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ASYMPTOTICS FOR THE HEAT EQUATION IN THE EXTERIOR OF A SHRINKING COMPACT SET IN THE PLANE VIA BROWNIAN HITTING TIMES

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ABSTRACT. Let $D_r = \{x \in R^2 : |x| \le r\}$ and let γ be a continuous, non-increasing function on $[0, \infty)$ satisfying $\lim_{t\to\infty} \gamma(t) = 0$. Consider the heat equation in the exterior of a time-dependent shrinking disk in the plane:

$$\begin{aligned} &u_t = \frac{1}{2}\Delta u,\ x \in R^2 - D_{\gamma(t)},\ t > 0,\\ &u(x, 0) = 0,\ x \in R^2 - D_{\gamma(t)},\\ &u(x, t) = 1,\ x \in D_{\gamma(t)}, t > 0. \end{aligned}$$

If there exist constants $0 < c_1 < c_2$ and a constant $k > 0$ such that $c_1 t^{-k} \le \gamma(t) \le c_2 t^{-k}$, for sufficiently large t, then $\lim_{t\to\infty} u(x, t) = \frac{1}{1+2k}$. The same result is also shown to hold when $D_{\gamma(t)}$ is replaced by $L_{\gamma(t)}$, where $L_r = \{(x_1, 0) \in R^2 : |x_1| \le r\}$. Also, a discrepancy is noted between the asymptotics for the above forward heat equation and the corresponding backward one. The method used is probabilistic.

1. Statement of results

Let

$$D_r = \{x \in R^d : |x| \le r\}$$

and consider the heat equation in the exterior domain $R^d - D_1$:

$$\begin{aligned} &w_t = \frac{1}{2}\Delta w,\ (x, t) \in (R^d - D_1) \times (0, \infty),\\ &w(x, 0) = 0,\ x \in R^d - D_1,\\ &w(x, t) = 1,\ (x, t) \in D_1 \times [0, \infty). \end{aligned} \tag{1.1}$$

The asymptotic behavior of the solution $w(x, t)$ for $x \in R^d - D_1$ as $t \to \infty$ is well known:

$$\lim_{t\to\infty} w(x, t) = \begin{cases} |x|^{2-d}, & \text{if } d \ge 3,\\ 1, & \text{if } d = 1, 2. \end{cases}$$

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In particular, this shows that a heat source in the shape of a ball does not fully heat three-dimensional space, whereas a heat source in the shape of a disk does fully heat the plane.

The question we address here is this: how effective will the heating be in the plane if the heat source shrinks in time? That is, instead of having a temporally constant heat source D_1, we use a heat source $D_{\gamma(t)}$, where γ is a continuous, positive function decreasing to 0 as $t \to \infty$. We have the following equation:

$$\begin{aligned} &u_t = \frac{1}{2}\Delta u,\ x \in R^2 - D_{\gamma(t)},\ t > 0, \\ &u(x,0) = 0,\ x \in R^2 - D_{\gamma(t)}, \\ &u(x,t) = 1,\ x \in D_{\gamma(t)},\ t \geq 0. \end{aligned} \tag{1.2}$$

If $\lim_{t\to\infty} u(x,t)$ exists, then it should be harmonic in $R^2 - \{0\}$, and since the only bounded harmonic functions in this domain are the constants, it follows that the above limit, if it exists, should be equal to a constant $c \in [0,1]$. Incidentally, the same type of argument shows that the corresponding problem in higher dimensions is uninteresting: if the limit exists, it should be harmonic in $R^d - 0$, and dominated by $|x|^{2-d}$; the only such bounded, nonnegative function is 0. A completely rigorous argument can easily be made using the maximum principle.

Before stating the theorem, we derive a probabilistic representation for the solution u to (1.2). We recall a fundamental property of Brownian motion: let P_x denote Wiener measure for a standard Brownian motion $B(t)$ in R^d starting from $x \in R^d$ and let E_x denote the corresponding expectation. Let $G \subset R^d$ be a domain and let $\sigma_{G^c} = \inf\{t \geq 0 : B(t) \in G^c\}$ denote the first entrance time into G^c. If $W(x,s)$ is twice continuously differentiable for $x \in G$, once continuously differentiable for $s \in (0,t]$, and continuous on $\bar{G} \times (0,t] \cup G \times [0,t]$, and if $\frac{1}{2}\Delta W + \frac{\partial W}{\partial s}$ is bounded on $G \times (0,t]$, then $W(B(s \wedge \sigma_{G^c}), s \wedge \sigma_{G^c}) - \int_0^{s\wedge\sigma_{G^c}} (\frac{1}{2}\Delta W + \frac{\partial W}{\partial s'})(B(s'), s')ds'$ is a martingale under P_x for $s \in [0,t]$. In particular, if it so happens that $\frac{1}{2}\Delta W + \frac{\partial W}{\partial s} \equiv 0$ in $G \times (0,t)$, then $W(B(s \wedge \sigma_{G^c}), s \wedge \sigma_{G^c})$ is a martingale. Since the expectation of a martingale is constant in time, equating the value of the expectation when $s = 0$ with the value for $s = t$, we obtain the formula

$$W(x,0) = E_x W(B(t \wedge \sigma_{G^c}), t \wedge \sigma_{G^c}),\ \text{for } x \in G. \tag{1.3}$$

Equation (1.3) leads to the well-known probabilistic representation for $w(x,t)$. Fix $t > 0$ and apply (1.3) to $W(x,s) = w(x,t-s)$ and $G = R^d - D_1$. From the boundary condition and the initial condition in (1.1) it follows that

$$W(B(t \wedge \sigma_{G^c}), t \wedge \sigma_{G^c}) = w(B(t \wedge \sigma_{D_1}), t - t \wedge \sigma_{D_1}) = 1$$

on the event $\{\sigma_{D_1} < t\}$, while

$$W(B(t \wedge \sigma_{G^c}), t \wedge \sigma_{G^c}) = w(B(t \wedge \sigma_{D_1}), t - t \wedge \sigma_{D_1}) = 0$$

on the event $\{\sigma_{D_1} > t\}$. Since $P_x(\sigma_{D_1} = t) = 0$, we obtain from (1.3) that

$$w(x,t) = P_x(\sigma_{D_1} \leq t) \text{ for } x \in R^d - D_1. \tag{1.4}$$

From (1.4), it follows that the dichotomy between $\lim_{t\to\infty} w(x,t) = 1$ for $d = 1,2$ and $\lim_{t\to\infty} w(x,t) < 1$ for $d \geq 3$ which we observed above is just the dichotomy between recurrence and transience of Brownian motion.

In order to apply (1.3) to the solution u of (1.2), we define for each $t > 0$ the stopping time

$$\sigma_\gamma^{(t)} = \inf\{s \in [0,t] : B(s) \in D_{\gamma(t-s)}\},$$

with the convention that $\sigma_\gamma^{(t)} = \infty$ if $\{s \in [0,t] : B(s) \in D_{\gamma(t-s)}\}$ is empty. Fixing a $t > 0$, consider the function $W(x,s) \equiv u(x,t-s),\ \ 0 \le s \le t,\ x \in R^2 - D_{\gamma(t-s)}$. By (1.2), we have $(\frac{1}{2}\Delta W + \frac{\partial W}{\partial s})(x,s) \equiv 0$ for $x \in R^2 - D_{\gamma(t-s)}$ and $0 \le s \le t$. We now apply (1.4) to this choice of $W(x,s)$ along with the stopping time $\sigma_\gamma^{(t)}$. The fact that the domain is time-dependent does not cause any problem. Note from the boundary condition and the initial condition in (1.2) that $W(B(t\wedge\sigma_\gamma^{(t)}), t\wedge\sigma_\gamma^{(t)}) = u(B(t\wedge\sigma_\gamma^{(t)}), t - t\wedge\sigma_\gamma^{(t)}) = 1$ on the event $\{\sigma_\gamma^{(t)} < t\}$, while $W(B(t\wedge\sigma_\gamma^{(t)}), t\wedge\sigma_\gamma^{(t)}) = u(B(t\wedge\sigma_\gamma^{(t)}), t-t\wedge\sigma_\gamma^{(t)}) = 0$ on the event $\{\sigma_\gamma^{(t)} > t\}$. Since $P_x(\sigma_\gamma^{(t)} = t) = 0$, we obtain the following probabilistic representation for the solution u of (1.2):

$$u(x,t) = P_x(\sigma_\gamma^{(t)} \le t) = P_x(B(s) \in D_{\gamma(t-s)} \text{ for some } s \in [0,t]). \tag{1.5}$$

We will prove the following theorem.

Theorem. *Let $u(x,t)$ be the solution to* (1.2)*, where $\gamma(t)$ is a continuous, nonincreasing function. Assume that there exist constants $0 < c_1 < c_2$ and a constant $k > 0$ such that $c_1 t^{-k} \le \gamma(t) \le c_2 t^{-k}$ for sufficiently large t. Then*

$$\lim_{t\to\infty} u(x,t) = \frac{1}{1+2k}. \tag{1.6}$$

Remark. By a basic monotonicity property which is an obvious consequence of the maximum principle, it follows that if $\gamma(t)$ decreases to 0 faster than any negative power of t, then $\lim_{t\to\infty} u(x,t) = 0$, while if $\gamma(t)$ decreases to 0 more slowly than any negative power of t, then $\lim_{t\to\infty} u(x,t) = 1$.

With just a little extra work, we will prove the same result when D_r is replaced by a *one*-dimensional set. Let

$$L_r = \{(x_1, 0) \in R^2 : |x_1| \le r\}.$$

Corollary. *Let $U(x,t)$ denote the solution to* (1.2) *with $D_{\gamma(t)}$ replaced by $L_{\gamma(t)}$. Then* (1.6) *holds with u replaced by U.*

Remark. By the basic monotonicity property, the same result also holds if $D_{\gamma(t)}$ or $L_{\gamma(t)}$ is replaced by $C_{\gamma(t)}$ where $r \to C_r$ is a continuous map from $[0,\infty)$ to the compact subsets of R^2, satisfying $L_r \subset C_r \subset D_r$.

There turns out to be an interesting discrepancy between the asymptotic behavior of the solution u to the forward heat equation (1.2) and the solution to the corresponding backward heat equation. Let $v^{(t)}(x,s)$ denote the solution to the backward heat equation

$$\begin{aligned}
&v_s^{(t)} + \frac{1}{2}\Delta v^{(t)} = 0,\ x \in R^2 - D_{\gamma(s)},\ 0 \le s \le t,\\
&v^{(t)}(x,t) = 0,\ x \in R^2 - D_{\gamma(t)},\\
&v^{(t)}(x,s) = 1,\ x \in D_{\gamma(s)}, 0 \le s \le t.
\end{aligned}$$

Consider the stopping time

$$\sigma_\gamma = \inf\{t \ge 0 : B(t) \in D_{\gamma(t)}\},$$

and define

$$v(x,t) = P_x(\sigma_\gamma \le t).$$

An analysis similar to that used to obtain the probabilistic representation of u in (1.5) shows that $v(x,t) = v^{(t)}(x,0)$. (Use (1.3) with $W(x,s) = v_s^t(x,s),\ 0 \le s \le t$, and use σ_γ in place of σ_{G^c}.) An old result of Spitzer [3] indicates that for $|x| > \gamma(0)$,

$$P_x\Big(\frac{|B(t)|}{\gamma(t)} \le 1 \text{ for some } t \ge 0\Big) = 1 \text{ if and only if } \int^\infty \frac{1}{t|\log\gamma(t)|}dt = \infty. \tag{1.7}$$

In terms of $v^{(t)}$, (1.7) states that

$$\begin{aligned} &\lim_{t\to\infty} v^{(t)}(x,0) = \lim_{t\to\infty} P_x(B(s) \in D_{\gamma(s)} \text{ for some } s \in [0,t]) = 1 \\ &\text{if and only if } \int^\infty \frac{1}{t|\log\gamma(t)|}dt = \infty. \end{aligned} \tag{1.8}$$

Note, for example, that the integral in (1.8) will be infinite if $\gamma(t) \ge t^{-\log\log t}$, but not if $\gamma(t) \le t^{-(\log\log t)^{1+\epsilon}}$. In contrast, by Theorem 1, it follows that

$$\begin{aligned} &\lim_{t\to\infty} u(x,t) \\ &= \lim_{t\to\infty} P_x(B(s) \in D_{\gamma(t-s)}, \text{ for some } s \in [0,t]) \begin{cases} = 1, & \text{if } \lim_{t\to\infty} t^k\gamma(t) = \infty, \\ & \text{for all } k > 0, \\ \neq 1, & \text{if } \lim_{t\to\infty} t^k\gamma(t) = 0, \\ & \text{for some } k > 0. \\ = 0, & \text{if } \lim_{t\to\infty} t^k\gamma(t) = 0, \\ & \text{for all } k > 0. \end{cases} \end{aligned} \tag{1.9}$$

Thus, there is a discrepancy between the asymptotic behavior of the solutions to the forward and backward heat equations. In probabilistic terms, there is a discrepancy in the asymptotic behavior of the hitting times of a shrinking disk, depending on whether the shrinking occurs in forward time or backward time. In particular, note that if $\gamma(t)$ decays to 0 faster than any negative power of t, but $\int^\infty \frac{1}{t|\log\gamma(t)|}dt = \infty$, as occurs for example if $\gamma(t) = t^{-\log\log t}$, then (1.8) is equal to 1 while (1.9) is equal to 0.

2. Proof of the Theorem and Corollary

Since everything concerning the Theorem is radially symmetric, we will use the notation $P_r,\ r \ge 0$, instead of $P_x,\ x \in R^2$, where $r = |x|$. When we turn to the proof of the Corollary, we will return to the notation P_x. We will need the following Lemma.

Lemma. *Let $\tau_a = \inf\{t \geq 0 : |X(t)| = a\}$. For l_1, l_2 satisfying $0 < l_1 < \frac{1}{2} < l_2$, and for $0 < a \leq b \leq \frac{1}{2}t^{l_2}$, there exists a universal constant $\lambda > 0$ such that*

$$\frac{\log b - \log a}{l_2 \log t - \log a} - \frac{1}{\lambda}\exp(-\lambda t^{2l_2-1}) \leq P_b(\tau_a > t) \leq \frac{\log b - \log a}{l_1 \log t - \log a} + \frac{1}{\lambda}\exp(-\lambda t^{1-2l_1}).$$

Proof. Since $P_b(\tau_a > \tau_c) = \frac{\log b - \log a}{\log c - \log a}$, for $a < b < c$ [2, p. 38], we have for $t, l > 0$ such that $b < t^l$,

$$P_b(\tau_a > t) \leq P_b(\tau_a > \tau_{t^l}) + P_b(\tau_{t^l} > t) = \frac{\log b - \log a}{l \log t - \log a} + P_b(\tau_{t^l} > t)$$

and

$$P_b(\tau_a > t) \geq P_b(\tau_a > \tau_{t^l}) - P_b(\tau_{t^l} < t) = \frac{\log b - \log a}{l \log t - \log a} - P_b(\tau_{t^l} < t).$$

The Brownian scaling property gives

$$P_b(\tau_{t^l} > t) = P_{bt^{-l}}(\tau_1 > t^{1-2l}) \text{ and } P_b(\tau_{t^l} < t) = P_{bt^{-l}}(\tau_1 < t^{1-2l}).$$

If $l \in (0, \frac{1}{2})$, then $P_{bt^{-l}}(\tau_1 > t^{1-2l}) \leq \frac{1}{\lambda}\exp(-\lambda t^{1-2l})$ for some $\lambda > 0$ [1, Theorem 3.6.1], while if $l > \frac{1}{2}$ and $b \leq \frac{1}{2}t^l$, then $P_{bt^{-l}}(\tau_1 < t^{1-2l}) \leq \frac{1}{\lambda}\exp(-\lambda t^{2l-1})$ for some $\lambda > 0$ [1, Theorem 2.2.2]. The Lemma follows from these estimates. □

Proof of the Theorem. By the basic monotonicity property alluded to in the Remark following the Theorem, we can assume without loss of generality that $\gamma(t) = c(1+t)^{-k}$ for some $c > 0$ and $k > 0$. In light of (1.5), to prove the Theorem we must show that

$$\lim_{t\to\infty} P_b(|B(s)| \leq c(1+t-s)^{-k} \text{ for some } s \in [0,t]) = \frac{1}{1+2k}. \tag{2.1}$$

Using the right-hand inequality in the Lemma and the monotonicity of $c(1+t)^{-k}$, we have for large t,

(2.2)

$$P_b(|B(s)| \leq c(1+t-s)^{-k} \text{ for some } s \in [0,t]) \geq P_b(\tau_{c(1+t)^{-k}} \leq t)$$
$$= 1 - P_b(\tau_{c(1+t)^{-k}} > t) \geq 1 - \frac{\log b - \log c + k\log(1+t)}{l_1 \log t - \log c + k\log(1+t)} - \frac{1}{\lambda}\exp(-\lambda t^{1-2l_1}).$$

Letting $t \to \infty$ in (2.2) and then letting l_1 increase to $\frac{1}{2}$, we obtain

$$\liminf_{t\to\infty} P_b(|B(s)| \leq c(1+t-s)^{-k} \text{ for some } s \in [0,t]) \geq \frac{1}{1+2k}. \tag{2.3}$$

Applying the Markov property at time $\frac{t}{2}$, and using the monotonicity of $c(1+t)^{-k}$ again, we have

(2.4)

$$P_b(|B(s)| > c(1+t-s)^{-k} \text{ for all } s \in [0,t]) \geq E_b(P_{|B(\frac{t}{2})|}(\tau_c > \frac{t}{2}); \tau_{c(1+\frac{t}{2})^{-k}} > \frac{t}{2}).$$

A direct calculation shows that there exists a constant $C > 0$, depending on b, such that for any $\epsilon > 0$,

$$P_b(|B(\frac{t}{2})| \leq \epsilon t^{\frac{1}{2}}) \leq C\epsilon^2 \text{ for all } t > 0. \tag{2.5}$$

Thus, since $P_b(\tau_c > \frac{t}{2})$ is increasing in b for $b > c$, we obtain from (2.4) and (2.5) that for any $\epsilon > 0$ and sufficiently large t

$$\begin{aligned}(2.6)\qquad & P_b(|B(s)| > c(1+t-s)^{-k} \text{ for all } s \in [0,t]) \\ & \geq P_b(\tau_{c(1+\frac{t}{2})^{-k}} > \frac{t}{2})P_{\epsilon t^{\frac{1}{2}}}(\tau_c > \frac{t}{2}) - C\epsilon^2.\end{aligned}$$

Using the lower bound in the Lemma to estimate the two probabilities on the right-hand side of (2.6), we have for sufficiently large t,

$$\begin{aligned}(2.7)\qquad & P_b(|B(s)| > c(1+t-s)^{-k}) \text{ for all } s \in [0,t]) \\ & \geq (\frac{\log b - \log c + k\log(1+\frac{t}{2})}{l_2 \log \frac{t}{2} - \log c + k\log(1+\frac{t}{2})} - \frac{1}{\lambda}\exp(-\lambda t^{2l_2-1})) \\ & \times (\frac{\log\epsilon + \frac{1}{2}\log t - \log c}{l_2\log\frac{t}{2} - \log c} - \frac{1}{\lambda}\exp(-\lambda t^{2l_2-1})) - C\epsilon^2.\end{aligned}$$

Letting $t \to \infty$ in (2.7), then letting ϵ decrease to 0 and l_2 decrease to $\frac{1}{2}$ gives

$$(2.8)\qquad \liminf_{t\to\infty} P_b(|B(s)| > c(1+t-s)^{-k} \text{ for all } s \in [0,t]) \geq \frac{2k}{1+2k}.$$

Now (2.1) follows from (2.3) and (2.8). □

Proof of the Corollary. We now return to the notation $P_x,\ x \in R^2$. Let $\tau_{L_a} = \inf\{t \geq 0 : B(t) \in L_a\}$. We will show that there exists a constant $K > 0$ such that for $0 < a < |x| < c$,

$$(2.9)\qquad \frac{\log|x| - \log a}{\log c - \log a} \leq P_x(\tau_{L_a} > \tau_c) \leq \frac{\log|x| - \log\frac{a}{2} + K}{\log c - \log\frac{a}{2}}.$$

Using (2.9), it follows immediately from the proof of the Lemma that

$$\begin{aligned}(2.10)\qquad & \frac{\log|x| - \log a}{l_2\log t - \log a} - \frac{1}{\lambda}\exp(-\lambda t^{2l_2-1}) \\ & \leq P_x(\tau_{L_a} > t) \leq \frac{\log|x| - \log\frac{a}{2} + K}{l_1\log t - \log\frac{a}{2}} + \frac{1}{\lambda}\exp(-\lambda t^{1-2l_1}).\end{aligned}$$

One now proves the Corollary just as the Theorem was proved, using the estimate (2.10) in place of the estimate in the Lemma.

It remains to prove (2.9). The left-hand inequality in (2.9) of course follows trivially since $P_x(\tau_{L_a} > \tau_c) \geq P_x(\tau_a > \tau_c) = \frac{\log|x|-\log a}{\log c - \log a}$. For the right-hand inequality, we begin by using the strong Markov property to write

$$(2.11)\qquad P_x(\tau_{L_a} < \tau_c) \geq E_x(P_{B(\tau_{\frac{a}{2}})}(\tau_{L_a} < \tau_c);\ \tau_{\frac{a}{2}} < \tau_c).$$

For any $t > 0$ and $y \in R^2$ with $|y| = \frac{1}{2}$, we have $P_y(\tau_{L_1} < t \wedge \tau_1) > 0$. This can be proved in any number of ways; for instance, by applying the Stroock-Varadhan support theorem [1, Theorem 2.6.1], or by applying the reflection principle [2] to the second coordinate of the Brownian motion and using the independence of the two components. Since Brownian motion is a Feller process, it follows that $P_y(\tau_{L_1} < t \wedge \tau_1)$ is continuous in y [1, Theorem 1.3.1]; thus,

$$\inf_{y\in R^2:|y|=\frac{1}{2}} P_y(\tau_{L_1} < t \wedge \tau_1) > 0,$$

and a fortiori there exists a $\rho > 0$ such that $\inf_{y\in R^2:|y|=\frac{1}{2}} P_y(\tau_{L_1} < \tau_1) \geq \rho$. By Brownian scaling it then follows that

$$P_y(\tau_{L_a} < \tau_a) \geq \rho > 0 \text{ for } |y| = \frac{a}{2} \text{ and all } a > 0. \tag{2.12}$$

Also,

$$P_y(\tau_{\frac{a}{2}} < \tau_c) = \frac{\log a - \log c}{\log \frac{a}{2} - \log c} \equiv q_{a,c} \text{ for } |y| = a. \tag{2.13}$$

We use (2.12), (2.13) and the strong Markov property to estimate $P_y(\tau_{L_a} < \tau_c)$ for $|y| = \frac{a}{2}$. Consider the event $\tau_{L_a} > \tau_c$ under P_y with $|y| = \frac{a}{2}$. In order for this event to occur, first of all, starting from $y \in \partial D_{\frac{a}{2}}$, $B(t)$ must hit ∂D_a before hitting L_a, and this event occurs with probability no greater than $1 - \rho$. Then, starting from ∂D_a, the Brownian motion has a probability $q_{a,c}$ of returning to $\partial D_{\frac{a}{2}}$ before reaching ∂D_c (during which time it may hit L_a, but we ignore this), in which case it gets another chance, starting from $\partial D_{\frac{a}{2}}$, to hit L_a before hitting ∂D_a, etc. This reasoning gives the estimate

(2.14)

$$P_y(\tau_{L_a} > \tau_c) \leq \sum_{n=0}^{\infty}(1-\rho)^{n+1} q_{a,c}^n (1 - q_{a,c}) = \frac{(1-q_{a,c})(1-\rho)}{1-(1-\rho)q_{a,c}} \equiv Q_{a,c} \text{ for } |y| = \frac{a}{2}.$$

From (2.11), (2.14), and the fact that $P_x(\tau_{\frac{a}{2}} < \tau_c) = \frac{\log|x| - \log c}{\log\frac{a}{2} - \log c}$, we have

(2.15)

$$P_x(\tau_{L_a} > \tau_c) \leq 1 - E_x(P_{B(\tau_{\frac{a}{2}})}(\tau_{L_a} < \tau_c);\ \tau_{\frac{a}{2}} < \tau_c) \leq 1 - (1 - Q_{a,c})\frac{\log|x| - \log c}{\log\frac{a}{2} - \log c}$$
$$= \frac{\log|x| - \log\frac{a}{2} + Q_{a,c}(\log c - \log|x|)}{\log c - \log\frac{a}{2}}.$$

From (2.13) and (2.14), we have

$$Q_{a,c} = \frac{(1-\rho)\log 2}{\log 2 + \rho \log\frac{c}{a}}. \tag{2.16}$$

It follows from (2.16) that there exists a constant $K > 0$ such that $Q_{a,c}\log c \leq K$. Substituting this estimate in the right-hand side of (2.15) proves (2.9). □

References

1. Pinsky, R.G., *Positive Harmonic Functions and Diffusion*, Cambridge Univ. Press, 1995. MR **96m:**60179
2. Rogers, L.C.G. and Williams, D., *Diffusions, Markov Processes and Martingales, Vol. 1, 2nd ed.*, Cambridge Univ. Press, 2000. CMP 2001:04
3. Spitzer, F., *Some theorems concerning two-dimensional Brownian Motion*, Trans. of the A.M.S. **87** (1958), 187-197. MR **21:**3051

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