sum

$$T = \sum_{l=1}^{L} T_l, \tag{3}$$

so that the fibers of the Lagrangian for each T_l are close to each other. Let $\{R_l\}_1^L$ be a decomposition of (y_1, y_2)-space $\mathbb{R}^2$ into sectors of equal angle $2\pi/L$ and all the lines starting from zero. Let α_l be a partition of unity, homogeneous of degree 0 and related to $R_l \cap S^1$. Define $T_l = T \circ \alpha_l$, where α_l means multiplication by it. Then (3) holds and it is enough to make estimates for some T_l. In view of this decomposition we will assume further that the symbol $b(x, y, \xi)$ of T in (2) is supported in some R_l with respect to y_1 and y_2. The set of $(0, 0, y_3)$ is of measure zero, so we can exclude it from the decomposition.

2. Now we make a dyadic decomposition in ξ-space. Let $\beta \in C_0^\infty((1/2, 2))$ satisfy $\sum_{-\infty}^{\infty} \beta(2^{-k}s) = 1$, $s > 0$. We define

$$b_\lambda(x, y, \xi) = \beta(|\xi|/\lambda)b(x, y, \xi)$$

and

$$T_\lambda u(x) = \int\int e^{i\Phi(x,y,\xi)} b_\lambda(x, y, \xi)u(y)d\xi dy.$$

The corresponding dyadic decomposition of T is now

$$T = \sum_{k \geq 1} T_{2^k}. \tag{4}$$

3. We will also need a further angular decomposition of T_λ. In order to accomplish it we make a partition of the unit sphere in ξ-space, related to the smooth factorization property. Let Γ be a narrow cone in ξ-space, containing the support of b. For each $y \in \overset{\circ}{R_l}$ there exists an r-dimensional submanifold $S_r(y)$, $r = 1$, of $S^2 \cap \Gamma$, such that $S^2 \cap \Gamma$ is parametrized by $\bar{\xi} = \bar{\xi}_y(u, v)$ for (u, v) in a bounded open set $U \times V$ near $(0, 0) \in \mathbb{R} \times \mathbb{R}$. Furthermore,

$$\bar{\xi}_y(u, v) \in S_r(y) \Leftrightarrow v = 0$$

and

$$\nabla_\xi \phi\left(y, \bar{\xi}_y(u,v)\right) = \nabla_\xi \phi\left(y, \bar{\xi}_y(u,0)\right),$$

so that v defines a parametrization of the fibers. The set $S_r(y)$ depends smoothly on $y \in \overset{\circ}{R}_l$ but has unbounded variation as $y_1^2 + y_2^2 \to 0$. However, the image set of $S_r(y)$ can be made small by choosing large L.

This implies that U is bounded uniformly with respect to $y \in \overset{\circ}{R}_l$ and for any $\lambda = 2^k$, $k > 0$ we can choose u_λ^ν, $\nu = 1, 2, \dots, N(\lambda)$, such that $|u_\lambda^\nu - u_\lambda^{\nu'}| \geq C_0 \lambda^{-1/2}$ for $\nu \neq \nu'$, and such that U is covered by balls with center u_λ^ν and radius $C_1 \lambda^{-1/2}$. Note that $N(\lambda) = O(\lambda^{1/2})$.

4. We introduce homogeneous partitions of unity of $\mathbb{R}^3 \backslash 0$ that depend on the scale λ of the dyadic decomposition. First let $\bar{\chi}_\lambda^\nu$ be a smooth partition of unity in U, satisfying $||D_u^\gamma \bar{\chi}_\lambda^\nu||_\infty = O(\lambda^{|\gamma|/2})$, and having the natural support properties associated to the partition u_λ^ν, namely $\bar{\chi}_\lambda^\nu(u_\lambda^\nu) = 1$ and $\bar{\chi}_\lambda^\nu(u) = 0$ if $|u - u_\lambda^\nu| \geq C\lambda^{-1/2}$. Then we define a corresponding partition of unity on Γ by $\chi_\lambda^\nu(s\bar{\xi}_y(u,v)) = \bar{\chi}_\lambda^\nu(u)$, $s > 0$.

The idea behind this decomposition is that the χ_λ^ν have the largest possible angular support so that $\xi \to \phi$ behaves like a linear function on supp b_λ^ν, where T_λ^ν is an operator with kernel

$$K_\lambda^\nu(x,u) = \int e^{i[\langle x,\xi\rangle - \phi(y,\xi)]} b_\lambda^\nu(x,y,\xi) d\xi, \; b_\lambda^\nu(x,y,\xi) = \chi_\lambda^\nu(\xi) b_\lambda(x,y,\xi).$$

5. On the support of $b_\lambda^\nu(x,y,\xi)$ the idea is to replace the function $\phi(y,\xi)$ by its linear approximation $\langle \nabla_\xi \phi(y, \bar{\xi}_y(u_\lambda^\nu, 0), \xi\rangle$. We define

$$r_\lambda^\nu(y,\xi) = \phi(y,\xi) - \langle \nabla_\xi \phi(y, \bar{\xi}_y(u_\lambda^\nu, 0), \xi\rangle.$$

Then for $N \geq 1$ and ξ in the support of $b_\lambda^\nu(x,y,\xi)$ the following holds:

$$|(\langle \nabla_\xi, \bar{\xi}_y(u_\lambda^\nu, 0)\rangle)^N r_\lambda^\nu(y,\xi)| \leq C_N \lambda^{-1} |\xi|^{1-N}, \tag{5}$$

$$D_\xi^\alpha r_\lambda^\nu(y,\xi)| \leq C_N \min\{\lambda^{-1/2}, |\xi|^{1-N}\}, \; |\alpha| = N. \tag{6}$$

Note that the term $|\xi|^{1-N}$ corresponds to the homogeneous behavior of $r_\lambda^\nu(y,\xi)$. So we need to show it for $\xi \in S^2 \cap$ supp χ_λ^ν. In view of Euler formula $r_\lambda^\nu(y, \bar{\xi}_y(u_\lambda^\nu, 0)) = 0$ and $\nabla_\xi r_\lambda^\nu(y, \bar{\xi}_y(u_\lambda^\nu, 0)) = 0$, so that the Taylor expansion of $r_\lambda^\nu(y,\xi)$ around $\bar{\xi}_y(u_\lambda^\nu, 0)$ implies (5) since $|\xi - \bar{\xi}_y(u_\lambda^\nu, 0)| \leq C\lambda^{-1/2}$ for $\xi \in S^2 \cap$ supp χ_λ^ν. Similarly, for (6) we observe that $|\xi|^{1-N} \leq C\lambda^{-1/2}$ if $N > 1$ and $\xi \in$ supp b_λ^ν and $\nabla_\xi r_\lambda^\nu(y,\xi) = \nabla_\xi \phi_\lambda^\nu(y,\xi) - \nabla_\xi \phi_\lambda^\nu(y, \bar{\xi}_y(u_\lambda^\nu, 0)) = O(\lambda^{-1/2})$.

6. Finally, we define

$$\tilde{b}_\lambda^\nu(x,y,\xi) = e^{ir_\lambda^\nu(y,\xi)} b_\lambda^\nu(x,y,\xi).$$

By a rotation, we assume that for every $\xi \in \Gamma$ there is the splitting $\xi = (\xi', \xi'') \in \mathbb{R} \times \mathbb{R}^2$, such that ξ'' is normal to $S_r(y)$ at $\bar{\xi}_y(u_k^\nu, 0)$. The stationary phase partial integrations are performed with a selfadjoint operator

$$L_\lambda^\nu = (I - \lambda\langle \nabla_\xi', \nabla_\xi'\rangle)(I - \lambda^2 \langle \nabla_\xi'', \nabla_\xi''\rangle).$$

The estimates for $\bar{\chi}_\lambda^\nu$ and the fact that $b(x,y,\xi)$ is of order $-1/2$ imply that

$$|(L_\lambda^\nu)^N b_\lambda^\nu(x,y,\xi)| \leq C_N \lambda^{-1/2}.$$

Furthermore, the same estimate holds for $\tilde{b}^\nu_\lambda(x,y,\xi)$ instead of $b^\nu_\lambda(x,y,\xi)$ in view of (5) and (6). Integration by parts gives

$$K^\nu_\lambda(x,y) = H^\nu_{N,\lambda}(x,y)\int e^{i\langle x-\nabla_\xi\phi(y,\bar{\xi}_y(u^\nu_\lambda,0),\xi\rangle}(L^\nu_\lambda)^N\tilde{b}^\nu_\lambda(x,y,\xi)d\xi, \tag{7}$$

where

$$H^\nu_{N,\lambda}(x,y) \\ = (1+\lambda|(x-\nabla_\xi\phi(y,\bar{\xi}_y(u^\nu_\lambda,0))'|^2)^{-N}(1+\lambda^2|(x-\nabla_\xi\phi(y,\bar{\xi}_y(u^\nu_\lambda,0))''|^2)^{-N}.$$

The estimate for $|(L^\nu_\lambda)^N\tilde{b}^\nu_\lambda(x,y,\xi)|$ and the fact that the support in the integral in (7) has volume $O(\lambda^{1/2}\lambda^2)$ imply that

$$|K^\nu_\lambda(x,y)| \le C_N\lambda^2 H^\nu_{N,\lambda}(x,y).$$

7. For fixed value of y this implies

$$\int_{\mathbb{R}^3}|K^\nu_\lambda(x,y)|dx \le C\lambda^2\lambda^{-1/2}\lambda^{-2} = C\lambda^{-1/2}.$$

The number of kernels $K^\nu_\lambda(x,y)$ was $N(\lambda)=O(\lambda^{1/2})$, so that we obtain

$$\int_{\mathbb{R}^3}|K_\lambda(x,y)|dx \le C.$$

Similarly, with the decompositions above, the rest of the proof is standard as in [7].

The statement of Theorem 1 can be clearly generalized to higher dimensions. The point of this paper is, however, to show that the optimal L^p estimates hold even in some cases when the factorization condition fails. Let us now give an example of a translation invariant operator for which the factorization condition fails. It is an interesting problem to investigate the L^p properties of such a Fourier integral operator.

By the invariant wave front we mean the wave front of a translation invariant distributional kernel $K(x,y)=K(x-y)$ which corresponds to the convolution operators. Let

$$\Phi(x,y,\xi) = \langle x,\xi\rangle - \phi(y,\xi) \tag{8}$$

be a non-degenerate phase function, smooth in y and positively homogeneous of degree one in ξ, where x,y,ξ are in some open subsets of $\mathbb{R}^n$, and $\det\phi''_{y\xi}\neq 0$. Let U be open in $\mathbb{R}^n\times\mathbb{R}^n$ and let $k=\max_{(y,\xi)\in U}\operatorname{rank}\phi''_{\xi\xi}(y,\xi)$. Denote by $U^{(k)}$ the set of $(y,\xi)\in U$ such that the rank is equal to k. By the implicit function theorem the mapping $\kappa : U^{(k)}\ni(y,\xi)\mapsto\ker\phi''_{\xi\xi}(y,\xi)\in\mathbb{G}_{n-k}(\mathbb{R}^n)$ is smooth. The factorization condition then means that κ extends smoothly from $U^{(k)}$ to U. Since Φ is homogeneous, the factorization condition always holds when $k=0$ or $k=n-1$. In terms of the phase function, translation invariance means that $\phi(y,\xi)=\langle y,\xi\rangle - H(\xi)$, where H is smooth and positively homogeneous of degree one. We will write $\kappa(\xi)$ for $\kappa(y,\xi)$. In [5], we have shown that if H is analytic, the smallest dimension when the factorization condition may fail, is $n=5$. Moreover, the factorization condition holds if $k\le 2$. There is a general explanation for it based on the fact that $\nabla\phi$ is constant along $\kappa(\xi)$ ([5]).

Thus, the smallest dimension when it may fail is $n=5$ with $k=3$. Let $m,l\ge 2$ and consider the function

$$\Phi(x,y,\xi) = \langle x-y,\xi\rangle + \xi_1\xi_2^l\xi_5^{-l} + (\xi_3\xi_5-\xi_2\xi_4)^m\xi_5^{1-2m} \tag{9}$$

in a conic neighborhood of $\xi_5 = 1$. At $\xi_5 = 1$, we get

$$\kappa(\xi) = \text{ span } \langle (\frac{l}{m} \frac{(\xi_3 - \xi_2\xi_4)^{m-1}}{\xi_2^{l-1}}, 0, \xi_2, 1) \rangle.$$

Therefore, at $\xi_2 = \xi_3 = 0$ and $\xi_5 = 1$, κ is discontinuous and the factorization condition fails. There is also an obvious generalization of (9) to higher dimensions (see [5]).

It is an interesting problem to verify the regularity properties of the corresponding Fourier integral operators $T \in I^\mu(\mathbb{R}^n, \mathbb{R}^n; \Lambda')$, where $\Lambda = \Lambda_\Phi$ is the graph of a canonical symplectic transformation from $T^*\mathbb{R}^n$ to $T^*\mathbb{R}^n$ (see [2], [7], [6] for the notation). If the singular support of T is not smooth, the factorization condition may still hold. In general, it is shown in [4] that if T is elliptic and continuous from L^p_{comp} to L^p_{loc}, $1 < p < \infty$, then $\mu \leq -k|1/p - 1/2|$. A standard argument for such negative results is based on the stationary phase method which is essentially contained in [1]. Namely, one observes that if $P \in \Psi^{-s}(\mathbb{R}^n)$ is an elliptic properly supported pseudodifferential operator of order $-s$ and type $(1,0)$, then $f = P\delta \in L^p(\mathbb{R}^n)$ when $s > n(1 - 1/p)$ and δ is a standard δ-function. It is sufficient to consider $1 < p < 2$ since the rest follows by taking adjoints. One can analyze $Tf = (T \circ P)\delta$ explicitly to see that μ has to be $\leq -k|1/p - 1/2|$ if T is L^p continuous. It is possible to check in a number of cases that if the phase function of an elliptic operator T is given by (9), then $\mu = -k|1/p - 1/2|$ implies $Tf \in L^p$.

Therefore, since the stationary phase test with functions with point singularities fails for T, it is reasonable to conjecture that if Φ is given by (9), the operators of order $-3|1/p - 1/2|$ are L^p continuous. Note that by the complex interpolation method this would follow from the boundedness from the Hardy space $H^1(\mathbb{R}^5)$ to $L^1(\mathbb{R}^5)$ of operators of order $-3/2$ with phase function (9).

References

1. M. Beals, *L^p boundedness of Fourier integrals,* Mem. Amer. Math. Soc., **264** (1982). MR **84m:**42026
2. J.J. Duistermaat, *Fourier integral operators,* Birkhäuser, Boston, 1996. MR **96m:**58245
3. M. Ruzhansky, *Analytic Fourier integral operators, Monge-Ampère equation and holomorphic factorization,* Arch. Mat., **72** (1999), 68-76. MR **99k:**58178
4. M. Ruzhansky, *On the sharpness of Seeger-Sogge-Stein orders,* Hokkaido Math. J., **28** (1999), 357-362. MR **2000e:**35247
5. M. Ruzhansky, *Singularities of affine fibrations in the regularity theory of Fourier integral operators,* Russian Math. Surveys, **55** (2000), 93-161. CMP 2000:11
6. M. Ruzhansky, *Regularity theory of Fourier integral operators with complex phases and singularities of affine fibrations,* CWI Tract, 131, Amsterdam, 2001.
7. A. Seeger, C.D. Sogge and E.M. Stein, *Regularity properties of Fourier integral operators,* Ann.of Math., **134** (1991), 231–251. MR **92g:**35252
8. C.D. Sogge, *Fourier integrals in classical analysis*, Cambridge Univ. Press, Cambridge, 1993. MR **94c:**35178
9. E.M. Stein, *Harmonic analysis*, Princeton Univ. Press, Princeton, 1993. MR **95c:**42002

Department of Mathematics, Imperial College, 180 Queen's Gate, London SW7 2BZ, United Kingdom

E-mail address: `ruzh@ic.ac.uk`