

## ON AN ADJOINT FUNCTOR TO THE THOM FUNCTOR

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ABSTRACT. We construct a right adjoint functor to the Thom functor, i.e., to the functor which assigns the Thom space  $T\xi$  to a vector bundle  $\xi$ .

### INTRODUCTION

Let  $\tau$  denote the functor which assigns the Thom space  $T\xi$  to a vector bundle  $\xi$ , and similarly for maps. The goal of this paper is to construct the right adjoint functor  $\lambda$  to the functor  $\tau$ .

To motivate this result, I remark that it is always nice to know whether a functor admits an adjoint one. However, here we have a more interesting motivation. Namely, it is useful to know when a space is the Thom space of a certain vector bundle (spherical fibration). For example, de-Thomification plays an important role in the theory of immersion of manifolds; see [BP], [C] and the survey [L]. In fact, Brown and Peterson [BP] de-Thomify a space, while Cohen [C] de-Thomifies a map. However, these de-Thomifications are very ad hoc. So, it is reasonable to want a de-Thomification machine, like the de-looping machine of May [M] or Boardman and Vogt [BV]. The following observation of Beck [B] plays the crucial role in the de-looping theory. The suspension functor  $S$  is the left adjoint to the loop functor  $\Omega$ , and so there is a monad  $M := \Omega S$ . Clearly, every loop space is a space over  $M$ . Conversely, if a space  $X$  is a space over  $M$ , then, using the simplicial resolution of the  $M$ -space  $X$ , one can provide a de-looping of  $X$  “at the simplicial level”, and then certain additional arguments enable us to lift this “simplicial de-looping” to the geometric level; see [B], [M].

Here we have a dual situation. As usual, the functor  $C := \tau\lambda$  is a comonad, and every Thom space is a space over  $C$ . Conversely, if  $X$  is a space over  $C$  then, dually to what we said above, one can take the cosimplicial resolution of  $X$  and provide a de-Thomification of  $X$  “at the cosimplicial level”. However, in order to do the next step, a lifting to the geometrical level, one must prove that the Thomification commutes with the functor Tot, and this problem looks quite complicated; cf. [Bo].

Summarizing, one can consider this paper as a first step in an attack on the de-Thomification problem.

Notice that the above arguments enable us to prove that a certain space is not a Thom space: it suffices to check that it is not a space over the comonad  $C$ . For

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example, we have (implicitly) used these arguments in [R1] in order to prove that the spectra  $k$  and  $kO$  are not Thom spectra.

THE CASE OF NON-ORIENTABLE BUNDLES

Let  $O_n$  be the group of orthogonal transformations of the Euclidean space  $\mathbb{R}^n$ , let  $BO_n$  denote its classifying space, and let  $\gamma$  denote the universal  $n$ -dimensional vector bundle over  $BO_n$ . Given a locally trivial bundle  $\xi$  with the fiber  $\mathbb{R}^n$  and structure group  $O_n$ , let  $T\xi$  denote the Thom space of  $\xi$ , i.e.,  $T\xi := D(\xi)/S(\xi)$ , where  $D(\xi)$  is the total space of the unit disc bundle and  $S(\xi)$  is the total space of the unit sphere subbundle of  $D(\xi)$ . We regard  $T\xi$  as a pointed space with the base point given by  $S(\xi)$ .

Let  $\mathcal{K}$  be the category whose objects are maps  $f : B \rightarrow BO_n$ , where  $B$  is a connected space and  $f$  is a map such that

$$\pi_1(B) \xrightarrow{f_*} \pi_1(BO_n) = \mathbb{Z}/2$$

is an epimorphism, and whose morphisms are commutative diagrams

$$\begin{array}{ccc} B & \xrightarrow{\varphi} & C \\ f \downarrow & & \downarrow g \\ BO_n & \xlongequal{\quad} & BO_n, \end{array}$$

where  $f$  and  $g$  are objects of  $\mathcal{K}$ . Let  $\mathcal{S}$  be the category whose objects are pointed spaces  $X$  with  $\pi_i(X) = 0$  for  $i < n$  and  $\pi_n(X) = \mathbb{Z}/2$ , and whose morphisms are maps  $f : X \rightarrow Y$  such that  $f_* : \pi_n(X) \rightarrow \pi_n(Y)$  is an isomorphism.

Let  $\tau : \mathcal{K} \rightarrow \mathcal{S}$  be the Thom functor which assigns the object  $\tau f := T(f^*\gamma) \in \mathcal{S}$  to the object  $f : X \rightarrow BO_n$  of  $\mathcal{K}$ .

**Theorem 1.** *The functor  $\tau$  admits a right adjoint functor  $\lambda : \mathcal{S} \rightarrow \mathcal{K}$ .*

*Proof.* We construct  $\lambda$  as follows. Choose any  $X \in \mathcal{S}$ . Given an integer  $k$ , let  $\Omega_k^n X$  be the component of  $\Omega^n X$  corresponding to  $k \in \pi_n(X) = \pi_0(\Omega^n X) = \mathbb{Z}/2$ . The standard  $O_n$ -action on  $\mathbb{R}^n$  yields the obvious  $O_n$ -action on  $S^n$ , which, in turn, induces a (right)  $O_n$ -action on  $\Omega^n X = (X, *)^{(S^n, *)}$ , and it is clear that every component  $\Omega_k^n X, k = 0, 1$ , is  $O_n$ -invariant. Convert the right  $O_n$ -action on  $\Omega_1^n X$  into a left  $O_n$ -action by setting  $ga = ag^{-1}, g \in O_n, a \in \Omega_1^n X$ . Consider the locally trivial bundle

$$p : EO_n \times_{O_n} \Omega_1^n X \rightarrow BO_n$$

which is associated with the universal principal  $O_n$ -bundle  $\Gamma := \{EO_n \rightarrow BO_n\}$ ; cf. [PS]. We define  $\lambda X$  to be the map  $p$ . The  $\lambda$ -action on morphisms is clear.

We prove that  $\lambda$  is right adjoint to  $\tau$ , i.e., that  $\mathcal{K}(f, \lambda X) = \mathcal{S}(T(f^*\gamma), X)$  for every  $f : B \rightarrow BO_n$ ; cf. [R2]. Indeed, consider the principal  $O_n$ -bundle

$$f^*\Gamma = \{q : E \rightarrow B\},$$

and let  $\xi$  be the  $\Omega_1^n X$ -bundle associated with  $f^*\Gamma$ , i.e.

$$\xi = \{E \times_{O_n} \Omega_1^n X \rightarrow B\}.$$

Then  $\xi$  is induced by  $f$  from the bundle  $\lambda X = \{p : EO_n \times_{O_n} \Omega_1^n X \rightarrow BO_n\}$ . So,  $\mathcal{K}(f, \lambda X) = \text{Sec } \xi$ , where  $\text{Sec } \xi$  denotes the set of all sections of  $\xi$ .

For every  $b \in B$  choose any  $O_n$ -equivariant map  $i_b : O_n \rightarrow E$  with  $qi_b(O_n) = b$ . We have (the first equality can be found e.g. in [H])

$$\begin{aligned} \text{Sec } \xi &= \{O_n\text{-equivariant maps } E \rightarrow \Omega_1^n X\} \\ &= \{O_n\text{-equivariant maps } f : E \rightarrow (X, *)^{(S^n, *)} \\ &\quad \text{such that } f(x) \in \Omega_1^n X \text{ for every } a \in E\} \\ &= \{\text{maps } f : E \times_{O_n} (S^n, *) \rightarrow (X, *) \text{ such that the map} \\ &\quad (S^n, *) = O_n \times_{O_n} (S^n, *) \xrightarrow{i_b} E \times_{O_n} (S^n, *) \xrightarrow{f} (X, *) \\ &\quad \text{belongs to } \Omega_1^n X \text{ for every } b\} \\ &= \mathcal{S}(T(f^*\gamma), X). \end{aligned}$$

□

THE CASE OF ORIENTABLE BUNDLES

Let  $BSO_n$  be the classifying space for the connected component  $SO_n$  of  $O_n$ . Let  $\mathcal{K}'$  be the category whose objects are maps  $f : B \rightarrow BSO_n$ , where  $B$  is a connected space, and whose morphisms are commutative diagrams

$$\begin{array}{ccc} B & \xrightarrow{\varphi} & C \\ f \downarrow & & \downarrow g \\ BSO_n & \xlongequal{\quad} & BSO_n, \end{array}$$

where  $f$  and  $g$  are objects of  $\mathcal{K}'$ . Let  $\mathcal{S}'$  be the category whose objects are pairs  $(X, a_X)$ , where  $X$  is a pointed space with  $\pi_i(X) = 0$  for  $i < n$  and  $a_X$  is a generator (one of two) of  $\pi_n(X) = \mathbb{Z}$ , and whose morphisms are maps  $\varphi : X \rightarrow Y$  with  $\varphi_*(a_X) = a_Y$ .

Let  $\gamma'$  be the universal oriented  $n$ -dimensional vector bundle over  $BSO_n$ . There is a unique element  $a \in \pi_n(T\gamma') = \mathbb{Z}$  such that  $\langle u, h(a) \rangle = 1$ , where  $u \in H^n(T\gamma') = \mathbb{Z}$  is the orientation of  $\gamma'$ ,  $h : \pi_n(T\gamma') \rightarrow H_n(T\gamma')$  is the Hurewicz homomorphism and  $\langle -, - \rangle$  is the Kronecker pairing.

Given an object  $f : X \rightarrow BSO_n$  of  $\mathcal{K}'$ , we have the canonical map  $F : T(f^*\gamma') \rightarrow T\gamma'$ , and  $F_* : \mathbb{Z} = \pi_n(T(f^*\gamma')) \rightarrow \pi_n(T\gamma) = \mathbb{Z}$  is an isomorphism. Now define the Thom functor  $\tau' : \mathcal{K}' \rightarrow \mathcal{S}'$  by setting  $\tau' f = (T(f^*\gamma), (F_*)^{-1}(a))$ .

**Theorem 2.** *The functor  $\tau'$  admits a right adjoint functor  $\lambda' : \mathcal{S}' \rightarrow \mathcal{K}'$ .*

*Proof.* Given an object  $(X, a_X)$  of  $\mathcal{K}'$ , consider the isomorphism  $\pi_n(X) \cong \pi_0(\Omega^n X)$ , and let  $\Omega_1^n X$  be the component of  $\Omega^n X$  which corresponds to  $a_X$ . As in §1, we have the left  $SO_n$ -action on  $\Omega^n X$ , and clearly the component  $\Omega_1^n X$  is invariant under the  $SO_n$ -action on  $\Omega^n X$ . We construct a fibre bundle  $p : ESO_n \times_{SO_n} \Omega_1^n X \rightarrow BSO_n$ , and we define  $\lambda'(X, a_X) := p$ . Now the proof can be completed similarly to that of Theorem 1. □

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