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TRIANGULAR DERIVATIONS RELATED TO PROBLEMS ON AFFINE n-SPACE

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Abstract. This paper studies the Cancellation Problem, the Embedding Problem, and the Linearization Problem. It shows how these problems can be related to a special class of locally nilpotent derivations.

1. Introduction

Locally nilpotent derivations have shown to be very useful in the study of various problems in algebra, algebraic geometry, and differential equations (see [6], [7], [11], [16], [1], [17], [18], [8], [9], and [10]).

In this paper we consider three of the "challenging problems on affine n-space" as described by Kraft in his Bourbaki Lecture in [14], 1995 and show how they can be related to the study of a special class of triangular (hence locally nilpotent) derivations. The problems we consider are the Cancellation Problem, the Embedding Problem, and the Linearization Problem.

The contents of this paper are arranged as follows. Section 2 recalls some more or less well-known results on embeddings and locally nilpotent derivations. Furthermore, it discusses results concerning the Cancellation Problem, the Embedding Problem, and the Linearization Problem.

Section 3 introduces a special class of triangular derivations and shows how embeddings of k^r in k^n can be characterized by these derivations. In case $r = 1$ this characterization asserts that a regular map $\alpha\colon k \to k^n$ is an embedding if and only if the derivation associated to α has a slice.

This enables us in Section 4 to establish a relation between the Cancellation Problem and the Embedding Problem. More precisely, if α is an embedding, then the associated locally nilpotent derivations (which all have a slice by the results of Section 3) satisfy the conclusion of the Cancellation Problem.

The relationship between the Cancellation Problem and the Embedding Problem leads us in Section 5 to a possible counterexample to the Cancellation Problem (Conjecture 5.1), as well as to the Embedding Problem and the Linearization Problem. All examples are in dimension five.

It should be noted that this paper is very much influenced by the elegant paper of Asanuma [3]. Several of the results in this paper are implicit in his work. In

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particular, we show in Section 5 that Asanuma's candidate counterexample to the Cancellation Problem (described in terms of Rees rings) is the same as the one given here in Conjecture 5.1.

2. Preliminaries

This section recalls some more or less well-known results concerning embeddings, locally nilpotent derivations, the Cancellation Problem, and the kernel algorithm of [9].

Throughout this paper k denotes a field of characteristic zero, $n \geq 2$ is a fixed natural number, and $k[X] := k[X_1, \dots, X_n]$ is the polynomial ring over k in n variables.

Embeddings. Let $r \in \{1, \dots, n-1\}$, $k[U] := k[U_1, \dots, U_r]$ and $f_1, \dots, f_n \in k[U]$. The map $\alpha\colon k^r \to k^n$ defined by $\alpha(u) := (f_1(u), \dots, f_n(u))$ for all $u \in k^r$ is called an *embedding* of k^r in k^n if $\mathrm{Im}(\alpha)$ is a closed subset of k^n (in the Zariski topology) and $\alpha\colon k^r \to \mathrm{Im}(\alpha)$ is an isomorphism of algebraic varieties over k. Denote the induced map from $k[X]$ to $k[U]$ by α^*. So α^* sends each X_i to f_i and $\mathrm{Im}(\alpha^*) = k[f_1, \dots, f_n] \subseteq k[U]$.

Proposition 2.1. *The map α is an embedding if and only if α^* is surjective.*

Proof. ($\Rightarrow$): Assume that α is an embedding. Then there is a polynomial map $\beta\colon k^n \to k^r$ such that $\beta \circ \alpha = 1_{k^r}$. So there exist $g_1, \dots, g_r \in k[X]$ such that $u_i = g_i(f_1(u), \dots, f_n(u))$ for all $i \in \{1, \dots, r\}$ and all $u = (u_1, \dots, u_r) \in k^r$. Because k is infinite, this implies that $U_i = g_i(f_1, \dots, f_n)$ for all i. So α^* is surjective.

($\Leftarrow$): Suppose that α^* is surjective, say $g_i \in k[X]$ is such that $g_i(f_1, \dots, f_n) = U_i$, for $i \in \{1, \dots, r\}$.

It follows from Lemma 2.2 below that $\mathrm{Im}(\alpha) = V(\mathrm{Ker}\,\alpha^*)$. So $\mathrm{Im}(\alpha)$ is closed. Write β for the restriction of the map $g\colon k^n \to k^r$ sending x to $(g_1(x), \dots, g_r(x))$ for all $x \in k^n$. Then obviously $\beta \circ \alpha(u) = u$ for all $u \in k^r$. Furthermore, if $x \in \mathrm{Im}(\alpha) = V(\mathrm{Ker}\,\alpha^*)$, then it follows from Lemma 2.2 also that $x_i = f_i(g_1(x), \dots, g_r(x))$ for all i, i.e., $x = \alpha \circ \beta(x)$. In other words, $\alpha\colon k^r \to \mathrm{Im}(\alpha)$ is an isomorphism with inverse β. $\square$

Lemma 2.2. *Let $g_1, \dots, g_r \in k[X]$ be such that $U_i = g_i(f_1, \dots, f_n)$ for all i. Then* $\mathrm{Ker}\,\alpha^* = (X_1 - f_1(g_1, \dots, g_r), \dots, X_n - f_n(g_1, \dots, g_r))$.

Proof. ($\subseteq$): Let $p \in k[X]$ with $p(f_1, \dots, f_r) = 0$. Write

$$p = p(f_1, \dots, f_n) + \sum_{i=1}^{n} a_i(X, U)(X_i - f_i(U))$$

for some $a_i(X, U) \in k[X, U]$. Substituting $U_i := g_i$ for all i, we find that $p \in (X_1 - f_1(g_1, \dots, g_r), \dots, X_n - f_n(g_1, \dots, g_r))$, since no U_i appears in p.

($\supseteq$) Easy. $\square$

An embedding $\alpha\colon k^r \to k^n$ is called *rectifiable* if there exists a k-automorphism φ of k^n such that $(\varphi \circ \alpha)(u) = (u_1, \dots, u_r, 0, \dots, 0)$ for all $u = (u_1, \dots, u_r) \in k^r$. Sometimes we write explicitely that α is rectifiable by φ.

The Embedding Problem. The Embedding Problem asks if every embedding of k^r in k^n is rectifiable. The case $r = 1$ and $n = 2$ was answered affirmatively by Abhyanker and Moh in [1] and Suzuki in [21]. A little later it was conjectured by Abhyankar in [2] that for $n \geq 3$ there do exist embeddings of k in k^n which are not rectifiable. However, Craighero showed in [5] that for $n \geq 4$ every embedding of k in k^n is rectifiable. The same result was obtained by Jelonek in [13]. In fact Jelonek showed that if $n \geq 2r + 2$, then every embedding of k^r in k^n is rectifiable, while Craighero showed this for all $n \geq 3r + 1$. See also the paper [20] of Srinivas for a generalization of this result.

The case $r = 1$ and $n = 3$ remains open. We discuss a possible counterexample in Section 5. For more results about embeddings of k in k^3 we refer to the paper [4] of Bhatwadekar and Roy.

The following easy argument, due to Jelonek in [13], shows that every embedding $\alpha\colon k^r \to k^n$ is *stably rectifiable*, i.e., there exists $m \geq 1$ such that $\tilde{\alpha}\colon k^r \to k^{n+m}$ defined by $\tilde{\alpha}(u) := (\alpha(u), 0, \ldots, 0)$ is rectifiable.

Proposition 2.3. *Let $\alpha = (f_1, \ldots, f_n)\colon k^r \to k^n$ be an embedding. Then $\tilde{\alpha} := (f_1, \ldots, f_n, 0, \ldots, 0)\colon k^r \to k^{n+r}$ is rectifiable.*

Proof. Let $g = (g_1, \ldots, g_r) \in k[X]^r$ be such that $g_i(f_1, \ldots, f_n) = U_i$ for all i. Then both $G := (X, U_1 + g_1, \ldots, U_r + g_r)$ and $H := (X_1 - f_1, \ldots, X_n - f_n, U)$ are k-automorphisms of k^n. Take $F := H \circ G$. Then one easily verifies that

$$F(\tilde{\alpha}(u)) = (0, \ldots, 0, u)$$

for all $u \in k^r$. $\square$

Locally nilpotent derivations. Let A be a commutative k-algebra and D a k-derivation on A. Then D is called *locally nilpotent* if for every $a \in A$ there exists a natural number m such that $D^m(a) = 0$. For all results concerning these derivations, see [16] or [10].

From now on D will denote a non-zero locally nilpotent derivation on A. Extend D to a k-derivation on $A[T]$ by putting $D(T) := 0$ and define $\varphi\colon A \to A[T]$ by

$$\varphi(a) := \sum_{i=0}^{\infty} \frac{1}{i!} D^i(a) T^i \quad \text{for all } a \in A.$$

Then φ is a ring homomorphism. If $b \in A$, then the substitution homomorphism from $A[T]$ to A sending T to b will be denoted by π_b. So the composition $\varphi_b := \pi_b \circ \varphi$ is an endomorphism of A.

An element $s \in A$ is called a *slice* of D if $D(s) = 1$. A non-zero locally nilpotent derivation does not always have a slice; however it always has a *preslice*, i.e., an element $p \in A$ such that $D^2(p) = 0$, but $d := D(p) \neq 0$. So if one considers the ring $\tilde{A} := A[d^{-1}]$ and extends D to $\tilde{A}$ in the obvious way, then the extended derivation on $\tilde{A}$, which we also denote by D, is again locally nilpotent and does have a slice, namely $s := d^{-1}p \in \tilde{A}$.

In case A is a domain the inclusion $A \subseteq \tilde{A}$ and the fact that locally nilpotent derivations having a slice have nice properties (see for instance Proposition 2.4 below) can be used to obtain useful information on arbitrary locally nilpotent derivations. Of particular interest will be the kernel of these derivations, also called the *ring of constants*. It is denoted by A^D.

Proposition 2.4. *Let D be a locally nilpotent derivation on A with a slice $s \in A$. Then*

(1) $A = A^D[s]$, *a polynomial ring in s over A^D. So $D = \frac{d}{ds}$ on A;*
(2) $A^D = \varphi_{-s}(A)$. *In particular, if A is a finitely generated k-algebra generated by some elements $a_i \in A$, then A^D is a finitely generated k-algebra generated by the elements $\varphi_{-s}(a_i)$.* □

In contrast with 2.4(2), A^D need not be finitely generated if D does not have a slice, even if A is. In fact this is strongly related to Hilbert's fourteenth problem. We refer to [7], [11], and [6] for more details.

There is, however, an algorithm to compute the kernel of A^D, provided that A is a domain and A^D is a finitely generated k-algebra. Moreover, if we do not know a priori that A^D is finitely generated, but find that the algorithm terminates, then it follows that A^D is finitely generated over k. Furthermore, the algorithm computes generators for A^D. Since we will use this algorithm several times in this paper, we now describe it (without proof) for the reader's convenience. For more details, we refer to [9].

The Kernel Algorithm. Let $A = k[a_1, \ldots, a_n]$ be a finitely generated k algebra without zero-divisors and let D be a non-zero locally nilpotent derivation on A.

Algorithm 2.5. (1) *Choose $p \in A$ with $D^2(p) = 0$ and $d := D(p) \neq 0$. Let $s := d^{-1}p \in A[d^{-1}]$ and define $b_i := \varphi_{-s}(a_i)$ for all $i \in \{1, \ldots, n\}$. Since D is locally nilpotent, there exist natural numbers e_i such that $r_i := d^{e_i} b_i \in A$. Define $R_0 := R_0(D, p) := k[r_1, \ldots, r_n, d]$. Then it can be shown that*

$$R_0 \subseteq A^D \subseteq R_0[d^{-1}]. \tag{1}$$

(2) *If B is a k-subalgebra of A and $f \in B$, then we write $B : f$ for the k-algebra generated by the elements $g \in A$ such that $fg \in B$. Now inductively define, for each $m \geq 1$, $R_m := R_m(D, p) := R_{m-1} : d$. We get*

$$R_0 \subseteq R_1 \subseteq \cdots \subseteq A^D = \bigcup_{m=0}^{\infty} R_m.$$

(3) *Each R_m is a finitely generated k-algebra and can be computed as follows. Suppose that $R_{m-1} = k[F_1, \ldots, F_l]$. Put*

$$I(F_1, \ldots, F_l) := \{p \in k[T_1, \ldots, T_l] \mid p(F_1, \ldots, F_l) \text{ is divisble by } d\}.$$

Then $I(F_1, \ldots, F_l)$ is an ideal in $k[T_1, \ldots, T_l]$ and it is generated by a finite number of elements, say $p_1(T), \ldots, p_s(T)$. This means, by definition, that $p_i(F_1, \ldots, F_l) = f_i d$ for some $f_i \in A$. Now it can be shown that $R_m = k[F_1, \ldots, F_l, f_1, \ldots, f_s]$.

(4) *If A^D is finitely generated as a k-algebra, then $A^D = R_r$ for some $r \in \mathbb{N}$ (and $R_r = R_n$ for all $n \geq r$). Conversely, if $R_r = R_{r+1}$ for some $r \in \mathbb{N}$, then $A^D = R_r$ and hence A^D is a finitely generated k-algebra.*

Remark 2.6. The k-algebra R_0 above satisfies (1). This is, in fact, the only property of R_0 that is used in the algorithm. This implies that one may replace R_0 by any finitely generated k-subalgebra R_0' of A^D which contains R_0, and start the algorithm again.

Example 2.7. Let A be the polynomial ring $k[T,U,X]$ and consider the locally nilpotent derivation $D := f'(U)\partial_X + c(T)\partial_U$ on A, where $f(U) \in k[U]$ and $c(T) \in k[T] \setminus \{0\}$. Let p be the preslice U of D and $d := D(p) = c(T)$. We compute the algebra $R_0 := R_0(D,p)$ from the first step of Algorithm 2.5. This computation will be useful in the proofs of Lemmas 3.2 and 4.5.

We use the notations from the algorithm. So $s = c(T)^{-1}U$ and we take $a_1 := T$, $a_2 := U$, and $a_3 := X$ as generators of A. Then $b_1 = \varphi_{-s}(a_1) = T$, $b_2 = \varphi_{-s}(a_2) = U + \frac{1}{2}c(T)(-c(T)^{-1}U) = 0$, and

$$\begin{aligned} b_3 &= \sum_{j\geq 0} \frac{1}{j!} D^j(X) \left(-\frac{U}{c(T)}\right)^j \\ &= X + \frac{1}{c(T)} \sum_{j\geq 1} \frac{1}{j!} f^{(j)}(U)(-U)^j, \end{aligned}$$

since $D^j(X) = c(T)^{j-1} f^{(j)}(U)$. Since the U-derivative of $\sum_{j\geq 1} 1/j! f^{(j)}(U)(-U)^j$ equals $-f'(U)$, it follows that

$$b_3 = X + \frac{1}{c(T)}(-f(U) + f(0)).$$

So we find $r_1 = T$, $r_2 = 0$, and $r_3 = c(T)X - f(U) + f(0)$. Consequently $R_0 = k[r_1, r_2, r_3, d] = k[T, c(T)X - f(U)]$.

The Cancellation Problem. Let $n \geq 2$ and let V be an affine variety over k such that $V \times k \cong k^n$ as algebraic varieties. The Cancellation Problem now asks if it follows that $V \cong k^{n-1}$.

The answer is affirmative if $n = 2, 3$ (see [12], [15], and [10]) and is open for all $n \geq 4$. We will discuss a possible counterexample for the case $n = 5$ in Section 5 (see also [3]). In order to do this, we need the following algebraic reformulation, which is a consequence of Proposition 2.4 (see [10]).

Proposition 2.8. *Let $n \geq 2$. The Cancellation Problem in dimension n has an affirmative answer if and only if $k[X]^D$ is generated by $n-1$ elements over k for every locally nilpotent derivation D on $k[X]$ with a slice, or, equivalently, if $k[X]^D \cong_k k^{[n-1]}$ for all such D.* □

The Linearization Problem. Let $F \in \mathrm{Aut}_k(k^n)$ such that $F^p = 1_{k^n}$ for some $p \geq 1$. The Linearization Problem asks if there exists a $\varphi \in \mathrm{Aut}_k(k^n)$ such that $\varphi^{-1} \circ F \circ \varphi$ is linear.

In case $n = 2$ the answer to this question is affirmative and follows immediately from the fact that $\mathrm{Aut}_k(k^2)$ is the amalgamated product of the affine subgroup and the subgroup of the De Jonquières transformations over their intersection (see [14]). If $n \geq 3$ the problem remains open. However, one has the following relation between the Linearization Problem and the Cancellation Problem.

Proposition 2.9. *If the Linearization Problem has an affirmative answer (in dimension n) for all automorphisms of order 2, then the Cancellation Problem has an affirmative answer (in dimension n) as well.*

Proof. Let $n \geq 2$ and let V be an algebraic variety over k. Assume that $\psi\colon V \times k \to k^n$ is an isomorphism of algebraic varieties. Let $F \in \mathrm{Aut}_k(V \times k)$ be defined by $F(v,t) := (v, -t)$ for all $v \in V$ and $t \in k$ and take $G := \psi \circ F \circ \psi^{-1} \in \mathrm{Aut}_k(k^n)$. Then obviously $G^2 = 1_{k^n}$ and $\mathrm{Fix}(G) := \{x \in k^n \mid G(x) = x\} \cong_k V$.

Now if the Linearization Problem has an affirmative answer for automorphisms of order two, then there exists an automorphism φ of k^n such that $\varphi^{-1} \circ G \circ \varphi = L$, a linear map. So $G = \varphi \circ L \circ \varphi^{-1}$, which implies that $\mathrm{Fix}(G) \cong_k \mathrm{Fix}(L)$. Because $\mathrm{Fix}(L) \cong_k k^d$ for some d, it follows that $V \cong_k k^d$. Since $V \times k \cong_k k^n$, it even follows that $d = n-1$. So $V \cong_k k^{n-1}$. □

3. Derivations related to embeddings

This section characterizes embeddings in terms of locally nilpotent derivations. We first introduce some terminology.

Let $D = (D_1, \dots, D_r)$ be a sequence of pairwise commuting derivations on a commutative ring A. We say that $s = (s_1, \dots, s_r) \in A^r$ is a *slice system* of D if $D_i(s_j) = \delta_{ij}$ for all i, j. If each D_i is locally nilpotent, then it follows easily from Proposition 2.4 that $A = A^D[s_1, \dots, s_r]$, a polynomial ring in $s_1, \dots, s_r$ over $A^D := \bigcap_{i=1}^r A^{D_i}$.

Now let $\alpha\colon k^r \to k^n$ be a polynomial map given by $\alpha(u) = (f_1(u), \dots, f_n(u))$ for all $u \in k^r$, where each f_i is an element of $k[U] := k[U_1, \dots, U_r]$. To each $i \in \{1, \dots, r\}$ we associate the triangular, hence locally nilpotent, derivation D_i given by

$$D_i := f_{1U_i}\partial_{X_1} + \cdots + f_{nU_i}\partial_{X_n} + T\partial_{U_i}$$

on the $n+r+1$ variable polynomial ring $A := k[T, U, X]$. One easily verifies that these derivations commute pairwise. Put $D := (D_1, \dots, D_r)$.

Theorem 3.1. *The map α is an embedding if and only if D has a slice system in A.*

The proof uses the following lemma.

Lemma 3.2. *Every element of A^D is equivalent, modulo (T, X), to an element of $k[f] := k[f_1, \dots, f_n]$.*

Proof. Let $i \in \{1, \dots, r\}$. Then U_i is a preslice of D_i and arguing as in Example 2.7 we get

$$R_0(D_i, U_i) = k[T, U_1, \dots, \hat{U}_i, \dots, U_r, f_1 - TX_1, \dots, f_n - TX_n].$$

By (1) it follows that

$$R_0(D_i, U_i) \subseteq A^{D_i} \subseteq R_0(D_i, U_i)[T^{-1}]$$

and taking the intersection over all i we obtain

$$R_0 \subseteq A^D \subseteq R_0[T^{-1}],$$

where $R_0 := k[T, f_1 - TX_1, \dots, f_n - TX_n]$.

Now let $q \in A^D$. Then in particular $q \in R_0[T^{-1}]$ so there is a $\rho \in \mathbb{N}$ and a polynomial p over k such that $T^\rho q = p(T, f_1 - TX_1, \dots, f_n - TX_n)$. Substituting $X_i := 0$ for all i and expanding the resulting right-hand side in powers of T gives

$$T^\rho q = \sum_{i \geq 0} p_i(f) T^i$$

for some $p_i(f) \in k[f] := k[f_1, \dots, f_n] \subseteq k[U]$. Since T^ρ divides the left-hand side, it divides the right-hand side as well. So $p_i(f) = 0$ for all $i < \rho$. Hence

$$q(X_1 := 0, \dots, X_n := 0) = \sum_{i \geq \rho} p_i(f) T^{i-\rho}.$$

Now substituting $T := 0$ we find $q \equiv p_0(f) \pmod{(T, X)}$. ☐

Proof of Theorem 3.1. ($\Rightarrow$): Assume that α is an embedding. Then each U_i can be written as $U_i = g_i(f_1, \dots, f_n)$ for some polynomial $g_i \in k[X]$. Define

$$s_i := \frac{U_i - g_i(f_1 - TX_1, \dots, f_n - TX_n)}{T}$$

for all $i \in \{1, \dots, r\}$. Since obviously each $f_j - TX_j$ belongs to A^D and $D_i(U_j) = \delta_{ij} T$ for all i, j, we get that $(s_1, \dots, s_r)$ is a slice system of D.

($\Leftarrow$): Suppose that D has a slice system $(s_1, \dots, s_r)$ in A^r. Then $D_i(U_i - Ts_i) = 0$ for all i. Also $D_j(U_i - Ts_i) = 0$ if $i \neq j$. So $U_i - Ts_i \in A^D$ for all i. Now use Lemma 3.2 and make the substitutions $T := 0$ and $X_i := 0$ for all i. It follows that each U_i is an element of $k[f]$. So by Proposition 2.1, α is an embedding. ☐

4. Rectifiability and the Cancellation Problem

Throughout this section, let $\alpha = (f_1, \dots, f_n)\colon k^r \to k^n$ denote an embedding. By the results of the previous section, we can associate to α a sequence $D = (D_1, \dots, D_r)$ of locally nilpotent derivations having a slice system in A^r, where A is the $n + r + 1$ variable polynomial ring $k[T, U, X]$.

In order to simplify the notations, we write $f - TX$ instead of $f_1 - TX_1, \dots, f_n - TX_n$. The main result of this section, Theorem 4.1, asserts that if α is rectifiable, then the kernel of each derivation D_i is a polynomial ring in $n - 1$ variables over k, which shows that the Cancellation Problem has an affirmative answer for these derivations. More precisely, we show the following.

Theorem 4.1. *If α is rectifiable by the k-automorphism F of k^n, then A^{D_i} is the $k[T]$-algebra generated by the $n + r - 1$ elements U_j, $j \neq i$, s_j, $j \neq i$, $F_i(f - TX)$, and $T^{-1}F_j(f - TX)$, $j \in \{r + 1, \dots, n\}$, where $s_j := T^{-1}(F_j(f - TX) - U_j)$ for all $j \in \{1, \dots, r\}$.*

Proof. Let us denote the $k[T]$-algebra described in the theorem by R_0.

We first claim that $R_0(D_i, U_i) \subseteq R_0$. Using the description of $R_0(D_i, U_i)$ given in the proof of Lemma 3.2, it is enough to show that each element $f_j - TX_j$ belongs to R_0. It is even sufficient to show that $F_j(f - TX)$ is an element of R_0, for all $j \in \{1, \dots, n\}$, for then we can apply the inverse of F to this n-tuple to obtain that all $f_j - TX_j$ belong to R_0.

Now $F_i(f - TX) \in R_0$ by definition and for $j \neq i$ the fact that $F_j(f - TX) \in R_0$ follows readily from the fact that $F_j(f - TX) = Ts_j + U_j$.

According to Remark 2.6 it is possible to replace $R_0(D_i, U_i)$ by R_0 in order to compute A^{D_i}. We claim that with this R_0 we get $R_1 := R_0 : T = R_0$ and hence $A^{D_i} = R_0$ as desired.

Observe that $s_j = F_{jX_1}(f)X_1 + \cdots + F_{jX_n}(f)X_n + T(\dots)$, for all $j \neq i$. So to find R_1, we need to find all relations between

$$U_1, \dots, \hat{U}_i, \dots, U_r, F_{1X_1}(f)X_1 + \cdots + F_{1X_n}(f)X_n, \dots, \\ U_i, \dots, F_{nX_1}(f)X_1 + \cdots + F_{nX_n}(f)X_n. \tag{2}$$

Since $\det(JF) \in k^*$ it follows that $\det(JF)(f) \in k^*$. Consequently the X-linear forms $F_{jX_1}(f)X_1 + \cdots + F_{jX_n}(f)X_n$, $j = 1, \ldots, n$, together with $U_1, \ldots, U_r$ form a coordinate system of $k[U, X]$. In particular, there are no relations between the elements of (2). Now the kernel algorithm gives $R_1 = R_0$. □

Corollary 4.2. *If α is rectifiable, then $A^{D_i} \cong_{k[T]} k[T]^{[n+r-1]} \cong_k k^{[n+r]}$.* □

As another consequence of Theorem 4.1, we will describe a new class of locally nilpotent derivations for which the Cancellation Problem has an affirmative answer.

In order to do this, let $A := k[T, U, X] := k[T, U, X_1, \ldots, X_n]$ be the $n+2$ variable polynomial ring over k. So $r = 1$ in the notation of the previous section. We will consider derivations of the form

$$D = a_1(U)\partial_{X_1} + \cdots + a_n(U)\partial_{X_n} + b(T)\partial_U$$

with $b(T) \neq 0$. Of course we can write such derivations as

$$D = f_1'(U)\partial_{X_1} + \cdots + f_n'(U)\partial_{X_n} + b(T)\partial_U$$

with $f_i(0) = 0$ for all i.

Theorem 4.3. *Let $n \neq 3$. If D has a slice in A, then $A^D \cong_{k[T]} k[T]^{[n]} \cong_k k^{[n+1]}$.*

In the case that $\deg b = 0$, $s := b^{-1}U$ is a slice of D and the result follows readily from Proposition 2.4. So from now on assume that $\deg b > 0$. To prove the theorem we use the following two lemmas. Let π denote the substitution homomorphism defined by $\pi(g(T, U, X)) = g(b(T), U, X)$ for all $g \in A$ and let D_1 be the derivation $D_1 := f_1'(U)\partial_{X_1} + \cdots + f_n'(U)\partial_{X_n} + T\partial_U$.

Lemma 4.4. *Let $N \geq 1$ and let $g_1(T, U, X), \ldots, g_N(T, U, X) \in k[T, U, X]$. Let I be the ideal of all polynomials $p \in k[Y_0, Y] := k[Y_0, Y_1, \ldots, Y_N]$ such that the polynomial $p(T, g_1(b(T), U, X), \ldots, g_N(b(T), U, X))$ is divisible by $b(T)$ and let J be the ideal of all polynomials $p \in k[Y]$ such that $p(g_1, \ldots, g_N)$ is divisible by T. Then $I = k[Y_0, Y]b(Y_0) + k[Y_0, Y]J$.*

Proof. ($\subseteq$): Let $p \in I$. Write $p = \sum_{i=0}^{(\deg b)-1} p_i(Y)Y_0^i + b(Y_0)\tilde{p}(Y_0, Y)$ for some polynomials $p_i \in k[Y]$ and $\tilde{p} \in k[Y_0, Y]$. Then by definition

$$b(T) \mid \sum_{i=0}^{(\deg b)-1} p_i(g_1(0, U, X), \ldots, g_N(0, U, X))T^i.$$

Since the T-degree of this sum is smaller than $\deg b$, it follows that

$$p_i(g_1(0, U, X), \ldots, g_N(0, U, X)) = 0 \text{ for all } i.$$

So $p_i \in J$.

($\supseteq$) Obvious. □

Lemma 4.5. *$R_i(D, U) = \pi(R_i(D_1, U))[T]$ for all $i \geq 0$.*

Proof. By induction on i.

($i = 0$): As in the Example 2.7, we get

$$R_0(D, U) = k[T, f_1 - b(T)X_1, \ldots, f_n - b(T)X_n]$$

and

$$R_0(D_1, U) = k[T, f_1 - TX_1, \ldots, f_n - TX_n].$$

($i > 0$): Let $i > 0$ and assume that

$$R_{i-1}(D, U) = k[T, g_1(b(T), U, X), \ldots, g_N(b(T), U, X)]$$

and

$$R_{i-1}(D_1, U) = k[T, g_1(T, U, X), \ldots, g_N(T, U, X)]$$

for some $N \geq 1$ and some polynomials g_j, $j \in \{1, \ldots, N\}$. Now the result follows from the previous lemma and the fact that the kernel algorithm constructs $R_i(D, U) = R_{i-1}(D, U) : b(T)$ and $R_i(D_1, U) = R_{i-1}(D_1, U) : T$. □

Corollary 4.6. *If D has a slice in A, then α is an embedding.*

Proof. Take $s \in A$ such that $D(s) = 1$. Then $D(U - b(T)s) = 0$, i.e., $U - b(T)s \in A^D$. Since $A^D = \bigcup_{i \geq 0} R_i(D, U)$ it follows from Lemma 4.5 that

$$U - b(T)s = P(T, g_1(b(T), U, X), \ldots, g_N(b(T), U, X))$$

for some $g_i(b(T), U, X) \in A^D$ and some polynomial P over k.

Let c be a root of $b(T)$ in the algebraic closure $\bar{k}$ of k. Substituting $T := c$ gives $U - p(c, g_1(0, U, X), \ldots, g_N(0, U, X)) = 0$. Now choosing a k-basis of $\bar{k}$ containing 1, one deduces that there exists a $p \in k[Y_1, \ldots, Y_N]$ such that $h := U - p(g_1(T, U, X), \ldots, g_N(T, U, X))$ is divisible by T. Since by Lemma 4.5 each $g_i(T, U, X)$ is an element of A^{D_1}, it follows that $s := T^{-1}h$ is a slice of D_1 in A. So by Theorem 4.1 α is an embedding. □

Proof of Theorem 4.3. By Corollary 4.6 α is an embedding. So α is rectifiable (the case $n = 1$ is obvious, the case $n = 2$ is the Abhyankar-Moh-Suzuki Theorem, and if $n \geq 4$ we can apply Craighero's and Jelonek's Theorem). Since by Lemma 4.5 $A^D = \pi(A^{D_1})[T]$, the desired result follows immediately from the $r = 1$ case of Theorem 4.1. □

5. Applications: Possible counterexamples to problems on affine space

In Theorem 4.1 we showed that if an embedding is rectifiable, the kernels of the corresponding derivations are polynomial rings. So in order to find a possible counterexample to the Cancellation Problem, it seems natural to look for non-rectifiable embeddings. A class of candidates of such embeddings was constructed by Shastri in [19].

More precisely, let $r = 1$ and $n = 3$. He showed that every (open) knot-type has a real polynomial representation which defines an embedding of $\mathbb{C}$ in $\mathbb{C}^3$. In particular, he obtained the following polynomial representation of the trefoil knot by putting $f(U) := U^3 - 3U$, $g(U) := U^4 - 4U^2$, and $h(U) := U^5 - 10U$ and $\alpha(u) := (f(u), g(u), h(u))$. Indeed this α gives an embedding of $\mathbb{C}$ in $\mathbb{C}^3$, since one easily verifies that $F(f(U), g(U), h(U)) = U$, with $F(X, Y, Z) := YZ - X^3 - 5XY + 2Z - 7X$. We call this embedding α the *Shastri map*.

Since $\alpha\colon \mathbb{R} \to \mathbb{R}^3$ represents the trefoil, it is not rectifiable over $\mathbb{R}$. This led Shastri to conjecture that $\alpha\colon \mathbb{C} \to \mathbb{C}^3$ is not rectifiable over $\mathbb{C}$ as well. So in light of Theorem 4.1, the following conjecture seems reasonable.

Conjecture 5.1. *Let $D := f'(U)\partial_X + g'(U)\partial_Y + h'(U)\partial_Z + T\partial_U$ on the polynomial ring $A := \mathbb{C}[T, U, X, Y, Z]$. Then $A^D \not\cong_{\mathbb{C}} \mathbb{C}^{[4]}$.*

Since D has a slice, namely $s := T^{-1}(U - F(f(U) - TX, g(U) - TY, h(U) - TZ))$, and $A^D \cong_{\mathbb{C}} A/(s)$, this conjecture is equivalent to the following conjecture.

Conjecture 5.2. $\mathbb{C}[T, U, X, Y, Z]/(s) \not\cong_{\mathbb{C}} \mathbb{C}^{[4]}$.

Obviously, by Proposition 2.8, an affirmative answer to these equivalent conjectures would give a negative answer to the Cancellation Problem and hence, by Proposition 2.9, a negative answer to the Linearization Problem as well. Also by Theorem 4.1 it would show that Shastri's embedding is indeed a counterexample to the Embedding Problem.

A similar conjecture was made by Asanuma in [3]. To relate his conjecture with Conjecture 5.1, we briefly recall some of Asanuma's results.

If I is an ideal in $R := k[X]$, then the *Rees ring* associated to I, denoted by $\mathcal{R}_R(I)$, is the $R[T]$-subalgebra of $R[T, T^{-1}]$ generated by the elements $T^{-1}i$ with $i \in I$.

Suppose now that $R/I \cong_k k^{[1]}$. In other words, suppose that $I = \operatorname{Ker} \alpha^*$ for some embedding α of k in k^n. Then it was shown in [3] that $\mathcal{R}_R(I)^{[1]} \cong_{k[T]} k[T]^{[n+1]} \cong_k k^{[n+2]}$.

Conjecture 5.3 (Asanuma's Conjecture)**.** *Let $I := \operatorname{Ker} \alpha^*$, where α is the Shastri map. Then $\mathcal{R}_R(I) \not\cong_{\mathbb{C}} \mathbb{C}^{[4]}$.*

The equivalence between Asanuma's Conjecture and Conjecture 5.1 follows immediately from Corollary 5.5 below. In order to establish this equivalence, let $\alpha = (f_1, \ldots, f_n)\colon k \to k^n$ be any embedding and let $\alpha^*\colon k[X] \to k[U]$ be the induced homomorphism of coordinate rings. Put $I := \operatorname{Ker} \alpha^*$, the ideal of relations between $f_1, \ldots, f_n$ over k. Let $A := k[T, U, X]$ and denote by $I(f - TX)$ the ideal in A obtained from I by making the substitutions $X_i := f_i - TX_i$ for all i. Finally, let D denote the derivation $f_1' \partial_{X_1} + \cdots + f_n' \partial_{X_n} + T\partial_U$ on A, corresponding to α.

Proposition 5.4. $A^D = k[T, f_1 - TX_1, \ldots, f_n - TX_n, T^{-1} I(f - TX)]$.

Proof. Let $\tilde{\alpha} := (f_1, \ldots, f_n, 0)\colon k \to k^{n+1}$. Let $\tilde{D}$ be the derivation on $A[X_{n+1}]$ corresponding to $\tilde{\alpha}$. Note that $\tilde{D}$ is the extension of D to $A[X_{n+1}]$ by sending X_{n+1} to 0. So

$$A[X_{n+1}]^{\tilde{D}} = A^D[X_{n+1}]. \tag{3}$$

By Proposition 2.3, $\tilde{\alpha}$ is rectifiable. Let $F = (F_1, \ldots, F_{n+1})$ be a k-automorphism of k^{n+1} rectifying $\tilde{\alpha}$. Then one easily verifies that $\tilde{I} := \operatorname{Ker} \tilde{\alpha}$ equals $(F_2, \ldots, F_{n+1})$ in $k[X, X_{n+1}]$. So, by Theorem 4.1 applied to the case $r = 1$, we get

$$\begin{aligned} A[X_{n+1}]^{\tilde{D}} &= k[T, F_1(f - TX, -TX_{n+1}), T^{-1}F_2(f - TX, -TX_{n+1}), \ldots, \\ &\qquad T^{-1}F_{n+1}(f - TX, -TX_{n+1})] \\ &= k[T, F_1(f - TX, -TX_{n+1}), \ldots, \\ &\qquad F_{n+1}(f - TX, -TX_{n+1}), T^{-1}\tilde{I}(f - TX, -TX_{n+1})] \\ &= k[T, f_1 - TX_1, \ldots, f_n - TX_n, -TX_{n+1}, T^{-1}\tilde{I}(f - TX, -TX_{n+1})] \\ &= k[T, f_1 - TX_1, \ldots, f_n - TX_n, -TX_{n+1}, T^{-1}I(f - TX), X_{n+1}], \end{aligned}$$

where $I := \operatorname{Ker} \alpha^*$ (since obviously $\tilde{I} = Ik[X, X_{n+1}] + X_{n+1}k[X, X_{n+1}]$). The desired result now follows using (3). □

Corollary 5.5. $A^D \cong_k \mathcal{R}_{k[X]}(I)$.

Proof. This follows readily from the previous proposition by sending X_i to $T^{-1}(f - X_i)$ for all i. □

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