

## ON ONE PROBLEM OF UNIQUENESS OF MEROMORPHIC FUNCTIONS CONCERNING SMALL FUNCTIONS

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ABSTRACT. In this paper, we show that if two non-constant meromorphic functions  $f$  and  $g$  satisfy  $\overline{E}(a_j, k, f) = \overline{E}(a_j, k, g)$  for  $j = 1, 2, \dots, 5$ , where  $a_j$  are five distinct small functions with respect to  $f$  and  $g$ , and  $k$  is a positive integer or  $\infty$  with  $k \geq 14$ , then  $f \equiv g$ . As a special case this also answers the long-standing problem on uniqueness of meromorphic functions concerning small functions.

### 1. INTRODUCTION AND MAIN RESULT

In this paper, by meromorphic function we shall always mean a meromorphic function in the complex plane  $\mathbb{C}$ . We adopt the standard notations in the Nevanlinna theory of meromorphic functions as explained in [1]. For any non-constant meromorphic function  $f(z)$ , we denote by  $S(r, f)$  any quantity satisfying

$$S(r, f) = o(T(r, f))$$

for  $r \rightarrow \infty$  except possibly a set of  $r$  of finite linear measure. A meromorphic function  $a(z)$  is called a small function with respect to  $f(z)$  if  $T(r, a) = S(r, f)$ . Let  $S(f)$  be the set of meromorphic functions in the complex plane  $\mathbb{C}$  which are small functions with respect to  $f$ . Note that  $\mathbb{C} \in S(f)$  and  $S(f)$  is a field (see [2]).

If  $f(z)$  is a non-constant meromorphic function,  $a(z) \in S(f) \cup \{\infty\}$ , and  $k$  is a positive integer or  $\infty$ , we denote by  $\overline{E}(a, k, f)$  the set of distinct zeros of  $f(z) - a(z)$  with multiplicities  $\leq k$ , where  $f(z) - \infty$  means  $1/f(z)$  (see [10, p.195]). In particular, we denote by  $\overline{E}(a, \infty, f)$  the set of distinct zeros of  $f(z) - a(z)$ , and we denote it simply by  $\overline{E}(a, f)$ .

Let  $f(z)$  and  $g(z)$  be non-constant meromorphic functions and let  $a(z) \in \{S(f) \cap S(g)\} \cup \{\infty\}$ . We denote by  $\overline{N}_0(r, a, f, g)$  the counting function of common zeros of  $f(z) - a(z) = 0$  and  $g(z) - a(z) = 0$  (ignoring multiplicities), each point counted only once. Let

$$(1.1) \quad \overline{N}_{12}(r, a, f, g) := \overline{N}(r, a, f) + \overline{N}(r, a, g) - 2\overline{N}_0(r, a, f, g).$$

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Then  $\overline{N}_{12}(r, a, f, g)$  denotes the counting function of different solutions to  $f(z) - a(z) = 0$  and  $g(z) - a(z) = 0$  (see [7, p.107]). If

$$\overline{N}(r, a, f) - \overline{N}_0(r, a, f, g) = 0 \quad \text{and} \quad \overline{N}(r, a, g) - \overline{N}_0(r, a, f, g) = 0,$$

we say  $f(z)$  and  $g(z)$  share  $a(z)$  **IM**. If

$$\overline{N}(r, a, f) - \overline{N}_0(r, a, f, g) = S(r, f) \quad \text{and} \quad \overline{N}(r, a, g) - \overline{N}_0(r, a, f, g) = S(r, g),$$

we say  $f(z)$  and  $g(z)$  share  $a(z)$  “**IM**” (see [10, p.254]). It is obvious that if  $f(z)$  and  $g(z)$  share  $a(z)$  **IM**, then  $\overline{E}(a, f) = \overline{E}(a, g)$ , and  $\overline{N}_{12}(r, a, f, g) = 0$ ; if  $f(z)$  and  $g(z)$  share  $a(z)$  “**IM**”, then  $\overline{N}_{12}(r, a, f, g) = S(r, f) + S(r, g)$ .

In the 1920s, R. Nevanlinna established the following famous second fundamental theorem:

**Theorem A** ([7, p.70]). *Let  $f(z)$  be a non-constant meromorphic function, and let  $a_1, a_2, \dots, a_q$  be  $q$  ( $\geq 3$ ) distinct elements in  $\mathbb{C} \cup \{\infty\}$ . Then*

$$(q - 2)T(r, f) \leq \sum_{j=1}^q \overline{N}(r, a_j, f) + S(r, f).$$

In 1929, R. Nevanlinna proved the following well-known theorem on the uniqueness of meromorphic functions by making use of Theorem A:

**Theorem B** ([7, p.109], see also [1, p.48]). *If  $f$  and  $g$  are meromorphic functions sharing  $a_j$  **IM** for  $j = 1, 2, \dots, 5$ , where  $a_1, a_2, \dots, a_5$  are five distinct elements in  $\mathbb{C} \cup \{\infty\}$ , then  $f \equiv g$ .*

In [7, p.111], R. Nevanlinna proved that Theorem B is sharp. It is natural to ask the following problem, which is the long-standing one:

**Problem A** (see [7, p.77 and p.109], see also [2]–[6], [8]–[11]). *Does Theorem B hold if  $a_1, a_2, \dots, a_5$  are five distinct elements in  $\{S(f) \cap S(g)\} \cup \{\infty\}$ ?*

In recent years, some partial results were obtained on Problem A (see [2]–[6], [8]–[11]). In this paper, we give a positive answer to Problem A. In fact, we prove more generally the following theorem:

**Theorem 1.** *Let  $f$  and  $g$  be non-constant meromorphic functions and let  $a_j$  ( $j = 1, 2, \dots, 5$ ) be five distinct elements in  $\{S(f) \cap S(g)\} \cup \{\infty\}$ . If*

$$(1.2) \quad \overline{E}(a_j, k, f) = \overline{E}(a_j, k, g) \quad (j = 1, 2, \dots, 5),$$

where  $k$  is a positive integer or  $\infty$  with  $k \geq 14$ , then  $f \equiv g$ .

By Theorem 1, we obtain the following corollary:

**Corollary 1.** *Let  $f$  and  $g$  be non-constant meromorphic functions and let  $a_j$  ( $j = 1, 2, \dots, 5$ ) be five distinct elements in  $\{S(f) \cap S(g)\} \cup \{\infty\}$ . If  $f$  and  $g$  share  $a_j$  **IM** ( $j = 1, 2, \dots, 5$ ), then  $f \equiv g$ .*

It is obvious that Corollary 1 answers the above Problem A in the affirmative.

2. SOME LEMMAS

**Lemma 1** ([11], see also [2, Lemma 2] or [10, p.185]). *Let  $f(z)$  be a non-constant meromorphic function and let  $a_1, a_2, \dots, a_5$  be five distinct elements in  $S(f) \cup \{\infty\}$ . Then*

$$(2.1) \quad 2T(r, f) \leq \sum_{j=1}^5 \overline{N}(r, a_j, f) + S(r, f).$$

Let  $f(z)$  be a non-constant meromorphic function,  $a(z) \in S(f) \cup \{\infty\}$ , and  $k$  be a positive integer. We denote by  $\overline{N}_k(r, a, f)$  the counting function of distinct zeros of  $f(z) - a(z)$  with multiplicities  $\leq k$ , by  $\overline{N}_{(k+1)}(r, a, f)$  the counting function of distinct zeros of  $f(z) - a(z)$  with multiplicities  $\geq k + 1$ , each point in these counting functions counted only once (see [10, p.190]).

**Lemma 2.** *Let  $f(z)$  be a non-constant meromorphic function and let  $a_1, a_2, \dots, a_5$  be five distinct elements in  $S(f) \cup \{\infty\}$ , and  $k$  a positive integer. Then*

$$(2.2) \quad \sum_{j=1}^5 \overline{N}_{(k+1)}(r, a_j, f) \leq \frac{3}{k} T(r, f) + S(r, f).$$

*Proof.* By Lemma 1, we can obtain (2.1). Noting that for  $j = 1, 2, \dots, 5$

$$k \overline{N}_{(k+1)}(r, a_j, f) + \overline{N}(r, a_j, f) \leq N(r, a_j, f) \leq T(r, f) + S(r, f),$$

we have from (2.1)

$$k \sum_{j=1}^5 \overline{N}_{(k+1)}(r, a_j, f) + 2T(r, f) \leq 5T(r, f) + S(r, f).$$

From this we get (2.2).

**Lemma 3.** *Let  $h$  be a non-constant meromorphic function, and let  $a \in S(h)$  and  $a \neq 0$ . Then*

$$(2.3) \quad m(r, \frac{a'h - ah'}{h - a}) = S(r, h),$$

$$(2.4) \quad m(r, \frac{a'h - ah'}{h(h - a)}) = S(r, h).$$

*Proof.* Noting

$$(2.5) \quad \frac{a'h - ah'}{h - a} = a' - \frac{a(h' - a')}{h - a},$$

$$(2.6) \quad \frac{a'h - ah'}{h(h - a)} = \frac{h'}{h} - \frac{h' - a'}{h - a},$$

and the lemma of the logarithmic derivative, we obtain Lemma 3.

The following lemma plays an important role in the proof of Theorem 1.

**Lemma 4.** *Let  $f$  and  $g$  be non-constant meromorphic functions and let  $a_1, a_2, \dots, a_5$  be five distinct elements in  $\{S(f) \cap S(g)\} \cup \{\infty\}$ . If  $f \neq g$ , then*

$$(2.7) \quad \overline{N}_0(r, a_5, f, g) \leq \sum_{j=1}^4 \overline{N}_{12}(r, a_j, f, g) + S(r, f) + S(r, g).$$

3. PROOF OF LEMMA 4

If  $\overline{N}_0(r, a_5, f, g) = S(r, f) + S(r, g)$ , (2.7) obviously holds. In the following we suppose

$$(3.1) \quad \overline{N}_0(r, a_5, f, g) \neq S(r, f) + S(r, g).$$

Set

$$(3.2) \quad L(w) := \frac{(w - a_1)(a_3 - a_2)}{(w - a_2)(a_3 - a_1)}.$$

Let  $F(z) := L(f(z)), G(z) := L(g(z)), b_j := L(a_j) \quad (j = 1, 2, \dots, 5)$ . From (3.2) we have  $b_1 = 0, b_2 = \infty, b_3 = 1$ ,

$$(3.3) \quad T(r, F) = T(r, f) + S(r, f), \quad T(r, G) = T(r, g) + S(r, g).$$

Since  $a_1, a_2, \dots, a_5$  are five distinct elements in  $\{S(f) \cap S(g)\} \cup \{\infty\}$ , from (3.2) we know that  $b_1, b_2, \dots, b_5$  are five distinct elements in  $\{S(F) \cap S(G)\} \cup \{\infty\}$ . Thus,  $b_4, b_5 \neq 0, 1, \infty$  and  $b_4 \neq b_5$ . Noting  $f \neq g$ , we have

$$(3.4) \quad F \neq G.$$

From (3.1) and (3.3), we get

$$(3.5) \quad \overline{N}_0(r, b_5, F, G) \neq S(r, F) + S(r, G).$$

Set

$$(3.6) \quad H := \frac{F'(a'G - aG')(F - G)}{F(F - 1)G(G - a)} - \frac{G'(a'F - aF')(F - G)}{G(G - 1)F(F - a)},$$

where  $a = b_4 (\neq 0, 1)$ . Then we have from (3.6)

$$(3.7) \quad H = \frac{(F - G)Q}{F(F - 1)(F - a)G(G - 1)(G - a)},$$

where

$$(3.8) \quad Q = F'(a'G - aG')(G - 1)(F - a) - G'(a'F - aF')(F - 1)(G - a).$$

By a simple computation,

$$(3.9) \quad \begin{aligned} Q &= a'FF'G^2 - a'FF'G - a(a - 1)FF'G' - aa'F'G^2 + aa'F'G \\ &\quad - a'F^2GG' + a'FGG' + a(a - 1)F'GG' + aa'F^2G' - aa'FG'. \end{aligned}$$

Suppose that  $H \equiv 0$ . From (3.4) and (3.6) we obtain

$$(3.10) \quad \frac{F'(a'G - aG')}{(F - 1)(G - a)} \equiv \frac{G'(a'F - aF')}{(G - 1)(F - a)}.$$

If  $a$  is a constant, noting  $a \neq 1$ , from (3.10) we get  $F \equiv G$ , which contradicts (3.4). Thus,  $a$  is not a constant. From (3.10) we have

$$\frac{F'(a'G - aG')}{G'(a'F - aF')} - 1 \equiv \frac{(F - 1)(G - a)}{(G - 1)(F - a)} - 1.$$

Thus,

$$\frac{a'[(F' - G')G - (F - G)G']}{G'(a'F - aF')} \equiv \frac{(1 - a)(F - G)}{(G - 1)(F - a)}.$$

From this we get

$$(3.11) \quad \frac{F' - G'}{F - G} \equiv \frac{(1 - a)G'(a'F - aF')}{a'G(G - 1)(F - a)} + \frac{G'}{G}.$$

By (3.5), we know that there is a point  $z_0$  such that  $z_0$  is a common zero of  $F - b_5$  and  $G - b_5$  that is not a zero or a pole of  $a, a', b_5, b_5 - 1, b_5 - a$ . It is obvious that  $z_0$  is a pole of the left-hand side of (3.11), and not a pole of the right-hand side of (3.11), which is a contradiction. Thus,

$$(3.12) \quad H \neq 0.$$

Suppose that  $z_n$  is a common zero of  $F - b_5$  and  $G - b_5$  that is not a zero or pole of  $a, b_5, b_5 - 1, b_5 - a$ . It is obvious that  $z_n$  is a zero of  $F - G$ , and  $z_n$  is not a pole of

$$\frac{Q}{F(F - 1)(F - a)G(G - 1)(G - a)},$$

where  $Q$  is given by (3.8). By (3.7), we know that  $z_n$  is a zero of  $H$ . Again by (3.12) we obtain

$$(3.13) \quad \begin{aligned} \overline{N}_0(r, b_5, F, G) &\leq N(r, 0, H) + S(r, F) + S(r, G) \\ &\leq m(r, H) + N(r, H) + S(r, F) + S(r, G). \end{aligned}$$

From (3.6) we have

$$(3.14) \quad \begin{aligned} H &= \frac{F'}{F - 1} \cdot \frac{a'G - aG'}{G(G - a)} - \left( \frac{F'}{F - 1} - \frac{F'}{F} \right) \cdot \frac{a'G - aG'}{G - a} \\ &\quad - \left( \frac{G'}{G - 1} - \frac{G'}{G} \right) \cdot \frac{a'F - aF'}{F - a} + \frac{G'}{G - 1} \cdot \frac{a'F - aF'}{F(F - a)}. \end{aligned}$$

Again by Lemma 3 and the lemma of the logarithmic derivative we obtain

$$(3.15) \quad m(r, H) = S(r, F) + S(r, G).$$

Substituting (3.15) into (3.13) we have

$$(3.16) \quad \overline{N}_0(r, b_5, F, G) \leq N(r, H) + S(r, F) + S(r, G).$$

Next we estimate on  $N(r, H)$ .

By (3.6), we know that the poles of  $H$  only possibly occur from the zeros of  $F, G, F - 1, G - 1, F - a$  and  $G - a$ , the poles of  $F, G$  and  $a$ . Let  $S_0$  be the set of all zeros, 1-points and poles of  $a$ , and let for  $j = 1, 2, 3, 4$

$$A_j := \{z | F(z) - b_j(z) = 0\} \setminus S_0, \quad B_j := \{z | G(z) - b_j(z) = 0\} \setminus S_0,$$

where  $b_1 = 0, b_2 = \infty, b_3 = 1$  and  $b_4 = a$ . Thus, the poles of  $H$  occur possibly only from the set

$$\bigcup_{1 \leq p \leq 4} A_p \bigcup_{1 \leq q \leq 4} B_q \bigcup S_0.$$

Let

$$\begin{aligned}
 S_1 &:= \bigcup_{1 \leq p \leq 4} \{A_p \cap B_p\}, \\
 S_2 &:= \left\{ \bigcup_{1 \leq p \leq 4} A_p \right\} \setminus \left\{ \bigcup_{1 \leq q \leq 4} B_q \right\}, \\
 S_3 &:= \left\{ \bigcup_{1 \leq q \leq 4} B_q \right\} \setminus \left\{ \bigcup_{1 \leq p \leq 4} A_p \right\}, \\
 S_4 &:= \bigcup_{\substack{1 \leq p \leq 4 \\ 1 \leq q \leq 4 \\ p \neq q}} \{A_p \cap B_q\}.
 \end{aligned}$$

It is clear that  $S_1$  is a set of the common zeros of  $F - b_j$  and  $G - b_j$  ( $j = 1, 2, 3, 4$ );  $S_2$  is a set of the zeros of  $F - b_j$  ( $j = 1, 2, 3, 4$ ) that is not the zeros of  $G - b_k$  ( $k = 1, 2, 3, 4$ );  $S_3$  is a set of the zeros of  $G - b_j$  ( $j = 1, 2, 3, 4$ ) that is not the zeros of  $F - b_k$  ( $k = 1, 2, 3, 4$ );  $S_4$  is a set of the zeros of  $F - b_j$  that is the zeros of  $G - b_k$ , where  $1 \leq j, k \leq 4$  and  $j \neq k$ . From this we have

$$\bigcup_{1 \leq j \leq 4} S_j = \bigcup_{1 \leq p \leq 4} A_p \bigcup_{1 \leq q \leq 4} B_q.$$

Thus, the poles of  $H$  occur possibly only from the set

$$\bigcup_{1 \leq j \leq 4} S_j \bigcup S_0.$$

Since  $b_1, b_2, b_3$  and  $b_4$  are four distinct elements in  $\{S(F) \cap S(G)\} \cup \{\infty\}$ , the contribution of  $S_0$  to  $N(r, H)$  is at most  $S(r, F) + S(r, G)$ . We next estimate the contribution of  $\bigcup_{1 \leq j \leq 4} S_j$  to  $N(r, H)$ . We discuss four cases:

Case 1. The contribution of  $S_1$  to  $N(r, H)$ .

We distinguish four subcases:

Subcase 1.1. Suppose that  $z_{11} \in A_1 \cap B_1$ , and assume that  $z_{11}$  is a zero of  $F$  of order  $p_1$  and  $G$  of order  $q_1$ . Then from (3.9), we know that  $z_{11}$  is a zero of  $Q$  of order at least  $p_1 + q_1 - 1$ . Noting that  $z_{11}$  is a zero of  $F - G$ , from (3.7) we deduce that  $z_{11}$  is not a pole of  $H$ .

Subcase 1.2. Suppose that  $z_{12} \in A_2 \cap B_2$ , and assume that  $z_{12}$  is a pole of  $F$  of order  $p_2$  and  $G$  of order  $q_2$ . From (3.9), we know that  $z_{12}$  is a pole of  $Q$  of order at most  $2p_2 + 2q_2 + 1$ . Noting that  $z_{12}$  is a pole of  $F - G$  of order at most  $\max\{p_2, q_2\}$ , from (3.7), we have that  $z_{12}$  is not a pole of  $H$ .

Subcase 1.3. Suppose that  $z_{13} \in A_3 \cap B_3$ . Noting that  $z_{13}$  is a zero of  $F - G$ , a simple pole of  $\frac{F'}{F-1}$  and  $\frac{G'}{G-1}$ , from (3.6) we have that  $z_{13}$  is not a pole of  $H$ .

Subcase 1.4. Suppose that  $z_{14} \in A_4 \cap B_4$ . From (2.6), we know that  $z_{14}$  is a simple pole of  $\frac{a'F - aF'}{F(F-a)}$  and  $\frac{a'G - aG'}{G(G-a)}$ . Noting that  $z_{14}$  is a zero of  $F - G$ , from (3.6) we see that  $z_{14}$  is not a pole of  $H$ .

From the above, it follows that the points in  $S_1$  are not poles of  $H$ . Thus, the contribution of  $S_1$  to  $N(r, H)$  is 0.

Case 2. The contribution of  $S_2$  to  $N(r, H)$ .

We distinguish four subcases:

Subcase 2.1. Suppose that  $z_{21} \in A_1$  and  $z_{21} \notin \bigcup_{1 \leq q \leq 4} B_q$ . Then  $z_{21}$  is a zero of  $F$ , not a zero of  $G, 1/G, G - 1$  and  $G - a$ . From (3.6), we have that  $z_{21}$  is a pole of  $H$  of order at most 1.

Subcase 2.2. Suppose that  $z_{22} \in A_2$  and  $z_{22} \notin \bigcup_{1 \leq q \leq 4} B_q$ . Then  $z_{22}$  is a pole of  $F$ , not a zero of  $G, 1/G, G - 1$  and  $G - a$ . From (3.6), we have that  $z_{22}$  is a pole of  $H$  of order at most 1.

Subcase 2.3. Suppose that  $z_{23} \in A_3$  and  $z_{23} \notin \bigcup_{1 \leq q \leq 4} B_q$ . Then  $z_{23}$  is a zero of  $F - 1$ , not a zero of  $G, 1/G, G - 1$  and  $G - a$ . From (3.6), we have that  $z_{23}$  is a pole of  $H$  of order at most 1.

Subcase 2.4. Suppose that  $z_{24} \in A_4$  and  $z_{24} \notin \bigcup_{1 \leq q \leq 4} B_q$ . Then  $z_{24}$  is a zero of  $F - a$ , not a zero of  $G, 1/G, G - 1$  and  $G - a$ . From (2.6) and (3.6), we have that  $z_{24}$  is a pole of  $H$  of order at most 1.

From the above, we know that the points in  $S_2$  are poles of  $H$  of order at most 1.

Case 3. The contribution of  $S_3$  to  $N(r, H)$ .

As with Case 2, we have that the points in  $S_3$  are poles of  $H$  of order at most 1.

Case 4. The contribution of  $S_4$  to  $N(r, H)$ .

Suppose that  $z_4 \in S_4$ . Then  $z_4 \in A_p$  and  $z_4 \in B_q$ , where  $1 \leq p, q \leq 4$  and  $p \neq q$ . Without loss of generality we can assume that  $z_4 \in A_1$  and  $z_4 \in B_2$ . Then,  $z_4$  is a zero of  $F$ , and a pole of  $G$ . From (2.6) and (3.6), we have that  $z_4$  is a pole of  $H$  of order at most 2. Thus, the points in  $S_4$  are poles of  $H$  of order at most 2.

Noting that each point of  $S_2$  and  $S_3$  is counted once, each point of  $S_4$  is counted twice in

$$\sum_{j=1}^4 \bar{N}_{12}(r, b_j, F, G).$$

From the above and (1.1) we obtain that the contribution of  $\bigcup_{1 \leq j \leq 4} S_j$  to  $N(r, H)$  is at most

$$\sum_{j=1}^4 \bar{N}_{12}(r, b_j, F, G).$$

Thus,

$$(3.17) \quad N(r, H) \leq \sum_{j=1}^4 \bar{N}_{12}(r, b_j, F, G) + S(r, F) + S(r, G).$$

Substituting (3.17) into (3.16) we have

$$\bar{N}_0(r, b_5, F, G) \leq \sum_{j=1}^4 \bar{N}_{12}(r, b_j, F, G) + S(r, F) + S(r, G),$$

i.e.,

$$\bar{N}_0(r, a_5, f, g) \leq \sum_{j=1}^4 \bar{N}_{12}(r, a_j, f, g) + S(r, f) + S(r, g),$$

and Lemma 4 is proved.

## 4. THEOREM 2 AND ITS PROOF

Using Lemma 4, we can prove the following Theorem:

**Theorem 2.** *Let  $f$  and  $g$  be non-constant meromorphic functions and let  $a_j$  ( $j = 1, 2, \dots, 5$ ) be five distinct elements in  $\{S(f) \cap S(g)\} \cup \{\infty\}$ . If  $f \neq g$ , then*

$$(4.1) \quad 2T(r, f) + 2T(r, g) \leq 9 \sum_{j=1}^5 \overline{N}_{12}(r, a_j, f, g) + S(r, f) + S(r, g).$$

*Proof.* By Lemma 4 we have

$$(4.2) \quad \overline{N}_0(r, a_5, f, g) \leq \sum_{j=1}^4 \overline{N}_{12}(r, a_j, f, g) + S(r, f) + S(r, g).$$

Noting

$$\overline{N}(r, a_5, f) + \overline{N}(r, a_5, g) = 2\overline{N}_0(r, a_5, f, g) + \overline{N}_{12}(r, a_5, f, g),$$

from this and (4.2) we get

$$(4.3) \quad \overline{N}(r, a_5, f) + \overline{N}(r, a_5, g) \leq 2 \sum_{j=1}^5 \overline{N}_{12}(r, a_j, f, g) - \overline{N}_{12}(r, a_5, f, g) + S(r, f) + S(r, g).$$

In the same manner as above, we have for  $i = 1, 2, 3, 4$

$$(4.4) \quad \overline{N}(r, a_i, f) + \overline{N}(r, a_i, g) \leq 2 \sum_{j=1}^5 \overline{N}_{12}(r, a_j, f, g) - \overline{N}_{12}(r, a_i, f, g) + S(r, f) + S(r, g).$$

By Lemma 1, we have

$$(4.5) \quad 2T(r, f) + 2T(r, g) \leq \sum_{i=1}^5 \overline{N}(r, a_i, f) + \sum_{i=1}^5 \overline{N}(r, a_i, g) + S(r, f) + S(r, g).$$

Substituting (4.3) and (4.4) into (4.5) we obtain (4.1), and Theorem 2 is thus proved.  $\square$

## 5. PROOF OF THEOREM 1

Suppose that  $f \neq g$ . By Theorem 2, we can obtain (4.1).

If  $k = \infty$ , by hypothesis we have  $\overline{E}(a_j, \infty, f) = \overline{E}(a_j, \infty, g)$  ( $j = 1, 2, \dots, 5$ ). Thus,

$$(5.1) \quad \overline{N}_{12}(r, a_j, f, g) = 0 \quad (j = 1, 2, \dots, 5).$$

Substituting (5.1) into (4.1) we obtain

$$2T(r, f) + 2T(r, g) = S(r, f) + S(r, g).$$

This is a contradiction. Next, we assume that  $k$  is a positive integer.

By hypothesis we have

$$\overline{E}(a_j, k, f) = \overline{E}(a_j, k, g) \quad (j = 1, 2, \dots, 5).$$

From this we obtain for  $j = 1, 2, \dots, 5$

$$(5.2) \quad \overline{N}_{12}(r, a_j, f, g) \leq \overline{N}_{(k+1)}(r, a_5, f) + \overline{N}_{(k+1)}(r, a_5, g) + S(r, f) + S(r, g).$$

Substituting (5.2) into (4.1) we obtain

$$(5.3) \quad 2T(r, f) + 2T(r, g) \leq 9 \sum_{j=1}^5 \overline{N}_{(k+1)}(r, a_j, f) + 9 \sum_{j=1}^5 \overline{N}_{(k+1)}(r, a_j, g) + S(r, f) + S(r, g).$$

By Lemma 2, we have

$$(5.4) \quad \sum_{j=1}^5 \overline{N}_{(k+1)}(r, a_j, f) \leq \frac{3}{k} T(r, f) + S(r, f),$$

$$(5.5) \quad \sum_{j=1}^5 \overline{N}_{(k+1)}(r, a_j, g) \leq \frac{3}{k} T(r, g) + S(r, g).$$

Substituting (5.4) and (5.5) into (5.3) we obtain

$$2T(r, f) + 2T(r, g) \leq \frac{27}{k} T(r, f) + \frac{27}{k} T(r, g) + S(r, f) + S(r, g),$$

which contradicts the assumption  $k \geq 14$ . Thus,  $f \equiv g$ .

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