

HÖLDER ESTIMATES OF SOLUTIONS TO A DEGENERATE DIFFUSION EQUATION

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ABSTRACT. This paper is concerned with the Hölder estimates of weak solutions of the Cauchy problem for the general degenerate parabolic equations

$$u_t = \Delta G(u) + \sum_{j=1}^N f_j(u)_{x_j} + h(u),$$

with the initial data $u(x, 0) = u_0(x_1, x_2, \dots, x_N)$, where the diffusion function $G(u)$ can be a constant on a nonzero measure set, such as the equations of two-phase Stefan type. Some explicit Hölder exponents of the composition function $G(u)$ with respect to the space variables are obtained by using the maximum principle.

This paper is concerned with the Hölder estimates of solutions of the Cauchy problem for the general degenerate parabolic equations

$$(1) \quad u_t = \Delta G(u) + \sum_{j=1}^N f_j(u)_{x_j} + h(u),$$

with initial data

$$(2) \quad u(x, 0) = u_0(x_1, x_2, \dots, x_N),$$

where $G(u)$ is a nondecreasing smooth function, and N denotes the space dimension. If $G(u) \equiv \text{const}$ and one of the $f_j(u)$ is nonlinear, (1) is the standard nonlinear hyperbolic conservation law with a source. In this case, the continuous solution of (1), in general, exists only locally in time. This means that shock waves appear in the solution in finite time.

It is well-known that degenerate parabolic equations do not possess classical solutions. When $g(u) = G'(u) = 0$ at some points, the solution u of the Cauchy problem (1) and (2), in general, fails to be smooth at the interface between a parabolic region and a region of parabolic degeneracy. Fortunately, if the diffusion function $G(u)$ has only one degenerate point, such as the porous media equation, i.e., $G(u) = u^m$ ($m > 1$), the regularity of weak solutions can be proved. We refer to the references in the survey paper [9] and the most recent papers [4, 8, 10, 11, 12, 14] for the details.

If $G(u)$ is degenerate on an interval, for instance the two-phase Stefan problem, the continuity of $G(u)$ was first proved in [3], and the Hölder continuity of $G(u)$

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with an implicit Hölder exponent was obtained in [6]. Some special cases for $G(u)$ having a degenerate interval were also studied in [13].

In this paper, we study the Hölder continuity of solutions, in $R^N \times [0, T]$ for any given time T , for the Cauchy problem (1), (2) in more general functions $G(u), f_j(u)$ and $h(u)$.

The main results in this paper are given in the following theorem:

Theorem 1. *Suppose that $G'(u) \geq 0$ and the solution u is bounded: $|u| \leq \bar{u}$. If*

$$(3) \quad \left| \frac{GG''}{G'^2} \right| \leq \beta, \quad \beta^2 \leq \frac{1}{2N},$$

and

$$(4) \quad |f_j''(u)| \leq c|G''(u)|, \quad |h'(u)| \leq cG'(u)$$

for a suitable positive constant c , then the solutions u of the Cauchy problem (1),(2) satisfy the global estimates

$$(5) \quad |\nabla G^\alpha(u)| \leq M \quad \text{on} \quad R^N \times R^+$$

if the initial data (2) satisfies the same regular estimate (5), where M is a positive constant depending only on the initial data, and

$$(6) \quad \alpha > 1 - \frac{1 + \sqrt{1 - 2N\beta^2}}{4}.$$

The equality in (6) is true if $h(u) \equiv 0$ and $f_j(u) \equiv 0$.

Proof of Theorem 1. Without loss of generality, let $G(u) \geq 0$. We consider the Cauchy problem for an equation related to (1),

$$(7) \quad u_t = \Delta G^\epsilon(u) + \sum_{j=1}^N f_j(u)_{x_j} + h(u),$$

with the initial data

$$(8) \quad u^\epsilon(x, 0) = u_0(x_1, x_2, \dots, x_N) + \epsilon_1,$$

where $G^\epsilon(u) = G(u) + \epsilon(u + \bar{u})$, with ϵ and ϵ_1 small nonnegative constants. For the porous medium equation, we choose $\epsilon = 0, \epsilon_1 > 0$; for other cases, we choose $\epsilon_1 = 0, \epsilon > 0$. Thus for fixed ϵ and ϵ_1 , the equation (7) is strictly parabolic. We can resolve the Cauchy problem (7), (8) and obtain the approximate solutions u^{ϵ, ϵ_1} . Since we assume u^{ϵ, ϵ_1} to be bounded, there exists a subsequence $u^{\epsilon^l, \epsilon_1^l}$ converging weakly to a bounded function u as $\epsilon^l, \epsilon_1^l \rightarrow 0^+$, on any bounded region. If we can prove the Hölder continuity of $G(u^{\epsilon, \epsilon_1})$, then $G(u^{\epsilon^l, \epsilon_1^l})$ converges weakly to $G(u)$ by applying the Minty device with the condition $G'(u) \geq 0$. From the conditions in (4), we can prove that $f_j(u^{\epsilon^l, \epsilon_1^l})$ and $h(u^{\epsilon^l, \epsilon_1^l})$ converge weakly to $f_j(u)$ and $h(u)$ respectively. Therefore the limit u is a weak solution of the Cauchy problem (1) and (2), and $G(u)$ satisfies the same Hölder estimates. This technique is standard, and the details can be found in [12]. Here we only give the proof of the uniform estimate (5).

Make the transformation

$$(9) \quad w = \frac{1}{2} \sum_{i=1}^N (G^\epsilon(u)_{x_i})^2$$

and set $P(u) = G^\epsilon(u)$. From (7), we obtain

$$(10) \quad P_t = P'(u)\Delta P + \sum_{j=1}^N f'_j(u)P_{x_j} + P'(u)h(u),$$

(11)

$$(P_{x_i})_t = P'(u)\Delta(P_{x_i}) + \frac{P''(u)}{P'(u)}P_{x_i}\Delta P + \left(\sum_{j=1}^N f'_j(u)P_{x_j}\right)_{x_i} + (P'(u)h(u))_{x_i}$$

and

$$(12) \quad \begin{aligned} w_t = P'(u)\Delta w - \sum_{i,j=1}^N P'(u)(P_{x_i x_j})^2 + \frac{2P''(u)}{P'(u)}w\Delta P + \sum_{j=1}^N f'_j(u)w_{x_j} \\ + \sum_{j=1}^N \frac{2f''_j(u)}{P'(u)}P_{x_j}w + 2\left(\frac{P''(u)h(u)}{P'(u)} + h'(u)\right)w. \end{aligned}$$

Let $z = P^s w$, with s a constant. By simple calculations,

$$(13) \quad \begin{aligned} \Delta w = \Delta(P^{-s}z) &= \sum_{j=1}^N (P^{-s}z)_{x_j x_j} \\ &= \sum_{j=1}^N [(P^{-s})_{x_j x_j}z + 2(P^{-s})_{x_j}z_{x_j} + P^{-s}z_{x_j x_j}] \\ &= \sum_{j=1}^N [(-sP^{-s-1}P_{x_j x_j} + s(s+1)P^{-s-2}P_{x_j}^2)z \\ &\quad + 2(P^{-s})_{x_j}z_{x_j} + P^{-s}z_{x_j x_j}] \\ &= (-sP^{-s-1}\Delta P + 2s(s+1)P^{-s-2}w)z \\ &\quad + \sum_{j=1}^N [-2sP^{-s-1}P_{x_j}z_{x_j} + P^{-s}z_{x_j x_j}]. \end{aligned}$$

Then we have by (10)–(13) that

$$(14) \quad \begin{aligned} z_t = P'(u)\Delta z - 2sP^{-1}P'(u)\sum_{i=1}^N P_{x_i}z_{x_i} \\ + 2s(s+1)P^{-s-2}P'(u)z^2 - \sum_{i,j=1}^N P^s P'(u)(P_{x_i x_j})^2 + \frac{2P''(u)}{P'(u)}z\Delta P \\ + \sum_{j=1}^N f'_j(u)z_{x_j} + \sum_{j=1}^N \frac{2f''_j(u)}{P'(u)}P_{x_j}z \\ + (sP^{-1}P'(u)h(u) + 2h'(u) + \frac{2P''(u)h(u)}{P'(u)})z. \end{aligned}$$

Since

$$(15) \quad \sum_{i,j=1}^N (P_{x_i x_j})^2 \geq \frac{1}{N}(\Delta P)^2,$$

we choose s such that

$$(16) \quad s(s+1) + \frac{N}{2} \left(\frac{P''P}{P'^2} \right)^2 < 0;$$

then $-1 < s < 0$ and there exists a suitable constant C such that

$$(17) \quad \begin{aligned} & 2s(s+1)P^{-s-2}P'(u)z^2 - \sum_{i,j=1}^N P^s P'(u)(P_{x_i x_j})^2 + \frac{2P''(u)}{P'(u)}z\Delta P \\ & \leq 2s(s+1)P^{-s-2}P'(u)z^2 - \frac{1}{N}P^s P'(u)(\Delta P)^2 + \frac{2P''(u)}{P'(u)}z\Delta P \\ & \leq -CP^{-s-2}P'(u)z^2. \end{aligned}$$

From (14) and (17), we have

$$(18) \quad \begin{aligned} z_t & \leq P'(u)\Delta z - 2sP^{-1}P'(u) \sum_{i=1}^N P_{x_i} z_{x_i} \\ & - CP^{-s-2}P'(u)z^2 + \sum_{j=1}^N f'_j(u)z_{x_j} + \sum_{j=1}^N \frac{2f''_j(u)}{P'(u)}P_{x_j}z \\ & + (sP^{-1}P'(u)h(u) + 2h'(u) + \frac{2P''(u)h(u)}{P'(u)})z. \end{aligned}$$

If the conditions in Theorem 1 are satisfied, then there exist constants $c_i > 0, i = 1, 2, 3, 4, 5$, such that

$$(19) \quad \begin{aligned} \left| \sum_{j=1}^N \frac{2f''_j(u)}{P'(u)}P_{x_j}z \right| & \leq 2c \frac{|G''(u)|}{P'(u)} \sum_{j=1}^N |P_{x_j}|z \\ & \leq c_1 \frac{|G''(u)|}{P'(u)} w^{\frac{1}{2}}z = c_1 \frac{P^{\frac{s}{2}+2}|G''(u)|}{P'(u)^2} P^{-s-2}P'(u)z^{\frac{3}{2}}, \end{aligned}$$

where

$$(20) \quad c_1 \frac{P^{\frac{s}{2}+2}|G''(u)|}{P'(u)^2} \leq c_2 \frac{P(u)|G''(u)|}{P'(u)^2} \leq c_2 M(\epsilon),$$

and $M(\epsilon) \rightarrow \beta$ as $\epsilon \rightarrow 0$. So $M(\epsilon)$ is uniformly bounded. Moreover,

$$(21) \quad |(sP^{-1}P'(u)h(u)z)| = |sP^{s+1}h(u)(P^{-s-2}P'(u)z)| \leq c_3 P^{-s-2}P'(u)z,$$

$$(22) \quad |2h'(u)z| = \left| \frac{2h'(u)P^{s+2}}{P'(u)}(P^{-s-2}P'(u)z) \right| \leq c_4 P^{-s-2}P'(u)z$$

and

$$(23) \quad \begin{aligned} \left| \frac{2P''(u)h(u)}{P'(u)}z \right| & = \left| 2P^{s+1}h(u) \left(\frac{P(u)G''(u)}{P'(u)^2} \right) (P^{-s-2}P'(u)z) \right| \\ & \leq c_5 \left(\frac{P(u)|G''(u)|}{P'(u)^2} \right) (P^{-s-2}P'(u)z) \leq c_5 M(\epsilon) (P^{-s-2}P'(u)z). \end{aligned}$$

We have from (18)–(23) that

$$(24) \quad \begin{aligned} z_t & \leq P'(u)\Delta z - 2sP^{-1}P'(u) \sum_{i=1}^N P_{x_i} z_{x_i} \\ & + (-Cz^2 + c_2 M(\epsilon)z^{\frac{3}{2}} + (c_3 + c_4)z + c_5 M(\epsilon)z) P^{-s-2}P'(u), \end{aligned}$$

where $M(\epsilon)$ is uniformly bounded. Thus we have $z \leq M$ for a suitable large positive constant M by using the maximum principle in (24).

From condition (3),

$$(25) \quad \left(\frac{P''P}{P'^2}\right)^2 = \left(\frac{G''(u)(G(u) + \epsilon(u + \bar{u}))}{(G'(u) + \epsilon)^2}\right)^2 \leq \beta^2 + \beta(\epsilon),$$

where $\beta(\epsilon) > 0$ is a function of ϵ and $\beta(\epsilon) \rightarrow 0$ as $\epsilon \rightarrow 0$. If we choose s^* to be the smaller root of the equation

$$(26) \quad s(s+1) + \frac{N}{2}(\beta^2 + \beta(\epsilon)) = 0,$$

then (16) is satisfied. Thus we have

$$(27) \quad P^{s^*} w \leq M,$$

where

$$(28) \quad s^* = \frac{-1 - \sqrt{1 - 2N(\beta^2 + \beta(\epsilon))}}{2}.$$

Letting $\epsilon \rightarrow 0$ in (28), we obtain the estimate (5), and with it the proof of Theorem 1. \square

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