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PRIMARY DECOMPOSITION: COMPATIBILITY, INDEPENDENCE AND LINEAR GROWTH

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Abstract. For finitely generated modules $N \subsetneq M$ over a Noetherian ring R, we study the following properties about primary decomposition: (1) The Compatibility property, which says that if $\mathrm{Ass}(M/N) = \{P_1, P_2, \ldots, P_s\}$ and Q_i is a P_i-primary component of $N \subsetneq M$ for each $i = 1, 2, \ldots, s$, then $N = Q_1 \cap Q_2 \cap \cdots \cap Q_s$; (2) For a given subset $X = \{P_1, P_2, \ldots, P_r\} \subseteq \mathrm{Ass}(M/N)$, X is an open subset of $\mathrm{Ass}(M/N)$ if and only if the intersections $Q_1 \cap Q_2 \cap \cdots \cap Q_r = Q'_1 \cap Q'_2 \cap \cdots \cap Q'_r$ for all possible P_i-primary components Q_i and Q'_i of $N \subsetneq M$; (3) A new proof of the 'Linear Growth' property, which says that for any fixed ideals $I_1, I_2, \ldots, I_t$ of R there exists a $k \in \mathbb{N}$ such that for any $n_1, n_2, \ldots, n_t \in \mathbb{N}$ there exists a primary decomposition of $I_1^{n_1} I_2^{n_2} \cdots I_t^{n_t} M \subset M$ such that every P-primary component Q of that primary decomposition contains $P^{k(n_1+n_2+\cdots+n_t)}M$.

0. Introduction

Throughout this paper R is a Noetherian ring and $M \neq 0$ is a finitely generated R-module unless explicitly stated otherwise. Let $N \subsetneq M$ be a proper R-submodule of M. By primary decomposition $N = Q_1 \cap Q_2 \cap \cdots \cap Q_s$ of N in M, we always mean an irredundant and minimal primary decomposition, where Q_i is a P_i-primary submodule of M, i.e. $\mathrm{Ass}(M/Q_i) = \{P_i\}$, for each $i = 1, 2, \ldots, s$, unless explicitly mentioned otherwise. Then $\mathrm{Ass}(M/N) = \{P_1, P_2, \ldots, P_s\}$ and we say that Q_i is a P_i-primary component of N in M. As a subset of $\mathrm{Spec}(R)$ with the *Zariski* topology, $\mathrm{Ass}(M/N)$ inherits a topology structure. For an ideal I in R, we use $(N :_M I^\infty)$ to denote $\bigcup_i (N :_M I^i)$.

Notation 0.1. Let $N \subsetneq M$ be finitely generated R-modules. We use $\Lambda_P(N \subsetneq M)$, or Λ_P if the R-modules $N \subsetneq M$ are clear from the context, to denote the set of all possible P-primary components of N in M for every $P \in \mathrm{Ass}(M/N)$.

We know that if $P \in \mathrm{Ass}(M/N)$ is an embedded prime ideal, then $\Lambda_P(N \subsetneq M)$ contains more than one element. (Also see the passage following Theorem 2.2 and the reference to [HRS].) Suppose that $N = Q_1 \cap Q_2 \cap \cdots \cap Q_s$ is a primary decomposition of $N \subsetneq M$ such that $Q_i \in \Lambda_{P_i}$ for $i = 1, 2, \ldots, s$. Then if we choose a P_i-primary submodule Q'_i of M such that $N \subseteq Q'_i \subseteq Q_i$ for each $i = 1, 2, \ldots, s$, we get a primary decomposition $N = Q'_1 \cap Q'_2 \cap \cdots \cap Q'_s$ of $N \subsetneq M$. For example

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we may choose $Q_i' = \ker(M \to (M/(P_i^{n_i}M + N))_{P_i})$ for all $n_i \gg 0$ to get primary decompositions $N = \bigcap_{1\le i\le s} \ker(M \to (M/(P_i^{n_i}M + N))_{P_i})$ for all $n_i \gg 0$. But given an arbitrary $Q_i'' \in \Lambda_{P_i}$ for each $i = 1, 2, \dots, s$, we do not know *a priori* if $N = Q_1'' \cap Q_2'' \cap \cdots \cap Q_s''$. This compatibility question is answered positively in Theorem 1.1:

Theorem 1.1 (Compatibility). *Let $N \subsetneq M$ be finitely generated R-modules and $\mathrm{Ass}(M/N) = \{P_1, P_2, \dots, P_s\}$. Suppose that for each $i = 1, 2, \dots, s$, Q_i is a P_i-primary component of N in M, i.e. $Q_i \in \Lambda_{P_i}$. Then $N = Q_1 \cap Q_2 \cap \cdots \cap Q_s$, which is necessarily an irredundant and minimal primary decomposition.*

Definition 0.2. Let $N \subsetneq M$ be finitely generated R-modules and X a subset of $\mathrm{Ass}(M/N)$, say $X = \{P_1, P_2, \dots, P_r\} \subseteq \mathrm{Ass}(M/N) = \{P_1, P_2, \dots, P_r, P_{r+1}, \dots, P_s\}$. We say that the primary decompositions of N in M are independent over X, or X-independent, if for any two primary decompositions, say, $N = Q_1 \cap Q_2 \cap \cdots \cap Q_s = Q_1' \cap Q_2' \cap \cdots \cap Q_s'$, of $N \subset M$ such that $\{Q_i, Q_i'\} \subseteq \Lambda_{P_i}(N \subset M)$ for $i = 1, 2, \dots, s$, we have $Q_1 \cap Q_2 \cap \cdots \cap Q_r = Q_1' \cap Q_2' \cap \cdots \cap Q_r'$. In this case, we denote the invariant intersection by $Q_X(N \subset M)$, or Q_X if $N \subset M$ is clear from the context.

It is well known that primary decompositions are independent over open subsets of $\mathrm{Ass}(M/N)$. (See Observations 0.3 below.) Actually it turns out that independence property characterizes open subsets of $\mathrm{Ass}(M/N)$:

Theorem 2.2. *Let $N \subsetneq M$ be finitely generated R-modules and $X \subseteq \mathrm{Ass}(M/N)$ be a subset of $\mathrm{Ass}(M/N)$. Then the primary decompositions of N in M are independent over X if and only if X is an open subset of $\mathrm{Ass}(M/N)$.*

In Section 3 we use *Artin-Rees numbers* to prove the following:

Theorem 3.3. *Let R be a Noetherian ring, M a finitely generated R-module and $I_1, I_2, \dots, I_t$ ideals of R. Then there exists a $k \in \mathbb{N}$ such that for all $n_1, n_2, \dots, n_t \in \mathbb{N}$ and for all ideals $J \subset R$,*

$$(J^{k|\underline{n}|}M + I_1^{n_1}I_2^{n_2}\cdots I_t^{n_t}M) \cap (I_1^{n_1}I_2^{n_2}\cdots I_t^{n_t}M :_M J^\infty) = I_1^{n_1}I_2^{n_2}\cdots I_t^{n_t}M,$$

where $|\underline{n}| := n_1 + n_2 + \cdots + n_t$.

As a corollary of Theorem 3.3, we have a new proof of the 'Linear Growth' property, which was first proved by I. Swanson [Sw] and then by R. Y. Sharp using different methods and in a more general situation [Sh2]:

Corollary 3.4 (Linear Growth; [Sw] and [Sh2]). *Let R be a Noetherian ring, M a finitely generated R-module and $I_1, I_2, \dots, I_t$ ideals of R. Then there exists a $k \in \mathbb{N}$ such that for any $n_1, n_2, \dots, n_t \in \mathbb{N}$, there exists a primary decomposition of $I_1^{n_1}I_2^{n_2}\cdots I_t^{n_t}M \subseteq M$*

$$I_1^{n_1}I_2^{n_2}\cdots I_t^{n_t}M = Q_{\underline{n}_1} \cap Q_{\underline{n}_1} \cap \cdots \cap Q_{\underline{n}_{r_{\underline{n}}}},$$

where the Q_i's are P_i-primary components of the primary decomposition such that $P_i^{k|\underline{n}|}M \subseteq Q_i$ for all $i = 1, 2, \dots, \underline{n}_{r_{\underline{n}}}$, where $\underline{n} = (n_1, n_2, \cdots, n_t)$ and $|\underline{n}| = n_1 + n_2 + \cdots + n_t$.

Before ending this introductory section, we make the following well-known observations, which is to the effect of saying that primary decompositions are independent over open subsets.

Observations on independence 0.3. Suppose $N = Q_1 \cap Q_2 \cap \cdots \cap Q_s$ is a primary decomposition of N in a finitely generated R-module M such that Q_i is P_i-primary for each $i = 1, 2, \ldots, s$.

(1) For any ideal $I \subseteq R$, the intersection $\bigcap_{I \not\subseteq P_i} Q_i = (N :_M I^\infty)$ is independent of the particular primary decomposition of N in M (cf. D. Eisenbud [Ei], page 101, Proposition 3.13). This means that the primary decompositions of $N \subsetneq M$ are independent over $X = \{P \in \operatorname{Ass}(M/N) \mid I \not\subseteq P\}$ and $Q_X = (N :_M I^\infty)$.

(2) Alternatively, for any multiplicatively closed set $W \subset R$, the intersection $\bigcap_{P_i \cap W = \emptyset} Q_i = \ker(M \to (M/N)_W)$ is independent of the particular primary decomposition (cf. D. Eisenbud [Ei], page 113, Exercise 3.12). That is to say that the primary decompositions of $N \subsetneq M$ are independent over $Y = \{P \in \operatorname{Ass}(M/N) \mid P \cap W = \emptyset\}$ and $Q_Y = \ker(M \to (M/N)_W)$.

1. Compatibility

The main theorem in this section is to show that all the primary components of R-modules $N \subsetneq M$ are totally compatible in forming the primary decompositions of $N \subsetneq M$.

Theorem 1.1 (Compatibility)**.** *Let $N \subsetneq M$ be finitely generated R-modules and $\operatorname{Ass}(M/N) = \{P_1, P_2, \ldots, P_s\}$. Suppose that for each $i = 1, 2, \ldots, s$, Q_i is a P_i-primary component of N in M, i.e. $Q_i \in \Lambda_{P_i}(N \subsetneq M)$. Then $N = Q_1 \cap Q_2 \cap \cdots \cap Q_s$, which is necessarily an irredundant and minimal primary decomposition.*

Proof. We induct on s, the cardinality of $\operatorname{Ass}(M/N)$.

If $s = 1$, then $N = Q_1$ and the claim is trivially true.

Suppose $s \geq 2$. By rearranging the order of $P_1, P_2, \ldots, P_s$, we may assume that P_s is a maximal prime ideal in $\operatorname{Ass}(M/N)$. Since $Q_i \in \Lambda_{P_i}$ for $i = 1, 2, \ldots, s$, we can find s specific primary decompositions

$$N = Q_{(i,1)} \cap Q_{(i,2)} \cap \cdots \cap Q_{(i,i)} \cap \cdots \cap Q_{(i,s)} \qquad \text{for } i = 1, 2, \ldots, s,$$

where $Q_{(i,j)} \in \Lambda_{P_j}$ and $Q_{(i,i)} = Q_i$ for all $i, j = 1, 2, \ldots, s$. Let $W = R \setminus \bigcup_{1 \leq i \leq s-1} P_i$. By Observation 0.3(2) and our assumption on P_s, we know that the primary decompositions of $N \subsetneq M$ are independent over $X = \{P \in \operatorname{Ass}(M/N) \mid P \cap W = \emptyset\} = \{P_1, P_2, \ldots, P_{s-1}\}$ with $Q_X = \ker(M \to (M/N)_W)$. That is to say,

$$Q_X = \ker(M \to (M/N)_W) = Q_{(i,1)} \cap Q_{(i,2)} \cap \cdots \cap Q_{(i,s-1)} \qquad \text{for } i = 1, 2, \ldots, s,$$

are all primary decompositions of $Q_X \subsetneq M$ and in particular $Q_i = Q_{(i,i)} \in \Lambda_{P_i}(Q_X \subset M)$ for $i = 1, 2, \ldots, s-1$. Since the cardinality of $\operatorname{Ass}(M/Q_X)$ is $s - 1$, we use the induction hypothesis to see that

$$Q_X = Q_1 \cap Q_2 \cap \cdots \cap Q_{s-1}.$$

But we already know that $Q_X = Q_{(s,1)} \cap Q_{(s,2)} \cap \cdots \cap Q_{(s,s-1)}$ by the X-independence of primary decompositions of $N \subsetneq M$. Hence we have

$$\begin{aligned} N &= Q_{(s,1)} \cap Q_{(s,2)} \cap \cdots \cap Q_{(s,s-1)} \cap Q_{(s,s)} \\ &= Q_X \cap Q_s \\ &= Q_1 \cap Q_2 \cap \cdots \cap Q_{s-1} \cap Q_s. \end{aligned}$$

∎

Remark 1.2. In [Bo, Chapter IV], the notion of primary decomposition is generalized to not necessarily finitely generated modules over not necessarily Noetherian rings. Let R be a (not necessarily Noetherian) ring and M be a (not necessarily finitely generated) R-module. A prime ideal P of R is said to be *weakly associated* with M if there exists an $x \in M$ such that P is minimal over the ideal $\mathrm{Ann}(x)$ and we denote by $\mathrm{Ass}_f(M)$ the set of prime ideals weakly associated with M (cf. [Bo, page 289, Chapter IV, § 1, Exercise 17]). We say that an element $r \in R$ is nearly nilpotent on M if for any $x \in M$, there exists an $n(x) \in \mathbb{N}$, such that $r^{n(x)}x = 0$ (cf. [Bo, page 267, Chapter IV, § 1.4, Definition 2]). Then for any R-submodule N of M, we define $r_M(N) := \{r \in R \,|\, r \text{ is nearly nilpotent on } M/N\}$ (cf. [Bo, page 292, Chapter IV, § 2, Exercise 11]). An R-submodule Q of M is said to be P-primary in M if $\mathrm{Ass}_f(M/Q) = \{P\}$, which is equivalent to the statement that every $r \in R$ is either a non-zerodivisor or nearly nilpotent on M/Q, and in this case we have $r_M(Q) = P$) (cf. [Bo, page 292, Chapter IV, § 2, Exercise 12(a)]). Then we say that an R-submodule N has a primary decomposition in M if there exist P_i-primary submodules $Q_i \subset M$, $i = 1, 2, \ldots, s$, such that $N = Q_1 \cap Q_2 \cap \cdots \cap Q_s$ (cf. [Bo, page 294, Chapter IV, § 2, Exercise 20]). Again we always assume primary decompositions to be irredundant and minimal (i.e. reduced) if they exist. If N has primary decompositions in M, then Observation 0.3(2) still holds (replace $\mathrm{Ass}(M/N)$ by $\mathrm{Ass}_f(M/N)$). Therefore the proof of compatibility, i.e. Theorem 1.1, also applies to the the case where $N \subsetneq M$ are not necessarily finitely generated R-modules over a not necessarily Noetherian ring R as long as the primary decompositions exist.

2. Independence over open subsets of $\mathrm{Ass}(M/N)$

Because of the compatibility property, i.e. Theorem 1.1, we have an equivalent statement to the definition of X-independence.

Lemma 2.1. *Let $N \subsetneq M$ be finitely generated R-modules and $X = \{P_1, P_2, \ldots, P_r\} \subseteq \mathrm{Ass}(M/N) = \{P_1, P_2, \ldots, P_r, P_{r+1}, \ldots, P_s\}$. Then the following are equivalent:*

(1) *The primary decompositions of N in M are independent over X.*
(2) *For any Q_i and Q_i' in Λ_{P_i}, where $i = 1, 2, \ldots, r$, the equality $Q_1 \cap Q_2 \cap \cdots \cap Q_r = Q_1' \cap Q_2' \cap \cdots \cap Q_r'$ holds.*

It turns out that the independence observed in Observations 0.3 actually exhausts all the possibilities.

Theorem 2.2. *Let $N \subsetneq M$ be finitely generated R-modules and $X \subseteq \mathrm{Ass}(M/N)$ be a subset of $\mathrm{Ass}(M/N)$. Then the primary decompositions of N in M are independent over X if and only if X is an open subset of $\mathrm{Ass}(M/N)$.*

Proof. Without loss of generality we assume $N = 0$.

The "if" part is just Observation 0.3(1). To prove the "only if" part, it suffices to show X is stable under specialization since $\mathrm{Ass}(M/N) = \mathrm{Ass}(M)$ is finite. Let P be an arbitrary prime ideal in $X \subseteq \mathrm{Ass}(M/N)$. All we need to show is that for any $P' \in \mathrm{Ass}(M)$ such that $P' \subset P$, we have $P' \in X$.

Say $X = \{P = P_1, P_2, \ldots, P_t, P_{t+1}, \ldots, P_r\}$ such that $P_i \subseteq P$ for $i = 1, 2, \ldots, t$ and $P_i \nsubseteq P$ for $i = t+1, \ldots, r$. Let

$$X_P := X \cap \mathrm{Ass}(M_P) = \{P_P = (P_1)_P, (P_2)_P, \ldots, (P_t)_P\}.$$

We first show that the primary decompositions of $0 \subsetneq M_P$ are independent over X_P: For any $L_i \in \Lambda_{(P_i)_P}(0 \subsetneq M_P), i = 1, 2, \ldots, t$, let Q_i be the the full pre-image of L_i under the map $M \to M_P$. Then choose $Q_i \in \Lambda_{P_i}(0 \subsetneq M)$ for $i = t+1, \ldots, r$. Then it is easy to see that $(Q_1 \cap Q_2 \cap \cdots \cap Q_r)_P = L_1 \cap L_2 \cap \cdots \cap L_t$. Then the X-independence assumption implies that the primary decompositions of $0 \subsetneq M_P$ are independent over $X_P = X \cap \operatorname{Ass}(M_P)$.

Hence by replacing M with M_P we may assume that (R, P) is local with the maximal ideal P and $P \in X = \{P = P_1, P_2, \ldots, P_t\} \subseteq \operatorname{Ass}(M)$. In this case to prove that X is stable under specialization is simply to prove that $X = \operatorname{Ass}(M)$. For each $i = 1, 2, \ldots, t$, choose a P_i-primary component Q_i of $0 \subsetneq M$. There exists a $k \in \mathbb{N}$ such that $P^k M \subseteq Q_1$ and therefore $P^n M \in \Lambda_P$ for all $n \geq k$. Set $L = Q_2 \cap Q_3 \cap \cdots \cap Q_t$. Then by Lemma 2.1 the assumption that the primary decompositions of 0 in M are independent over X simply means that $Q_1 \cap L = P^n M \cap L$ for all $n \geq k$, which implies $Q_1 \cap L = 0$ by Krull Intersection Theorem. This forces $0 = Q_1 \cap Q_2 \cap \cdots \cap Q_t$ to be a primary decomposition of 0 in M. In particular it means that $\operatorname{Ass}(M) = \{P = P_1, P_2, \ldots, P_t\} = X$. □

In particular, if $P \in \operatorname{Ass}(M/N)$ is not minimal over $\operatorname{Ann}(M/N)$, then the P-primary components of N in M are not unique. In fact, in [HRS], W. Heinzer, L. J. Ratliff, Jr. and K. Shah showed that if $P \in \operatorname{Ass}(M/N)$ is an embedded prime ideal, then there are infinitely many maximal P-primary components of N in M with respect to containment. See [HRS] and their following papers for more information about the embedded primary components.

3. 'Linear Growth' property

In this section we give a new proof of the 'Linear Growth' property using *Artin-Rees numbers* and compatibility. The 'Linear Growth' property was first proved by I. Swanson [Sw] and then by R. Y. Sharp using different methods and in a more general situation [Sh2].

We first give a definition of *Artin-Rees numbers*, $\operatorname{AR}(J, N \subseteq M)$, of a pair of finitely generated R-modules $N \subseteq M$ with respect to an ideal J of R. These numbers have been studied in [Hu], where a set of ideals is considered instead of one single ideal.

Definition 3.1. Let $N \subseteq M$ be finitely generated R-modules over a Noetherian ring R and J an ideal of R. We define $\operatorname{AR}(J, N \subseteq M) := \min\{\, k \mid J^n M \cap N \subseteq J^{n-k} N \text{ for all } n \geq k \,\}$.

Remark 3.2. If $K \subseteq L \subseteq M$, then

$$\operatorname{AR}(J, K \subseteq M) \leq \operatorname{AR}(J, K \subseteq L) + \operatorname{AR}(J, L \subseteq M).$$

If $J^n M \subseteq N$ for some n, then $\operatorname{AR}(J, N \subseteq M) \leq n$.

Theorem 3.3. *Let R be a Noetherian ring, M a finitely generated R-module and $I_1, I_2, \ldots, I_t$ ideals of R. Then there exists a $k \in \mathbb{N}$ such that for all $n_1, n_2, \ldots, n_t \in \mathbb{N}$ and for all ideals $J \subset R$,*

$$\begin{aligned} I_1^{n_1} I_2^{n_2} \cdots I_t^{n_t} M &\supseteq J^{k|\underline{n}|} M \cap (I_1^{n_1} I_2^{n_2} \cdots I_t^{n_t} M :_M J^\infty), \qquad \textit{i.e.} \\ I_1^{n_1} I_2^{n_2} \cdots I_t^{n_t} M &= (J^{k|\underline{n}|} M + I_1^{n_1} I_2^{n_2} \cdots I_t^{n_t} M) \cap (I_1^{n_1} I_2^{n_2} \cdots I_t^{n_t} M :_M J^\infty), \end{aligned}$$

where $|\underline{n}| := n_1 + n_2 + \cdots + n_t$.

Proof. It is enough to prove the theorem for

$$\begin{aligned}
\mathcal{R} &= R[I_1T_1, I_2T_2, \ldots, I_tT_t, T_1^{-1}, T_2^{-1}, \ldots, T_t^{-1}],\\
\mathcal{M} &= \bigoplus_{n_1,n_2,\ldots,n_t\in\mathbb{Z}} I_1^{n_1}I_2^{n_2}\cdots I_t^{n_t}MT_1^{n_1}T_2^{n_2}\cdots T_t^{n_t},\\
\mathcal{I}_i &= T_i^{-1}\mathcal{R} \qquad \text{for each } i = 1,2,\ldots,t, \qquad \text{and}\\
\mathcal{J} &= J\mathcal{R}.
\end{aligned}$$

That is because if we contract the result for $\mathcal{R}$ back to R, we get the desired result. Hence without loss of generality we assume $I_i = (x_i)$ is generated by an M-regular element $x_i \in R$ for each $i = 1, 2, \ldots, t$. The same technique is also used in [Sw] and [Sh2].

And it also suffices to prove the theorem for one fixed ideal J. The reason is that for every J in R, we have

$$J \subseteq J' := \bigcap_{\substack{P\in Y\\ J\subseteq P}} P, \quad \text{where} \quad Y = \bigcup_{(n_1,n_2,\ldots,n_t)\in\mathbb{Z}^t} \operatorname{Ass}(M/I_1^{n_1}I_2^{n_2}\cdots I_t^{n_t}M),$$

and, furthermore, there are only finitely many such J' to deal with since the set $Y = \bigcup_{(n_1,n_2,\ldots,n_t)} \operatorname{Ass}(M/I_1^{n_1}I_2^{n_2}\cdots I_t^{n_t}M)$ is finite (cf. [Mc, page 125, Lemma 13.1]).

For each $i = 1, 2, \ldots, t$, let $N_i = x_iM :_M J^\infty \subseteq M$, $k_i' = \operatorname{AR}(J, N_i \subseteq M)$ and k_i'' be such that $J^{k_i''}N_i \subseteq x_iM$. Then $\operatorname{AR}(J, x_iM \subseteq N_i) \le k_i''$.

Let $k' = \max\{k_i' \mid 1 \le i \le t\}$, $k'' = \max\{k_i'' \mid 1 \le i \le t\}$ and $k = k'+k''$. It is easy to see by Remark 3.2 that $\operatorname{AR}(J, x_iM \subseteq M) \le k_i' + k_i'' \le k$ for all $i = 1, \ldots, t$. Since each x_i is regular on M, we have $\operatorname{AR}(J, x_1^{m_1}x_2^{m_2}\cdots x_{i-1}^{m_{i-1}}x_i^{m_i+1}x_{i+1}^{m_{i+1}}\cdots x_t^{m_t}M \subseteq x_1^{m_1}x_2^{m_2}\cdots x_t^{m_t}M) = \operatorname{AR}(J, x_iM \subseteq M) \le k$ because of the R-linear isomorphism $M \cong x_1^{m_1}x_2^{m_2}\cdots x_t^{m_t}M$ induced by multiplication by $x_1^{m_1}x_2^{m_2}\cdots x_t^{m_t}$. Therefore we have $\operatorname{AR}(J, x_1^{n_1}x_2^{n_2}\cdots x_t^{n_t}M \subseteq M) \le k(n_1 + n_2 + \cdots + n_t) = k|\underline{n}|$ by the same Remark 3.2 applied to the filtration

$$x_1^{n_1}x_2^{n_2}\cdots x_t^{n_t}M \subseteq x_1^{n_1-1}x_2^{n_2}\cdots x_t^{n_t}M \subseteq \cdots \subseteq x_t^2M \subseteq x_tM \subseteq M$$

of $x_1^{n_1}x_2^{n_2}\cdots x_t^{n_t}M \subseteq M$ so that each quotient is isomorphic to M/x_iM for some $i = 1, 2, \ldots, t$.

We prove the theorem by induction on $|\underline{n}| = n_1 + n_2 + \cdots + n_t$. If $|\underline{n}| = 0$, the claim is trivially true.

Now suppose $|\underline{n}| \ge 1$. By symmetry we assume $n_1 \ge 1$. Notice, by the induction hypothesis, that

$$(*)\qquad \begin{aligned}
&J^{k|\underline{n}|}M \cap (x_1^{n_1}x_2^{n_2}\cdots x_t^{n_t}M :_M J^\infty)\\
&\subseteq J^{k(|\underline{n}|-1)}M \cap (x_1^{n_1-1}x_2^{n_2}\cdots x_t^{n_t}M :_M J^\infty)\\
&\subseteq x_1^{n_1-1}x_2^{n_2}\cdots x_t^{n_t}M.
\end{aligned}$$

Therefore, using the definition of integers $k,\ k',\ k''$ and the fact that

$$\begin{aligned}
&\operatorname{AR}\left(J,\ (x_1^{n_1}x_2^{n_2}\cdots x_t^{n_t}M :_{x_1^{n_1-1}x_2^{n_2}\cdots x_t^{n_t}M} J^\infty) \subseteq x_1^{n_1-1}x_2^{n_2}\cdots x_t^{n_t}M\right)\\
&\qquad\qquad = \operatorname{AR}(J,\ x_1M :_M J^\infty \subseteq M) \qquad \text{and}\\
&(x_1^{n_1}x_2^{n_2}\cdots x_t^{n_t}M :_{x_1^{n_1-1}x_2^{n_2}\cdots x_t^{n_t}M} J^\infty)/x_1^{n_1}x_2^{n_2}\cdots x_t^{n_t}M\\
&\qquad\qquad \cong (x_1M :_M J^\infty)/x_1M,
\end{aligned}$$

we have

$$\begin{aligned}
J^{k|\underline{n}|}M \cap & (x_1^{n_1}x_2^{n_2}\cdots x_t^{n_t}M :_M J^\infty) \\
=&(x_1^{n_1-1}x_2^{n_2}\cdots x_t^{n_t}M)\cap J^{k|\underline{n}|}M\cap(x_1^{n_1}x_2^{n_2}\cdots x_t^{n_t}M :_M J^\infty) \qquad \text{by } (*) \\
=&(x_1^{n_1-1}x_2^{n_2}\cdots x_t^{n_t}M)\cap J^{k|\underline{n}|}M\cap(x_1^{n_1}x_2^{n_2}\cdots x_t^{n_t}M :_{x_1^{n_1-1}x_2^{n_2}\cdots x_t^{n_t}M} J^\infty) \\
=&\big(J^{k|\underline{n}|}M\cap(x_1^{n_1-1}x_2^{n_2}\cdots x_t^{n_t}M)\big)\cap(x_1^{n_1}x_2^{n_2}\cdots x_t^{n_t}M :_{x_1^{n_1-1}x_2^{n_2}\cdots x_t^{n_t}M} J^\infty) \\
\subseteq&\big(J^{k}(x_1^{n_1-1}x_2^{n_2}\cdots x_t^{n_t}M)\big)\cap(x_1^{n_1}x_2^{n_2}\cdots x_t^{n_t}M :_{x_1^{n_1-1}x_2^{n_2}\cdots x_t^{n_t}M} J^\infty) \\
\subseteq&J^{k''}(x_1^{n_1}x_2^{n_2}\cdots x_t^{n_t}M :_{x_1^{n_1-1}x_2^{n_2}\cdots x_t^{n_t}M} J^\infty) \\
\subseteq&x_1^{n_1}x_2^{n_2}\cdots x_t^{n_t}M.
\end{aligned}$$

□

Corollary 3.4 (Linear Growth; [Sw] and [Sh2]). *Let R be a Noetherian ring, M a finitely generated R-module and $I_1, I_2, \ldots, I_t$ ideals of R. Then there exists a $k\in\mathbb{N}$ such that for any $n_1, n_2, \ldots, n_t \in \mathbb{N}$, there exists a primary decomposition of $I_1^{n_1}I_2^{n_2}\cdots I_t^{n_t}M \subseteq M$,*

$$I_1^{n_1}I_2^{n_2}\cdots I_t^{n_t}M = Q_{\underline{n}_1}\cap Q_{\underline{n}_1}\cap\cdots\cap Q_{\underline{n}_{r_{\underline{n}}}},$$

where the Q_i's are P_i-primary components of the primary decomposition such that $P_i^{k|\underline{n}|}M\subseteq Q_i$ for all $i=1,2,\ldots,\underline{n}_{r_{\underline{n}}}$, where $\underline{n}=(n_1,n_2,\ldots,n_t)$ and $|\underline{n}| = n_1+n_2+\cdots+n_t$.

Proof. Let k be as in Theorem 3.3. By Theorem 1.1 (Compatibility), it suffices to show that for each $\underline{n}\in\mathbb{N}^t$ and each $P\in\operatorname{Ass}(M/I_1^{n_1}I_2^{n_2}\cdots I_t^{n_t}M)$, there is a P-primary component Q of $I_1^{n_1}I_2^{n_2}\cdots I_t^{n_t}M\subset M$ such that $P^{k|\underline{n}|}M\subseteq Q$. So we fix $\underline{n}$ and $P\in\operatorname{Ass}(M/I_1^{n_1}I_2^{n_2}\cdots I_t^{n_t}M)$. Let

$$\begin{aligned}
(P^{k|\underline{n}|}M + I_1^{n_1}I_2^{n_2}\cdots I_t^{n_t}M) &= Q_1\cap Q_2\cap\cdots\cap Q_r \qquad\text{and}\\
(I_1^{n_1}I_2^{n_2}\cdots I_t^{n_t}M :_M P^\infty) &= Q_{r+1}\cap Q_{r+2}\cap\cdots\cap Q_s
\end{aligned}$$

be irredundant and minimal primary decompositions of the corresponding submodules of M, where Q_i is a P_i-primary submodule of M for each $i=1,2,\ldots,s$. As $P\notin\operatorname{Ass}(M/(I_1^{n_1}I_2^{n_2}\cdots I_t^{n_t}M :_M P^\infty))$, we may assume that $P_1=P$. By Theorem 3.3, $(P^{k|\underline{n}|}M + I_1^{n_1}I_2^{n_2}\cdots I_t^{n_t}M)\cap(I_1^{n_1}I_2^{n_2}\cdots I_t^{n_t}M :_M P^\infty) = I_1^{n_1}I_2^{n_2}\cdots I_t^{n_t}M$. Hence

$$I_1^{n_1}I_2^{n_2}\cdots I_t^{n_t}M = Q_1\cap Q_2\cap\cdots\cap Q_r\cap Q_{r+1}\cap Q_{r+2}\cap\cdots\cap Q_s.$$

Although the above intersection may not necessarily be irredundant and minimal, we know that Q_1 is a $P_1=P$-primary component of $I_1^{n_1}I_2^{n_2}\cdots I_t^{n_t}M\subset M$ since $P\in\operatorname{Ass}(M/I_1^{n_1}I_2^{n_2}\cdots I_t^{n_t}M)$ and Q_1 is the only P-primary submodule in the above intersection. Evidently $P^{k|\underline{n}|}M\subseteq Q_1$. □

Actually Theorem 3.3 can be stated in a more general situation: The filtration $\{I_1^{n_1}I_2^{n_2}\cdots I_t^{n_t}M \mid (n_1,n_2,\ldots,n_t)\in\mathbb{Z}^t\}$ may be replaced by a 'multi-graded' filtration $\{M_{(n_1,n_2,\ldots,n_t)} \mid (n_1,n_2,\ldots,n_t)\in\mathbb{Z}^t\}$ of M such that

$$\mathcal{M} = \bigoplus_{(n_1,n_2,\ldots,n_t)\in\mathbb{Z}^t} M_{(n_1,n_2,\ldots,n_t)}T_1^{n_1}T_2^{n_2}\cdots T_t^{n_t}$$

naturally forms a multi-graded Noetherian module over a multi-graded sub-ring $\mathcal{R}$ in $R[T_1,T_2,\ldots,T_t,T_1^{-1},T_2^{-1},\ldots,T_t^{-1}]$ with the usual grading such that T_1^{-1},

$T_2^{-1}, \dots, T_t^{-1}$ are all contained in $\mathcal{R}$ and the $(0,0,\dots,0)$ part of $\mathcal{R}$ is R. We call such a filtration 'Noetherian'. To simplify notation, we use $\underline{n}$ to denote $(n_1, n_2, \dots, n_t)$ and use $|\underline{n}|$ to denote $n_1 + n_2 + \cdots + n_t$. And $\mathbb{N}^t := \{(n_1, n_2, \dots, n_t) \,|\, n_i \geq 0,\ i = 1, 2, \dots, t\}$.

The next theorem and its corollary look apparently more general than Theorem 3.3 and Corollary 3.4, although in essence they are the same.

Theorem 3.5. *Let R be a Noetherian ring, M a finitely generated R-module and $\{M_{(n_1 n_2 \dots n_t)} \,|\, (n_1, n_2, \dots, n_t) \in \mathbb{Z}^t\}$ a Noetherian filtration of M. Then:*

1. *There exists a $k \in \mathbb{N}$ such that for all $\underline{m} \in \mathbb{Z}^t$, for all $\underline{n} \in \mathbb{N}^t$ and for all ideals $J \subset R$, $J^{k|\underline{n}|} M_{\underline{m}} \cap (M_{\underline{m}+\underline{n}} :_{M_{\underline{m}}} J^\infty) \subseteq M_{\underline{m}+\underline{n}}$, i.e. $(J^{k|\underline{n}|} M_{\underline{m}} + M_{\underline{m}+\underline{n}}) \cap (M_{\underline{m}+\underline{n}} :_{M_{\underline{m}}} J^\infty) = M_{\underline{m}+\underline{n}}$.*
2. *The set $\bigcup_{\underline{m} \in \mathbb{Z}^t, \underline{n} \in \mathbb{N}^t} \operatorname{Ass}(M_{\underline{m}}/M_{\underline{m}+\underline{n}})$ is finite.*

Proof. The proof of (1) may be carried out in almost the same way as the proof of Theorem 3.3. But here we choose to use Theorem 3.3 and provide a sketch of the proof: Simply apply Theorem 3.3 to the Noetherian $\mathcal{R}$ module $\mathcal{M}$ and ideals $\mathcal{I}_i = T_i^{-1}\mathcal{R}$ and then restrict the results to each of the homogeneous pieces. Theorem 3.3 gives results for all the ideals of $\mathcal{R}$, but here we are only interested in the ideals $J\mathcal{R}$, the ideals extended from ideals $J \subset R$.

To prove (2), we notice that the set

$$\bigcup_{\underline{n} \in \mathbb{N}^t} \operatorname{Ass}_{\mathcal{R}}(\mathcal{M}/T_1^{-n_1} T_2^{-n_2} \cdots T_t^{-n_t} \mathcal{M})$$

is finite. Then (2) follows by contracting to each of the homogeneous pieces. □

Corollary 3.6. *Let R be a Noetherian ring, M a finitely generated R-module and $\{M_{(n_1 n_2 \dots n_t)} \,|\, (n_1, n_2, \dots, n_t) \in \mathbb{Z}^t\}$ a Noetherian filtration of M. Then there exists a $k \in \mathbb{N}$ such that for any $\underline{m} \in \mathbb{Z}^t$, $\underline{n} \in \mathbb{N}^t$ and $P \in \operatorname{Ass}(M_{\underline{m}}/M_{\underline{m}+\underline{n}})$, there exists a $Q \in \Lambda_P(M_{\underline{m}+\underline{n}} \subseteq M_{\underline{m}})$ such that $P^{k|\underline{n}|} M_{\underline{m}} \subseteq Q$.*

Example 3.7 (Compare with [Sh1])**.** Assume that R is Nagata (e.g. R is excellent) and M is a finitely generated R-module and $I_1, I_2, \dots, I_t$ ideals of R. Then we have a multi-graded filtration $\{\overline{I_1^{n_1} I_2^{n_2} \cdots I_t^{n_t} M} \,|\, \underline{n} \in \mathbb{Z}^t\}$. In order to see if the filtration satisfies the Linear Growth property, we may mod out the nil-radical and hence assume that R is reduced. Then it is straightforward to see that the associated graded module is finite over $\mathcal{R} = R[I_1 T_1, I_2 T_2, \dots, I_t T_t, T_1^{-1}, T_2^{-1}, \dots, T_t^{-1}]$. Hence the filtration satisfies the Linear Growth property. Similarly we can show the Linear Growth property of the filtration $\{\overline{I_1^{n_1}} \cdot \overline{I_2^{n_2}} \cdots \overline{I_t^{n_t}} M \,|\, \underline{n} \in \mathbb{Z}^t\}$ provided R is reduced and Nagata.

In [Sh1] R. Y. Sharp proved the Linear Growth property of the filtration $\{\overline{I^n} \,|\, n \in \mathbb{Z}\}$ of Noetherian ring R without the Nagata assumption. The argument there also works for the filtration $\{\overline{I_1^{n_1} I_2^{n_2} \cdots I_t^{n_t}} \,|\, \underline{n} \in \mathbb{Z}^t\}$ of any Noetherian ring R. That is because the set $\bigcup_{\underline{n} \in \mathbb{Z}^t} \operatorname{Ass}(R/\overline{I_1^{n_1} I_2^{n_2} \cdots I_t^{n_t}})$ is finite (cf. [Ra]) and hence we can localize and then complete. In fact, if we know in advance the set

$$\bigcup_{\underline{n} \in \mathbb{Z}^t} \operatorname{Ass}(M/\overline{I_1^{n_1} I_2^{n_2} \cdots I_t^{n_t} M})$$

is finite for a finitely generated faithful R-module M, we can localize, complete and then contract the result of Example 3.7 for $\hat{M}$ back to M to deduce that the filtration $\{\overline{I_1^{n_1} I_2^{n_2} \cdots I_t^{n_t}} M \mid \underline{n} \in \mathbb{Z}^t\}$ satisfies the Linear Growth property. We need M to be faithful so that the process of contraction works.

Example 3.8. Assume R is Nagata and has characteristic p, where p is a prime number and M is a finitely generated R-module. Then for any ideal I in R, tight closure of I, denoted by I^*, is defined [HH]. It is shown that $\sqrt{0} \subseteq I^* \subseteq \overline{I}$ for any ideal I in R [HH]. By the same argument as in Example 3.7 we can deduce that the filtration $\{(I_1^{n_1} I_2^{n_2} \cdots I_t^{n_t})^* M \mid \underline{n} \in \mathbb{Z}^t\}$ is Noetherian and hence has the Linear Growth property. If, furthermore, R is reduced, then the filtration $\{I_1^{n_1 *} I_2^{n_2 *} \cdots I_t^{n_t *} M \mid \underline{n} \in \mathbb{Z}^t\}$ satisfies the Linear Growth property.

In [Ra] it is shown that $\operatorname{Ass}(R/\overline{I^n})$ is non-decreasing and eventually stabilizes for any ideal I in a Noetherian ring R. For any finitely generated R-module M, a result of [Br] says that $\operatorname{Ass}(M/I^n M)$ also stabilizes for large n. If R is Nagata and of characteristic $p > 0$, then it follows from Example 3.8 and Theorem 3.5 that the set $\bigcup_{\underline{n} \in \mathbb{Z}^t} \operatorname{Ass}(M/(I_1^{n_1} I_2^{n_2} \cdots I_t^{n_t})^* M)$ is finite. In the case of $t = 1$, we would like to study the stability of $\operatorname{Ass}(M/(I^{n*} M))$. Since $\bigoplus_{n \in \mathbb{Z}} I^{n*} M T^n$ is finite over $R[IT, T^{-1}]$ (see Example 3.8), we know the filtration $\{I^{n*} M \mid n \in \mathbb{N}\}$ of M is eventually stable, i.e. $I^{n+1*} M = I I^{n*} M$ for all large n. Hence the argument in [Br] can be applied to show that $\operatorname{Ass}(M/I^{n*} M)$ stabilizes for large n.

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