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NORMAL CHARACTERISTIC NUMBERS

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Abstract. This paper computes the greatest common divisor of certain normal Pontrjagin numbers and Chern numbers. These calculations are of interest in studying singularities.

1. Introduction

Let M^{4n} be a closed smooth manifold of dimension $4n$, and let $\bar{\wp}$ denote the Pontrjagin class of the stable normal bundle of M. One then obtains an integer $\bar{\wp}_n[M^{4n}]$ by evaluation of the top dimensional Pontrjagin class on the fundamental homology class of M. The goal of this paper is to prove the following result.

Proposition. *The greatest common divisor of the integers* $\bar{\wp}_n[M^{4n}]$ *is* 3^k *where* k *is the smallest integer* j *for which* $\alpha_3(2n+j) \leq 3j$.

For a prime p, $\alpha_p(x)$ is the sum of the digits in the p-adic expansion of x, or equivalently the minimum number of powers of p needed to express x.

That $\sigma_n = gcd\{\bar{\wp}_n[M^{4n}]\}$ is of interest follows from the work of András Szücs [2]. He shows that the number of cusps of any stable smooth map of M^{4n} in R^{6n-1} is $\bar{\wp}_n[M^{4n}]$, so the possible numbers of cusps is the group $\sigma_n Z$. He also describes ([2], Remark 5) the method for calculating these numbers.

In an analogous way, one has

Proposition. *The greatest common divisor of the normal Chern numbers* $\bar{c}_n[M^{2n}]$ *of stably almost complex manifolds of dimension* $2n$ *is* 2^k *where* k *is the smallest integer* j *for which* $\alpha_2(n+j) \leq 2j$.

Szücs has indicated that these numbers give the number of singular points for appropriate complex maps.

The general background information needed about characteristic classes may be found in [1].

2. The method

The function assigning to M^{4n} the value $\bar{\wp}_n[M^{4n}]$ defines a ring homomorphism $\phi : \Omega_*^{SO} \to Z$ from the oriented cobordism ring to the integers. The torsion subgroup must go to zero under the homomorphism, so this gives $\phi : \Omega_*^{SO}/\mathrm{Tor} \to Z$. The ring $\Omega_*^{SO}/\mathrm{Tor}$ is a polynomial ring $Z[x_{4n}]$ on generators of dimension $4n$.

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(Similarly, sending M^{2n} to $\bar{c}_n[M^{2n}]$ gives a ring homomorphism $\phi : \Omega_*^U \to Z$ from complex cobordism to the integers, and Ω_*^U is a polynomial ring $Z[x_{2n}]$.)

The generator x_4 may be taken to be the class of the complex projective plane $\mathbb{C}P^2$, for which $\bar{\wp}_1(\mathbb{C}P^2) = -3$. Then $\phi(\underbrace{\mathbb{C}P^2 \times \cdots \times \mathbb{C}P^2}_{n}) = (-1)^n 3^n$, so σ_n must be a power of 3. (Similarly x_2 may be taken to be the class of $\mathbb{C}P^1$, with $\bar{c}_1[\mathbb{C}P^1] = -2$, so the greatest common divisor must be a power of 2.)

Because the problem is 3-primary, it suffices to consider 3-primary generators for $\Omega_*^{SO}/\mathrm{Tor}$ and these generators are characterized by the fact that $s_n(\wp)[x_{4n}]$ is nonzero mod 3 if $2n+1$ is not a power of 3 and is divisible by 3 but not by 9 if $2n+1$ is a power of 3. For such 3-primary indecomposables one computes the values of $\bar{\wp}_n[x_{4n}]$, and σ_n will be the greatest common divisor of the values $\bar{\wp}_n[x_{4i_1} \cdots x_{4i_r}] = \bar{\wp}_{i_1}[x_{4i_1}] \cdots \bar{\wp}_{i_r}[x_{4i_r}]$ for $i_1 + \cdots + i_r = n$. Here $s_n(\wp)$ is the primitive class with $s_n(\wp) = \sum y_i^{2n}$ if the Pontrjagin class has the form $\wp = \pi(1+y_i^2)$ with $\dim y_i = 2$. (Similarly, 2-primary generators for Ω_*^U are characterized by having $s_n(c)[x_{2n}]$ nonzero mod 2 if $n+1$ is not a power of 2 and being 2 mod 4 if $n+1$ is a power of 2. Here $s_n(c) = \sum y_i^n$ if $c = \pi(1+y_i)$ with $\dim y_i = 2$.)

The remainder of the paper is devoted to finding appropriate generators and then finding the greatest common divisor for monomials in these generators.

3. Generators

This section will establish the following two results.

Lemma. *There are* 3*-primary generators* x_{4n} *for* $\Omega_*^{SO}/\mathrm{Tor}$ *for which the power of* 3 *dividing* $\bar{\wp}_n[x_{4n}]$ *is*

$$\begin{cases} 1 & \text{if } 2n+1 \text{ is a power of } 3, \\[2ex] \dfrac{1}{2}\{\alpha_3(2n+1)-1\} & \text{if } 2n+1 \text{ is not a power of } 3. \end{cases}$$

Lemma. *There are* 2*-primary generators* x_{2n} *for* Ω_*^U *for which the power of* 2 *dividing* $\bar{c}_n[x_{2n}]$ *is*

$$\begin{cases} 1 & \text{if } n+1 \text{ is a power of } 2, \\[2ex] \alpha_2(n+1)-1 & \text{if } n+1 \text{ is not a power of } 2. \end{cases}$$

Notice that for x_4, $2n+1 = 3$ and $\bar{\wp}_1[\mathbb{C}P^2] = -3$, and for x_2, $n+1 = 2$ and $\bar{c}_1[x_2] = -2$. Thus, one has the result for $n = 1$, and more importantly, for $2n+1 = 3^1$, $n+1 = 2^1$.

The easiest case is for $n+1$ not divisible by 2 and $2n+1$ not divisible by 3. For this consider the manifold $\mathbb{C}P^n$. The cohomology of $\mathbb{C}P^n$ is $Z[\alpha]/(\alpha^{n+1} = 0)$ and the Chern class is $c = (1+\alpha)^{n+1}$. Then $s_n(c) = (n+1)\alpha^n$, so $s_n(c)[\mathbb{C}P^n] = n+1$ which is not divisible by 2 if $n+1$ is not divisible by 2. Then $\bar{c} = \frac{1}{(1+\alpha)^{n+1}}$ and $\bar{c}_n[\mathbb{C}P^n]$ is the coefficient of α^n in

$$\frac{1}{(1+\alpha)^{n+1}} = \sum_{i=0}^{\infty} \binom{n+i}{i} (-\alpha)^i,$$

which is $(-1)^n \binom{2n}{n}$.

Now, it is well known that the power of p dividing $x!$ is $(x-\alpha_p(x))/(p-1)$, so the power of p dividing the binomial coefficient $\binom{x}{y} = \frac{x!}{y!(x-y)!}$ is

$$\begin{aligned}
&\frac{1}{(p-1)}\{x-\alpha_p(x)-(y-\alpha_p(y))-(x-y-\alpha_p(x-y))\}\\
&= \quad \frac{1}{(p-1)}\{\alpha_p(y)+\alpha_p(x-y)-\alpha_p(x)\}.
\end{aligned}$$

The power of 2 dividing $\bar{c}_n[\mathbb{C}P^n] = (-1)^n\binom{2n}{n}$ is then $2\alpha_2(n)-\alpha_2(2n) = \alpha_2(n)$ since $\alpha_2(2n) = \alpha_2(n)$. Since $n+1$ is odd, n is even and $\alpha_2(n+1) = \alpha_2(n)+1$ or $\alpha_2(n) = \alpha_2(n+1)-1$, as in the lemma.

For $\mathbb{C}P^{2n}$, the Chern class is $c = (1+\alpha)^{2n+1}$ and the Pontrjagin class is $\wp = (1+\alpha^2)^{2n+1}$. Then $s_n(\wp) = s_{2n}(c) = (2n+1)\alpha^{2n}$ so $s_n(\wp)[\mathbb{C}P^{2n}] = 2n+1$ which is not divisible by 3 if $2n+1$ is not divisible by 3. Then $\bar{\wp}_n[\mathbb{C}P^{2n}]$ is the coefficient of α^{2n} in

$$\bar{\wp} = \frac{1}{(1+\alpha^2)^{2n+1}} = \sum_{i=0}^{\infty}\binom{2n+i}{i}(-\alpha^2)^i,$$

which is $(-1)^n\binom{2n+n}{n} = (-1)^n\binom{3n}{n}$. The power of 3 dividing $\binom{3n}{n}$ is

$$\frac{1}{2}\{\alpha_3(n)+\alpha_3(2n)-\alpha_3(3n)\} = \frac{1}{2}\alpha_3(2n)$$

since $\alpha_3(3n) = \alpha_3(n)$. Since $2n+1$ is not divisible by 3, $\alpha_3(2n+1) = \alpha_3(2n)+1$, so $\frac{1}{2}\alpha_3(2n) = \frac{1}{2}\{\alpha_3(2n+1)-1\}$, as in the lemma.

If $2n+1$ is divisible by 3, but is not a power of 3, then $2n+1 = a_1\cdot 3 + a_2\cdot 3^2 + \cdots + a_r\cdot 3^r$ is not just a single power of 3. Because $2n+1$ is odd and 3 is odd, at least one of the coefficients a_i must be odd, hence is 1, so fix i with $a_i = 1$, and let $m = 2n+1-3^i$ (nonzero since $2n+1 \neq 3^i$).

Let $M^{4n} \subset \mathbb{C}P^{3^i}\times\mathbb{C}P^m$ be the submanifold dual to $\alpha+\beta$ where $H^*(\mathbb{C}P^{3^i}) = Z[\alpha]/(\alpha^{3^i+1} = 0)$ and $H^*(\mathbb{C}P^m) = Z[\beta]/(\beta^{m+1} = 0)$. The Chern class of M is $\frac{(1+\alpha)^{3^i+1}(1+\beta)^{m+1}}{1+\alpha+\beta}$ and the s-class is

$$s_n(\wp) = s_{2n}(c) = (3^i+1)\alpha^{2n} + (m+1)\beta^{2n} - (\alpha+\beta)^{2n} = -(\alpha+\beta)^{2n},$$

with $s_n(\wp)[M] = -(\alpha+\beta)^{2n}[M] = -(\alpha+\beta)^{2n+1}[\mathbb{C}P^{3^i}\times\mathbb{C}P^m]$, which is the coefficient of $\alpha^{3^i}\beta^m$ in $-(\alpha+\beta)^{2n+1}$. This is $-\binom{2n+1}{3^i}$ and is nonzero mod 3. Thus M^{4n} is a 3-primary generator for $\Omega_*^{SO}/\mathrm{Tor}$.

The Pontrjagin class of M is $\wp = \frac{(1+\alpha^2)^{3^i+1}(1+\beta^2)^{m+1}}{1+(\alpha+\beta)^2}$, so

$$\begin{aligned}
\bar{\wp}[M] &= \bar{\wp}[M] = \frac{1+(\alpha+\beta)^2}{(1+\alpha^2)^{3^i+1}(1+\beta^2)^{m+1}}[M]\\
&= \frac{(1+(\alpha+\beta)^2)(\alpha+\beta)}{(1+\alpha^2)^{3^i+1}(1+\beta^2)^{m+1}}\,[\mathbb{C}P^{3^i}\times\mathbb{C}P^m],
\end{aligned}$$

which is the coefficient of $\alpha^{3^i}\beta^m$ in

$$\frac{\{\alpha+\beta+\alpha^3+3\alpha^2\beta+3\alpha\beta^2+\beta^3\}}{(1+\alpha^2)^{3^i+1}(1+\beta^2)^{m+1}}.$$

Since 3^i is odd and m is even, only terms $\alpha^{\text{odd}}\beta^{\text{even}}$ in the numerator can contribute to $\alpha^{3^i}\beta^m$ and this is the coefficient of $\alpha^{3^i}\beta^m$ in

$$\begin{aligned}
&\{\alpha+\alpha^3+3\alpha\beta^2\}\left(\sum_{j=0}^{\infty}\binom{3^i+j}{j}(-\alpha^2)^j\right)\left(\sum_{\ell=0}^{\infty}\binom{m+\ell}{\ell}(-\beta^2)^{\ell}\right)\\
&\quad = \quad (-1)^n\binom{3^i+\frac{3^i-1}{2}}{\frac{3^i-1}{2}}\binom{m+\frac{m}{2}}{\frac{m}{2}}+(-1)^{n-1}\binom{3^i+\frac{3^i-1}{2}-1}{\frac{3^i-1}{2}-1}\binom{m+\frac{m}{2}}{\frac{m}{2}}\\
&\qquad + (-1)^{n-1}3\binom{3^i+\frac{3^i-1}{2}}{\frac{3^i-1}{2}}\binom{m+\frac{m}{2}-1}{\frac{m}{2}-1}.
\end{aligned}$$

Now

$$\binom{m+\frac{m}{2}}{\frac{m}{2}}=\binom{3\left(\frac{m}{2}\right)}{\frac{m}{2}}=\frac{3\left(\frac{m}{2}\right)}{\left(\frac{m}{2}\right)}\binom{3\left(\frac{m}{2}\right)-1}{\frac{m}{2}-1}=3\binom{m+\frac{m}{2}-1}{\frac{m}{2}-1},$$

so the first and last terms cancel, and

$$\binom{3^i+a}{a}\not\equiv 0 \bmod 3 \text{ if } a<3^i.$$

Thus the power of 3 dividing $\bar{\wp}_n[M]$ is the power of 3 dividing $\binom{m+\frac{m}{2}}{\frac{m}{2}} = \binom{3\left(\frac{m}{2}\right)}{\frac{m}{2}}$ which is $\frac{1}{2}\{\alpha_3\left(\frac{m}{2}\right)+\alpha_3(m)-\alpha_3\left(\frac{3m}{2}\right)\}=\frac{1}{2}\alpha_3(m)$, since $\alpha_3\left(\frac{m}{2}\right)=\alpha_3\left(\frac{3m}{2}\right)$. Now $2n+1=m+3^i$, with 3^i not appearing in the 3-adic expansion of m, so $\alpha_3(2n+1)=\alpha_3(m)+1$ and $\frac{1}{2}\alpha_3(m)=\frac{1}{2}\{\alpha_3(2n+1)-1\}$, as in the lemma.

If $n+1$ is divisible by 2, but is not a power of 2, one may choose i for which 2^i appears with coefficient 1 in the 2-adic expansion of $n+1$ and let $m=n+1-2^i$. (For example, $n+1=2^i(2v+1)$ and one may let $m=2^{i+1}v$.)

Let $M^{2n}\subset\mathbb{C}P^{2^i}\times\mathbb{C}P^m$ be the submanifold dual to $\alpha+\beta$. The Chern class of M is $\frac{(1+\alpha)^{2^i+1}(1+\beta)^{m+1}}{1+\alpha+\beta}$ and the s-class is $s_n(c)=(2^i+1)\alpha^n+(m+1)\beta^n-(\alpha+\beta)^n=-(\alpha+\beta)^n$, so $s_n(c)[M]=-(\alpha+\beta)^n[M]=-(\alpha+\beta)^{n+1}[\mathbb{C}P^{2^i}\times\mathbb{C}P^m]=-\binom{n+1}{2^i}$, which is odd. Also

$$\begin{aligned}
\bar{c}_n[M] \quad &= \quad \bar{c}[M]=\frac{(1+\alpha+\beta)}{(1+\alpha)^{2^i+1}(1+\beta)^{m+1}}[M]\\
&= \quad \frac{(1+\alpha+\beta)(\alpha+\beta)}{(1+\alpha)^{2^i+1}(1+\beta)^{m+1}}\,[\mathbb{C}P^{2^i}\times\mathbb{C}P^m],
\end{aligned}$$

which is the coefficient of $\alpha^{2^i}\beta^m$ in

$$(\alpha+\beta+\alpha^2+2\alpha\beta+\beta^2)\left(\sum_{j=0}^{\infty}\binom{2^i+j}{j}(-\alpha)^j\right)\left(\sum_{\ell=0}^{\infty}\binom{m+\ell}{\ell}(-\beta)^{\ell}\right),$$

which is

$$
\begin{aligned}
&(-1)^n \binom{2^i+2^i-1}{2^i-1}\binom{2m}{m} \\
&\qquad + (-1)^n \binom{2^i+1}{2^i}\binom{2m-1}{m-1} + (-1)^{n-1}\binom{2^i+2^i-2}{2^i-2}\binom{2m}{m} \\
&\qquad + (-1)^{n-1} 2 \binom{2^i+2^i-1}{2^i-1}\binom{2m-1}{m-1} + (-1)^{n-1}\binom{2^{i+1}}{2^i}\binom{2m-2}{m-2} \\
= \quad &(-1)^n \binom{2^{i+1}-1}{2^i-1}\binom{2m-1}{m-1} \\
&\qquad \times \left[\frac{2m}{m} + \frac{2^{i+1}}{2^i} - \frac{2^i-1}{2^{i+1}-1}\cdot\frac{2m}{m} - 2 - \frac{2^{i+1}}{2^i}\cdot\frac{m-1}{2m-1}\right].
\end{aligned}
$$

The sum is

$$
\begin{aligned}
2+2-\frac{2^{i+1}-2}{2^{i+1}-1}-2-\frac{2m-2}{2m-1} \quad &= \quad 2-\left(1-\frac{1}{2^{i+1}-1}\right)-\left(1-\frac{1}{2m-1}\right) \\
&= \quad \frac{1}{2^{i+1}-1}+\frac{1}{2m-1} = \frac{2m+2^{i+1}-2}{(2^{i+1}-1)(2m-1)} \\
&= \quad \frac{2(n+1)-2}{(2^{i+1}-1)(2m-1)},
\end{aligned}
$$

so

$$
\begin{aligned}
\bar{c}_n[M] \quad &= \quad (-1)^n \binom{2^{i+1}-1}{2^i-1}\binom{2m-1}{m-1} 2\, \frac{n}{(2^{i+1}-1)(2m-1)} \\
&= \quad (-1)^n \binom{2^{i+1}-1}{2^i-1}\binom{2m}{m} \frac{n}{(2^{i+1}-1)(2m-1)}.
\end{aligned}
$$

Since $\binom{2^{i+1}-1}{2^i-1}$, n, $2^{i+1}-1$, and $2m-1$ are all odd, the power of 2 dividing $\bar{c}_n[M]$ is the power of 2 dividing $\binom{2m}{m}$ which is $\alpha_2(m)$. Since $n+1 = m+2^i$, $\alpha_2(m) = \alpha_2(n+1)-1$, as in the lemma.

For $2n+1 = 3^r$ with $r > 1$, let $M^{4n} \subset \mathbb{C}P^{3^r}$ be the degree 3 hypersurface (submanifold dual to 3α). The Chern class of M is $\frac{(1+\alpha)^{3^r+1}}{1+3\alpha}$ so $s_n(\wp)[M] = s_{2n}(c)[M] = \{(3^r+1)\alpha^{2n} - (3\alpha)^{2n}\}[M] = \{(3^r+1)\alpha^{2n} - (3\alpha)^{2n}\}3\alpha[\mathbb{C}P^{2n+1}] \equiv -3\alpha^{2n+1}[\mathbb{C}P^{2n+1}] \equiv -3 \bmod 9$. Thus M is a 3-primary generator of $\Omega_*^{SO}/\mathrm{Tor}$. The Pontrjagin class of M is $\frac{(1+\alpha^2)^{3^r+1}}{1+(e\alpha)^2}$, so $\bar{\wp}_n[M]$ is the coefficient of α^{3^r} in

$$
\frac{1+(3\alpha)^2}{(1+\alpha^2)^{3^r+1}}(3\alpha) = (3\alpha+27\alpha^3)\left(\sum_{k=0}^{\infty}\binom{3^r+k}{k}(-\alpha^2)^k\right),
$$

which is

$$3\binom{3^r+\frac{3^r-1}{2}}{\frac{3^r-1}{2}}(-1)^{\frac{3^r-1}{2}}+27\binom{3^r+\frac{3^r-3}{2}}{\frac{3^r-3}{2}}(-1)^{\frac{3^r-3}{2}} \text{ and } \binom{3^r+\frac{3^r-1}{2}}{\frac{3^r-1}{2}}\not\equiv 0,$$

modulo 3, so this is divisible by 3, but not by 9. Thus the power of 3 dividing $\bar{\wp}_n[M]$ is 1, as in the lemma.

For $n+1=2^r$ with $r>1$, let $M^{2n}\subset \mathbb{C}P^{2^r}$ be the degree 2 hypersurface (submanifold dual to 2α). The Chern class of M is $\frac{(1+\alpha)^{2^r+1}}{1+2\alpha}$ so

$$s_n(c)[M]=\{(2^r+1)\alpha^n-(2\alpha)^n\}[M]=\{(2^r+1)\alpha^n-(2\alpha)^n\}(2\alpha)[\mathbb{C}P^{2n+1}],$$

and this is 2 mod 4. Thus M is a 2-primary generator of Ω_*^U. Then $\bar{c}_n[M]$ is the coefficient of α^{n+1} in

$$\frac{1+2\alpha}{(1+\alpha)^{2^r+1}}\cdot(2\alpha)=(2\alpha+4\alpha^2)\left(\sum_{j=0}^{\infty}\binom{2^r+j}{j}(-\alpha)^j\right),$$

which is $-2\binom{2^{r+1}-1}{2^r-1}+4\binom{2^{r+1}-2}{2^r-2}\equiv 2 \bmod 4$ since both binomial coefficients are odd. Thus the power of 2 dividing $\bar{c}_n[M]$ is 1, as in the lemma.

4. THE GREATEST COMMON DIVISOR

The argument for finding the greatest common divisor is essentially the same in the two cases, so the proof will be given for Pontrjagin classes.

Let $k_n=\min\{j \mid \alpha_3(2n+j)\le 3j\}$ and

$$3^{g_n}=gcd\{\bar{\wp}_{n_1}[x_{4n_1}]\cdots\bar{\wp}_{n_r}[x_{4n_r}] \mid n_1+\cdots+n_r=n\}.$$

One has certain easy facts.

Fact. *The power of* 3 *dividing* $\bar{\wp}_n[x_{4n}]$ *is* 1 *if and only if* $2n+1$ *can be written as the sum of three powers of* 3.

If $2n+1=3^r$, then $2n+1=3^{r-1}+3^{r-1}+3^{r-1}$, and if $\frac{1}{2}\{\alpha_3(2n+1)-1\}=1$, then $\alpha_3(2n+1)=3$, so $2n+1$ is a sum of 3 powers of 3.

Fact. $\alpha_3(x)\equiv x \bmod 2$.

Fact. $\alpha_3(2n+n)\le 3n$, *so* $k_n\le n$.

Obviously, $3n=2n+n$ is the sum of $3n$ ones.

Fact. *If* $\alpha_3(2n+j)\le 3j$, *then* $\alpha_3(2n+(j+1))\le 3(j+1)$.

Since $\alpha_3(2n+(j+1))\le \alpha_3(2n+j)+1$, this is clear.

Fact. *If* $\alpha_3(2n+j)\le 3j$ *and* $j\le n$, *then* $2n+j$ *can be written as the sum of* $3j$ *powers of* 3.

If $\alpha_3(2n+j)=r\le 3j$, then $2n+j$ is the sum of r powers of 3. Replacing 3^x by $3^{x-1}+3^{x-1}+3^{x-1}$ increases the number of powers of 3 by 2, and one may write $2n+j$ as the sum of any number of powers of 3 congruent to r mod 2 up to a maximum of $2n+j$ powers of 3 (all 1's). Now $\alpha_3(2n+j)\equiv 2n+j\equiv r\equiv j\equiv 3j$ mod 2 and $3j\le 2n+j$, so $2n+j$ can be written as a sum of $3j$ powers of 3.

Corollary. *If* $\alpha_3(2n+j) \le 3j$ *with* $j \le n$*, then* $g_n \le j$*. Thus* $g_n \le k_n$*.*

Proof. If $\alpha_3(2n+j) \le 3j$ and $j \le n$, then $2n+j$ is the sum of $3j$ powers of 3. One can then write $2n+j = (2n_1+1)+\cdots+(2n_j+1)$ where each $2n_i+1 = 3^{a_i}+3^{b_i}+3^{c_i}$ uses three of the powers of 3. Then $2n = 2n_1+\cdots+2n_j$, and the power of 3 dividing $\bar{\wp}_{n_1}[x_{4n_1}]\cdots\bar{\wp}_{n_j}[x_{4n_j}]$ is j. Thus $g_n \le j$.

Now suppose that $g_m = k_m$ for $m < n$, and that g_n is the power of 3 dividing $\bar{\wp}_{n_1}[x_{4n_1}]\cdots\ \bar{\wp}_{n_r}[x_{4n_r}]$ with $2n = 2n_1+\cdots+2n_r$.

If $r = 1$, then the power of 3 dividing $\bar{\wp}_n[x_{4n}]$ is g_n. Then either $g_n = 1$ $(2n+1 = 3^j)$ so $\alpha_3(2n+1) = 1 \le 3\cdot 1$ or $g_n = \frac{1}{2}\{\alpha_3(2n+1)-1\}$ giving $\alpha_3(2n+1) = 1+2g_n$ so $\alpha_3(2n+g_n) = \alpha_3(2n+1+(g_n-1)) \le \alpha_3(2n+1)+(g_n-1) = 3g_n$. Thus $\alpha_3(2n+g_n) \le 3g_n$ and $k_n \le g_n$.

If $r > 1$ and g_n is the power of 3 dividing $\bar{\wp}_{n_1}[x_{4n_1}]\cdots\bar{\wp}_{n_r}[x_{4n_r}]$, then the power of 3 dividing each $\bar{\wp}_{n_i}[x_{4n_i}]$ must be g_{n_i} (otherwise one can replace x_{4n_i} by a decomposable class to decrease the power of 3), and so $g_{n_i} = k_{n_i}$ inductively and $g_n = g_{n_1}+\cdots+g_{n_r}$. Thus $\alpha_3(2n_i+g_{n_i}) \le 3g_{n_i}$ and $2n+g_n = (2n_1+g_{n_1})+\cdots+(2n_r+g_{n_r})$, so

$$\alpha_3(2n+g_n) \le \alpha_3(2n_1+g_{n_1})+\cdots+\alpha_3(2n_r+g_{n_r}) \le 3g_{n_1}+\cdots+3g_{n_r} = 3g_n.$$

Thus $k_n \le g_n$.

Since $k_n \le g_n$ and $g_n \le k_n$, one obtains $g_n = k_n$ completing the induction and establishing the proposition.

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