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A NEW STATISTIC FOR THE $3x+1$ PROBLEM

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ABSTRACT. A finite $(3x+1)$–trajectory is a sequence $\underline{a} = a_1, \ldots, a_n$ of positive integers such that $a_{i+1} = 3a_i + 1$ if a_i is odd, $a_{i+1} = a_i/2$ if a_i is even, $a_i > 1$ if $i < n$ and $a_n = 1$. For such a sequence $\underline{a}$ we define $C(\underline{a}) = (a_1a_2+\cdots+a_{n-1}a_n+a_na_1)/(a_1^2+\cdots+a_n^2)$. We prove that $9/13 < C(\underline{a}) < 5/7$ if a_1 is odd and $a_1 \geq 3$. Histograms suggest that C may have an interesting limiting distribution.

1. Introduction

For an integer $a > 1$, define $S(a) = 3a+1$ if a is odd, and $S(a) = a/2$ if a is even. We call the sequence $a, S(a), S^2(a), \ldots$ the $(3x+1)$–trajectory, or simply the trajectory, of a. As soon as some $S^i(a) = 1$, the sequence is deemed to terminate; we then say that the trajectory of a is finite. The famous $3x+1$ Conjecture predicts that there are no infinite trajectories. For a comprehensive survey of this conjecture up to 1985, see [1]. Some more recent developments are surveyed in [3, Chapter 1].

The $3x+1$ Conjecture is widely believed to be intractable at present. However, some results have been obtained regarding certain statistics associated with trajectories. The stopping time of an integer $a > 1$ is defined to be the smallest integer i such that $S^i(a) < a$; if no such i exists, then we say that a has infinite stopping time. A notable result, first proved by R. Terras [2], asserts that the set of integers with finite stopping time has asymptotic density 1; see [1, p. 6].

The stopping time is a local statistic, in the sense that it involves only a segment of the trajectory. When it comes to global statistics, i.e. those involving the entire trajectory, the results have been meager. Two global statistics mentioned in [1] are the total stopping time and the expansion factor. The first of these is simply the length of the trajectory, while the second is the ratio of the largest term in the trajectory to the initial term.

In this paper, we introduce a new global statistic, which is more subtle than the total stopping time or the expansion factor. If $\underline{a} = a_1, \ldots, a_n$ is a finite $(3x+1)$–trajectory, with $a_1 = m > 1$, we define

$$C(m) = C(\underline{a}) = \frac{a_1a_2 + \cdots + a_{n-1}a_n + a_na_1}{a_1^2 + \cdots + a_n^2}.$$

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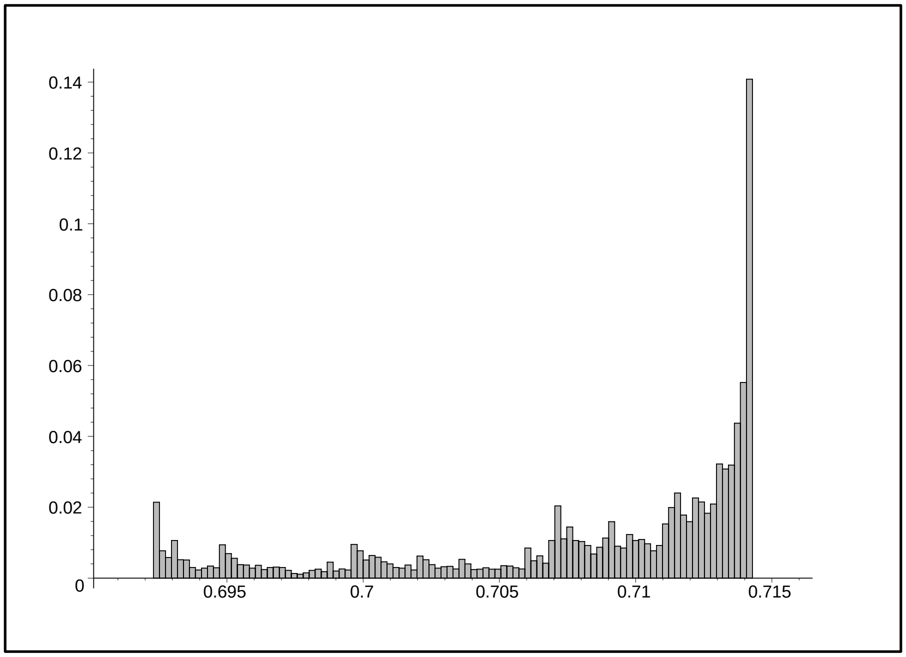

FIGURE 1. The values taken by $C(i)$, for $2^{20}+1 \leq i \leq 2^{20}+20001$, and i odd. The height of each bar in the histogram is the fraction of values falling in the associated range.

We denote by $N(m) = N(\underline{a})$ and $D(m) = D(\underline{a})$ the numerator and denominator, respectively, of this fraction.

The values of C for $10,000$ consecutive odd numbers, starting with $m = 2^{20} + 1$, are displayed in Figure 1. This histogram, produced using `Maple`'s statistical package, suggests that C may have an interesting limiting distribution.

The main results of this paper are sharp upper and lower bounds for the values of C. If $m \geq 3$ is an odd integer whose trajectory is finite, we show in Proposition 2.1 and Theorem 3.8 that $9/13 < C(m) < 5/7$. In Proposition 2.1 and Proposition 3.10, we find explicit sequences $\{n_k\}$ and $\{m_k\}$ such that $C(n_k) \to 9/13$ and $C(m_k) \to 5/7$ as $k \to \infty$.

Our methods are elementary, but some of the proofs are not easy. We conclude the paper with some informal and inexpert remarks on the distribution of C and the distribution of its randomized counterpart.

2. The lower bound

In Proposition 2.1, we show that $C(m) > 9/13$ when $m \geq 3$ is an odd integer whose trajectory is finite. Our argument is local, in that we need only work with an initial segment $m, 3m+1, (3m+1)/2, \ldots, (3m+1)/2^k$ of the trajectory. This local approach would not be of any use in proving the more difficult inequality $C(m) < 5/7$.

Proposition 2.1. *Suppose that m is an odd integer, $m \geq 3$, and the trajectory of m is finite. Then $C(m) > 9/13$. Moreover, $\lim_{n\to\infty} C((4^n-1)/3) = 9/13$.*

Proof. We proceed by induction on the length of the trajectory of m. Let $(3m+1)/2^k$ be the first odd number after m in this trajectory.

First suppose that $(3m+1)/2^k > 1$. Let $C((3m+1)/2^k) = N/D$, with $N = N(\frac{3m+1}{2^k})$ and $D = D(\frac{3m+1}{2^k})$, as above. Then $C(m)$ equals

$$\frac{m(3m+1) + \frac{(3m+1)^2}{2} + \cdots + \frac{(3m+1)^2}{2^{2k-1}} + N + m - \frac{3m+1}{2^k}}{m^2 + (3m+1)^2 + \frac{(3m+1)^2}{4} + \cdots + \frac{(3m+1)^2}{2^{2k-2}} + D}$$
$$= \frac{m(3m+2) + (3m+1)^2\left(\frac{1}{2} + \frac{1}{8} + \cdots + \frac{1}{2^{2k-1}}\right) - \frac{3m+1}{2^k} + N}{m^2 + (3m+1)^2\left(1 + \frac{1}{4} + \cdots + \frac{1}{2^{2k-2}}\right) + D}.$$

Let $\alpha = 1/2 + 1/8 + \cdots + 1/2^{2k-1}$. We must show that

$$\frac{3m^2 + 2m + \alpha(3m+1)^2 - \frac{3m+1}{2^k} + N}{m^2 + 2\alpha(3m+1)^2 + D} > \frac{9}{13}.$$

Since $N/D > 9/13$ by the inductive hypothesis, it suffices to show that

$$13\left(3m^2 + 2m + \alpha(3m+1)^2 - \frac{3m+1}{2^k}\right) > 9(m^2 + 2\alpha(3m+1)^2).$$

Clearing the denominators, this reduces to

$$30m^2 + 26m - \frac{13(3m+1)}{2^k} > 5\alpha(3m+1)^2. \tag{1}$$

Now $\alpha < 2/3$. If $k \geq 3$, it suffices to prove

$$30m^2 + 26m - \frac{13(3m+1)}{8} > \frac{10}{3}(3m+1)^2$$

for all odd $m \geq 5$. After simplification, the last inequality becomes

$$6m > \frac{39}{8}m + \frac{13}{8} + \frac{10}{3}$$

which indeed holds for $m \geq 5$.

If $k \leq 2$, we use the exact values of α, namely $1/2$ and $5/8$ for k respectively equal to 1 and 2, in Equation (1). After simplification, (1) reduces to $15m^2 > 20m + 51$ when $k = 2$ and to $15m^2 > 17m + 18$ when $k = 1$. These inequalities hold for all $m \geq 3$.

In the base case we have $(3m+1)/2^k = 1$. Equivalently, k is even and $m = (4^{k/2} - 1)/3$. Suppose $n = k/2$, with $n \geq 2$. A short computation shows that

$$C(m) = \frac{2^{4n} - 1}{\left(\frac{2^{2n}-1}{3}\right)^2 + \frac{2^{4n+2}-1}{3}} = \frac{9 \cdot 2^{4n} - 9}{13 \cdot 2^{4n} - 2^{2n+1} - 2}.$$

Observing that $C((4^n - 1)/3) > 9/13$ for all $n \geq 2$ completes the induction and clearly $C((4^n - 1)/3) \to 9/13$ as $n \to \infty$. □

3. The upper bound

Our main goal in this section is to prove that $C(m) < 5/7$ whenever $m \geq 3$ is an integer whose trajectory is finite. To achieve this goal, we need to replace $(3x+1)$–trajectories by $3x$–trajectories, and we need to replace C by an "unwrapped" variant C' defined below. We shall see that C' is easier to work with than C, but C' does not satisfy the inequality of Proposition 2.1; e.g. $C'(5) < 9/13$.

Definition 3.1. Let $\underline{a} = a_1, \ldots a_{n+1}$ be a sequence of positive real numbers. Let $N'(\underline{a}) = a_1a_2 + a_2a_3 + \cdots + a_na_{n+1}$. Let $D'(\underline{a}) = a_1^2 + \cdots + a_n^2$, and let $C'(\underline{a}) = N'(\underline{a})/D'(\underline{a})$.

Definition 3.2. Let $\underline{a} = a_1, \ldots, a_{n+1}$ be a sequence of positive numbers. We say that $\underline{a}$ is a $(3x, x/2)$–sequence if, for each $i \leq n$, either $a_{i+1} = 3a_i$ or $a_{i+1} = a_i/2$. We say that $\underline{a}$ is a $(3x+1, x/2)$–sequence if, for each $i \leq n$, either $a_{i+1} = 3a_i + 1$ or $a_{i+1} = a_i/2$.

The following formula makes C' and C less mysterious.

Proposition 3.3. *If $\underline{a} = a_1, \ldots, a_{n+1}$ is a $(3x, x/2)$–sequence, then*

$$5D'(\underline{a}) - 7N'(\underline{a}) = 2(a_1^2 - a_{n+1}^2) \ ,$$

or equivalently,

$$C'(\underline{a}) = \frac{5}{7} - \frac{2}{7} \cdot \frac{a_1^2 - a_{n+1}^2}{a_1^2 + \cdot + a_n^2}.$$

Proof. It suffices to prove the first equality. We proceed by induction on n. Direct computation shows that the result is valid when $n = 1$, so we suppose that $n > 1$.

For $1 \leq i \leq n$, let $c_i = a_{i+1}$ and let $\underline{c} = c_1, \ldots, c_n$. Let $\underline{b} = a_1, a_2$. Now

$$5D'(\underline{a}) - 7N'(\underline{a}) = (5D'(\underline{b}) - 7N'(\underline{b})) + (5D'(\underline{c}) - 7N'(\underline{c})).$$

The inductive hypothesis implies that the right-hand side of the last equation equals

$$2(b_1^2 - b_2^2) + 2(c_1^2 - c_n^2) = 2(a_1^2 - a_2^2 + a_2^2 - a_{n+1}^2) = 2(a_1^2 - a_{n+1}^2)$$

as desired. □

The next few results keep track of C' when we pass from $(3x, x/2)$–sequences to $(3x+1, x/2)$–sequences.

Definition 3.4. Let $\underline{a} = a_1, \ldots, a_{n+1}$ be a $(3x, x/2)$–sequence. Let $I = \{i \leq n : a_{i+1} = 3a_i\}$. Define a sequence $\underline{b} = b_1, \ldots, b_{n+1}$ by setting $b_1 = a_1$, by letting $b_{i+1} = 3b_i + 1$ when $i \in I$, and by letting $b_{i+1} = b_i/2$ if $i \leq n$ and $i \notin I$. We call $\underline{b}$ the $(3x+1, x/2)$–sequence associated with $\underline{a}$. Each $(3x+1, x/2)$–sequence is the associate of a unique $(3x, x/2)$–sequence.

Lemma 3.5. *Let $\underline{a} = a_1, \ldots, a_{n+1}$ be a sequence of positive numbers. Fix k with $1 \leq k \leq n$. Suppose that $a_{k+1} = 3a_k$ and that $a_{n+1}/a_{k+1} \leq 1/2$. Let $\mu = 1 + 1/a_{k+1}$. Suppose that*

$$5D'(a_{k+1}, \ldots, a_{n+1}) - 7N'(a_{k+1}, \ldots, a_{n+1}) \geq 2(a_{k+1}^2 - a_{n+1}^2).$$

Then the sequence

$$\underline{b} = a_1, \ldots, a_k, \mu a_{k+1}, \ldots, \mu a_{n+1} \tag{2}$$

satisfies $5D'(\underline{b}) - 7N'(\underline{b}) > 5D'(\underline{a}) - 7N'(\underline{a})$.

Proof. We have

$$\begin{aligned}
&5(D'(\underline{b}) - D'(\underline{a})) - 7(N'(\underline{b}) - N'(\underline{a})) \\
&= -7(\mu-1)a_k a_{k+1} \\
&\quad +(\mu^2-1)[5(a_{k+1}^2+\cdots+a_n^2) - 7(a_{k+1}a_{k+2}+\cdots+a_n a_{n+1})] \\
&\geq -7a_k + 2(\mu^2-1)(a_{k+1}^2 - a_{n+1}^2) \\
&= \frac{-7a_{k+1}}{3} + 2\left(\frac{2}{a_{k+1}} + \frac{1}{a_{k+1}^2}\right)(a_{k+1}^2 - a_{n+1}^2) \\
&= \left[\frac{-7}{3} + 2\left(2+\frac{1}{a_{k+1}}\right)\left(1-\left(\frac{a_{n+1}}{a_{k+1}}\right)^2\right)\right] a_{k+1} \\
&\geq \left(\frac{-7}{3} + 2(2)\left(\frac{3}{4}\right)\right) a_{k+1} = \frac{2}{3}a_{k+1} > 0 \ ,
\end{aligned}$$

as desired. □

Lemma 3.6. *Let $\underline{a} = a_1, \ldots, a_{n+1}$ be a $(3x, x/2)$–sequence and let $\underline{b} = b_1, \ldots, b_{n+1}$ be the $(3x+1, x/2)$–sequence associated with $\underline{a}$, as in Definition 3.4. If $b_{n+1}/b_i \leq 1/2$ for all $i \leq n$, then we have*

$$5D'(\underline{b}) - 7N'(\underline{b}) \geq 5D'(\underline{a}) - 7N'(\underline{a}).$$

Proof. Let $I = \{i \leq n : a_{i+1} = 3a_i\}$. Let $r = |I|$ and suppose $I = \{i_1, \ldots, i_r\}$ with $i_1 < \cdots < i_r$. Let $\underline{b}^{(0)} = \underline{a}$. Let $\underline{b}^{(1)}$ be the sequence derived from $\underline{b}^{(0)}$ as in Lemma 3.5, equation (2) with $k = i_1$. Let $\underline{b}^{(2)}$ be the sequence derived from $\underline{b}^{(1)}$ as in Lemma 3.5, with $k = i_2$. Continuing in this way, we reach a sequence $\underline{b}^{(r)}$ derived from $\underline{b}^{(r-1)}$ as in Lemma 3.5, with $k = i_r$. Since $b_1 = a_1$, $\underline{b}_{k+1}^{(r)} = 3\underline{b}_k^{(r)} + 1$ for $k \in I$, and $\underline{b}_{k+1}^{(r)} = \underline{b}_k^{(r)}/2$ for $k \leq n$ and $k \notin I$, we have $\underline{b}^{(r)} = \underline{b}$.

For $j \leq r$, we claim that

$$5D'(\underline{b}^{(j)}) - 7N'(\underline{b}^{(j)}) \geq 5D'(\underline{b}^{(j-1)}) - 7N'(\underline{b}^{(j-1)}) \ .$$

To establish this claim, we need to verify the three hypotheses of Lemma 3.5 for $k = i_j$. The fact that $k \in I$ and the definitions of $\underline{b}^{(0)}, \ldots, \underline{b}^{(j-1)}$ imply that $\underline{b}_{k+1}^{(j-1)} = 3\underline{b}_k^{(j-1)}$. Since all r μ-factors, in the notation of Lemma 3.5, are greater than 1, we have

$$\frac{b_{n+1}^{(0)}}{b_i^{(0)}} \leq \frac{b_{n+1}^{(1)}}{b_i^{(1)}} \leq \cdots \leq \frac{b_{n+1}^{(r)}}{b_i^{(r)}} = \frac{b_{n+1}}{b_i} \leq \frac{1}{2}$$

for all i. In particular, $b_{n+1}^{(j-1)}/b_{k+1}^{(j-1)} \leq 1/2$. To verify the third and final hypothesis of Lemma 3.5, we observe that $b_{k+1}^{(j-1)}, \ldots, b_{n+1}^{(j-1)}$ is a $(3x, x/2)$–sequence. Hence Proposition 3.3 implies that

$$5D'(b_{k+1}^{(j-1)}, \ldots, b_{n+1}^{(j-1)}) - 7N'(b_{k+1}^{(j-1)}, \ldots, b_{n+1}^{(j-1)}) = 2((b_{k+1}^{(j-1)})^2 - (b_{n+1}^{(j-1)})^2).$$

Thus the claim is established, and so Lemma 3.5 yields

$$5D'(\underline{b}^{(r)}) - 7N'(\underline{b}^{(r)}) > \cdots > 5D'(\underline{b}^{(0)}) - 7N'(\underline{b}^{(0)}).$$

Hence $5D'(\underline{b}) - 7N'(\underline{b}) \geq 5D'(\underline{a}) - 7N'(\underline{a})$, provided that $r > 0$. If $r = 0$, then $\underline{a} = \underline{b}$ and of course $5D'(\underline{b}) - 7N'(\underline{b}) = 5D'(\underline{a}) - 7N'(\underline{a})$. □

Corollary 3.7. *Let $\underline{b} = b_1, \ldots, b_{n+1}$ be a $(3x+1, x/2)$–sequence. Suppose that $b_{n+1}/b_i \leq 1/2$ for all $i \leq n$. Then $C'(\underline{b}) < 5/7$.*

Proof. Let $\underline{a}$ be the $(3x, x/2)$–sequence associated with $\underline{b}$, as in Definition 3.4. Then $a_1 = b_1$ and $a_{n+1} \leq b_{n+1}$. By Lemma 3.6,

$$5D'(\underline{b}) - 7N'(\underline{b}) \geq 5D'(\underline{a}) - 7N'(\underline{a}).$$

By Proposition 3.3,

$$5D'(\underline{a}) - 7N'(\underline{a}) = 2(a_1^2 - a_{n+1}^2) \geq 2(b_1^2 - b_{n+1}^2).$$

Hence

$$\frac{5}{7} - C'(\underline{b}) \geq \frac{2(b_1^2 - b_{n+1}^2)}{7(b_1^2 + \cdots + b_n^2)} > 0.$$

Thus $C'(\underline{b}) < 5/7$, as desired. □

Theorem 3.8. *Let m be a not necessarily odd integer. Suppose that $m \geq 3$ and the $(3x+1)$–trajectory of m is finite. Then $C(m) < 5/7$.*

Proof. Let $\underline{c} = c_1, \ldots, c_n$ be the $(3x+1)$–trajectory of m. Define $\underline{b} = b_1, \ldots, b_{n+1}$ by letting $b_i = c_i$ for $i \leq n$ and setting $b_{n+1} = 1/2$. Let $\underline{a} = a_1, \ldots, a_{n+1}$ be the $(3x, x/2)$–sequence associated with $\underline{b}$, as in Definition 3.4. Then $a_1 = b_1 = m$, $b_n = 1$, and $a_{n+1} \leq b_{n+1} = 1/2$.

We have $C(m) < 5/7$ if and only if

$$5D'(\underline{b}) - 7(N'(\underline{b}) - b_n b_{n+1} + b_n b_1) > 0.$$

By Lemma 3.6, the left-hand side of the last inequality is at least $5D'(\underline{a}) - 7N'(\underline{a}) + 7b_n b_{n+1} - 7b_n b_1$. Since $\underline{a}$ is a $(3x, x/2)$–sequence, Proposition 3.3 yields

$$5D'(\underline{a}) - 7N'(\underline{a}) = 2(a_1^2 - a_{n+1}^2) \geq 2(b_1^2 - b_{n+1}^2).$$

Hence

$$\begin{aligned} 5D'(\underline{b}) - 7(N'(\underline{b}) - b_n b_{n+1} + b_n b_1) &\geq 2(b_1^2 - b_{n+1}^2) + 7b_n b_{n+1} - 7b_n b_1 \\ &= 2b_1^2 - 1/2 + 7/2 - 7b_1 \\ &= 2b_1^2 - 7b_1 + 3 \\ &= 2m^2 - 7m + 3. \end{aligned}$$

The last quantity is positive when $m > 3$, and so $C(m) < 5/7$ when $m > 3$. One can check directly that $C(3) < 5/7$. This completes the proof. □

Our next task is to exhibit a sequence $\{m(k)\}_{k=2}^{\infty}$ of odd integers such that $C(m(k)) \to 5/7$ as $k \to \infty$. Both Figure 1 and Proposition 3.3 suggest that such sequences exist in abundance, but it is not trivial to find one explicitly.

Lemma 3.9. *Let $\{m(k)\}_{k=2}^{\infty}$ be a sequence of positive odd integers. Let $\underline{b} = \underline{b}(k) = b_1, \ldots, b_{n+1}$ be the $(3x+1)$–trajectory of $m(k)$. Suppose that:*

(1) *$b_1, b_3, \ldots, b_{2k-1}$ are odd,*
(2) *b_{2k+1} is a power of 2,*
(3) *$\lim_{k\to\infty} k/m(k) = 0$.*

Then $\lim_{k\to\infty} C(m(k)) = 5/7$.

Proof. Let $m = m(k)$. We have

$$C(\underline{b}) = \frac{N'(\underline{b}) + m}{D'(\underline{b}) + 1} \geq \frac{N'(\underline{b})}{D'(\underline{b})} = C'(\underline{b}).$$

Since $C(\underline{b}) < 5/7$ by Theorem 3.8, it suffices to show that $C'(\underline{b}) \to 5/7$ as $k \to \infty$.

Let $\underline{a} = a_1, \ldots, a_{n+1}$ be the $(3x, x/2)$–sequence associated with $\underline{b}$, as in Definition 3.4. Let $\underline{a} = \underline{b}^{(0)}, \underline{b}^{(1)}, \ldots, \underline{b}^{(r)} = \underline{b}$ be as in the proof of Lemma 3.6. In the notation of that proof we have $r = k$ and $I = I(\underline{a}) = \{1, 3, \ldots, 2k-1\}$. If $1 \leq j \leq k$, then passing from $\underline{b}^{(j-1)}$ to $\underline{b}^{(j)}$ involves multiplication by a factor $\mu = \mu_j$ as in Lemma 3.5, and $\mu_j \leq 1 + 1/a_1 = 1 + 1/m$ for all j. It follows that

$$1 \leq \frac{b_i}{a_i} \leq \left(1 + \frac{1}{m}\right)^k$$

for $1 \leq i \leq n+1$. Hence both $N'(\underline{b})/N'(\underline{a})$ and $D'(\underline{b})/D'(\underline{a})$ lie between 1 and $(1+1/m)^{2k}$, and so

$$\frac{C'(\underline{b})}{C'(\underline{a})} = \frac{N'(\underline{b})/N'(\underline{a})}{D'(\underline{b})/D'(\underline{a})}$$

lies between $(1+1/m)^{2k}$ and $(1+1/m)^{-2k}$. Since $\lim_{k\to\infty} k/m = 0$, we have $\lim_{k\to\infty}(1+1/m)^{2k} = 1$.

Thus it suffices to show that $C'(\underline{a}) \to 5/7$ as $k \to \infty$. By Proposition 3.3, we have

$$C'(\underline{a}) = \frac{5}{7} - \frac{2}{7}\,\frac{(a_1^2 - a_{n+1}^2)}{(a_1^2 + \cdots + a_n^2)}.$$

Since $a_1 = m$, $a_{n+1} = 1$, and $a_{2k+1} = (3/2)^k m$, it follows that $C'(\underline{a}) \to 5/7$ as $k \to \infty$, as desired. □

It is well known (see [1]) and easily proved by induction that for any positive integer c, the trajectory of $m = 2^k c - 1$ satisfies condition (1) of Lemma 3.9 and $b_{2k+1} = 3^k c - 1$. The following proposition selects a sequence $c(k)$ such that $m(k) = 2^k c(k) - 1$ satisfies the remaining conditions of Lemma 3.9.

Proposition 3.10. *Let $k \geq 2$, and let*

$$c = c(k) = \frac{2^{3^{k-1}} + 1}{3^k}.$$

Then c is an integer. If we set $m = m(k) = 2^k c - 1$, then $C(m) \to 5/7$ as $k \to \infty$.

Proof. Let $m(k) = 2^k c - 1$, with c to be determined. We must choose c so that the hypotheses of Lemma 3.9 are satisfied. Let $\underline{b} = b_1, \ldots, b_{n+1}$ be the trajectory of $m(k)$. For $0 \leq i \leq k$, we have $b_{2i+1} = 2^{k-i}3^i c - 1$ hence hypothesis (1) of Lemma 3.9 is satisfied. We must choose c so that $b_{2k+1} = 3^k c - 1$ is a power of 2, say $3^k c - 1 = 2^d$. Thus we must find a solution d to the congruence $2^d \equiv -1 \pmod{3^k}$.

Recall that the unit group $(\mathbb{Z}/3^k\mathbb{Z})^*$ is cyclic of order $2(3^{k-1})$ and that the order 3^{k-1} subgroup is generated by 4. This can be directly checked by an inductive argument using the binomial theorem to verify that $(1+3)^{3^{j-1}} \equiv 1+3^j \pmod{3^{j+1}}$. Hence we find that 2 generates $(\mathbb{Z}/3^k\mathbb{Z})^*$ so $d = 3^{k-1}$ is the smallest positive integral solution to $2^d \equiv -1 \pmod{3^k}$. Thus if we let $c = (2^{3^{k-1}}+1)/3^k$, then c is an integer.

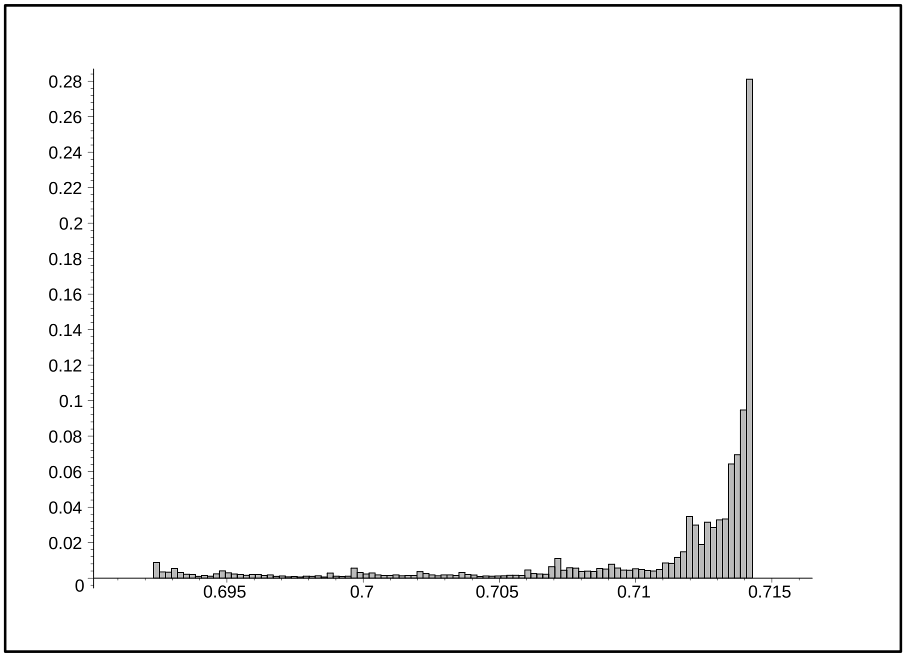

FIGURE 2. The values taken by $C(\underline{a})$, for $10,000$ random simulated trajectories each starting at $2^{20}+1$. The height of each bar in the histogram is the fraction of values falling in the associated range.

We therefore set $m = 2^k c - 1$, so that b_{2k+1} is a power of 2, fulfilling hypothesis (2) of Lemma 3.9.

Obviously $k/m(k) \to 0$ as $k \to \infty$. Thus all hypotheses of Lemma 3.9 are satisfied, and so $C(m(k)) \to 5/7$, as desired. □

4. The distribution

Generating a $(3x+1)$–trajectory is, to quote from [1, p. 20], a "deterministic process that simulates random behavior". To make this more explicit, we consider a random sequence which begins with a positive number $a_1 > 1$ and is defined inductively as follows. Suppose the sequence has been generated up to its ith term a_i. We then select one of two alternatives, each with probability 1/2. The first alternative is to set $a_{i+1} = a_i/2$. The second alternative is to set $a_{i+1} = 3a_i + 1$ and $a_{i+2} = a_{i+1}/2$. Thus, at each step, the length of the sequence increases by either one or two terms. This random sequence is deemed to terminate as soon as a term equal to or less than 1 is reached; of course infinite sequences are possible. Since we wish to simulate $(3x+1)$–trajectories that begin with odd numbers, we further require that the first step of the random process go from the one-term sequence a_1 to the three-term sequence $a_1, 3a_1+1, (3a_1+1)/2$.

For such a random sequence $\underline{a} = a_1, \ldots, a_n$, we can define $C(\underline{a})$ as in Section 1. In Figure 2, we display the values of $C(\underline{a})$ for $10,000$ random sequences $\underline{a}$, each beginning with $a_1 = 2^{20}+1$.

One can then ask how the limiting distributions suggested by Figure 1 and Figure 2 are related. We must leave this as an open problem; presumably it would be easier to prove results about the distribution of C for random sequences than it would be to prove results about the distribution of C for actual $(3x+1)$–trajectories.

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