PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 130, Number 6, Pages 1743–1753
S 0002-9939(01)06409-7
Article electronically published on November 9, 2001

EQUIVALENCE OF DOMAINS WITH ISOMORPHIC SEMIGROUPS OF ENDOMORPHISMS

SERGEI MERENKOV

(Communicated by Steven R. Bell)

ABSTRACT. For two bounded domains $\Omega_1,\ \Omega_2$ in $\mathbb{C}$ whose semigroups of analytic endomorphisms $E(\Omega_1),\ E(\Omega_2)$ are isomorphic with an isomorphism $\varphi :\ E(\Omega_1) \to E(\Omega_2)$, Eremenko proved in 1993 that there exists a conformal or anticonformal map $\psi :\ \Omega_1 \to \Omega_2$ such that $\varphi f = \psi \circ f \circ \psi^{-1}$, for all $f \in E(\Omega_1)$.

In the present paper we prove an analogue of this result for the case of bounded domains in $\mathbb{C}^n$.

1. INTRODUCTION

A classical theorem of L. Bers says that every $\mathbb{C}$-algebra isomorphism $H(A) \to H(B)$ of algebras of holomorphic functions in domains A and B in the complex plane has the form $f \mapsto f \circ \theta$, where $\theta :\ B \to A$ is a conformal isomorphism, or $f \mapsto \overline{f} \circ \theta$ with anticonformal θ. In particular, the algebras $H(A)$ and $H(B)$ are isomorphic if and only if the domains A and B are conformally equivalent. H. Iss'sa [9] obtained a similar theorem for fields of meromorphic functions on Stein spaces. A good reference for these results is [5].

Likewise, the question of recovering a topological space from the algebraic structure of its semigroup of continuous self-maps has been extensively studied [12].

In 1990, L. Rubel asked whether similar results hold for semigroups (under composition) $E(D)$ of holomorphic endomorphisms of a domain D. A. Hinkkanen constructed examples [6] which show that even non-homeomorphic domains in $\mathbb{C}$ can have isomorphic semigroups of endomorphisms. The reason is that the semigroup of endomorphisms of a domain can be too small to characterize this domain.

However, in 1993, A. Eremenko [4] proved that for two Riemann surfaces D_1, D_2, which admit bounded nonconstant holomorphic functions, and such that the semigroups of analytic endomorphisms $E(D_1)$ and $E(D_2)$ are isomorphic with an isomorphism $\varphi :\ E(D_1) \to E(D_2)$, there exists a conformal or anticonformal map $\psi :\ D_1 \to D_2$ such that $\varphi f = \psi \circ f \circ \psi^{-1}$, for all $f \in E(D_1)$. In the present paper we investigate the analogue of this result for the case of bounded domains in $\mathbb{C}^n$. The theorems of Bers and Iss'sa, mentioned above, do not extend to arbitrary domains in $\mathbb{C}^n$.

Received by the editors December 12, 2000.
2000 *Mathematics Subject Classification.* Primary 32A10, 08A35.
This research was supported by NSF, DMS 0072197.

For a bounded domain Ω in $\mathbb{C}^n$ we denote by $E(\Omega)$ the semigroup of analytic endomorphisms of Ω under composition. In what follows, we say that a map is *(anti-) biholomorphic*, if it is biholomorphic or antibiholomorphic. We prove the following theorem.

Theorem 1. *Let Ω_1, Ω_2 be bounded domains in $\mathbb{C}^n$, $\mathbb{C}^m$ respectively, and suppose that there exists $\varphi:\ E(\Omega_1) \to E(\Omega_2)$, an isomorphism of semigroups. Then $n = m$ and there exists an (anti-) biholomorphic map $\psi:\ \Omega_1 \to \Omega_2$ such that*

$$\varphi f = \psi \circ f \circ \psi^{-1}, \quad \text{for all } f \in E(\Omega_1). \tag{1}$$

The existence of a homeomorphism ψ satisfying (1) follows from simple general considerations (Section 2). The hard part is proving that ψ is (anti-) biholomorphic. In dimension 1 this is done by linearization of holomorphic germs of $f \in E(\Omega)$ near an attracting fixed point. In several dimensions such linearization theory exists ([1], pp. 192–194), but it is too complicated (many germs with an attracting fixed point are non-linearizable, even formally). In Sections 3, 4 we show how to localize the problem. In Sections 5, 6 we describe, using only the semigroup structure, a large enough class of linearizable germs. Linearization of these germs permits us to reduce the problem to a matrix functional equation, which is solved in Section 7. In Section 8 we complete the proof that ψ is (anti-) biholomorphic.

Theorem 1 can be slightly generalized, namely one may assume that φ is an epimorphism. In Section 9 we prove the following theorem.

Theorem 2. *If $\varphi:\ E(\Omega_1) \to E(\Omega_2)$ is an epimorphism between semigroups, where Ω_1, Ω_2 are bounded domains in $\mathbb{C}^n$, $\mathbb{C}^m$ respectively, then φ is an isomorphism.*

The author is grateful to A. Eremenko for his guidance and numerous suggestions concerning the paper. He also thanks L. Avramov, S. Bell and A. Gabrielov for valuable discussions and their interest in this work.

2. Topology

For a bounded domain Ω in $\mathbb{C}^n$ we denote by $C(\Omega)$ the subsemigroup of $E(\Omega)$ consisting of constant maps. An endomorphism c_z is constant if it sends Ω to a point $z \in \Omega$. The subset $C(\Omega) \subset E(\Omega)$ can be described using only the semigroup structure as follows:

$$c \in C(\Omega) \text{ iff } \ \forall (f \in E(\Omega)), \quad (c \circ f = c). \tag{2}$$

It is clear that we have a bijection between constant endomorphisms of Ω and points of this domain as a set: to each z corresponds a unique $c_z \in C(\Omega)$ and vice versa, so we can identify the two. Under this identification, a subset of Ω corresponds to a subsemigroup of $C(\Omega)$.

Having defined points of a domain in terms of its semigroup structure of analytic endomorphisms, we can construct a map ψ between Ω_1 and Ω_2 as follows:

$$\psi(z) = w \ \text{ iff } \ \varphi c_z = c_w. \tag{3}$$

So defined, ψ satisfies (1). Indeed, let $f \in E(\Omega_1)$, $f(z) = \zeta$. This is equivalent to

$$f \circ c_z = c_\zeta. \tag{4}$$

Applying φ to both sides of (4), we have

$$\varphi f \circ c_{\psi(z)} = c_{\psi(\zeta)}. \tag{5}$$

But (5) is equivalent to $\varphi f(\psi(z)) = \psi(\zeta) = \psi(f(z))$, which is (1).

We describe the topology of a domain Ω using its injective endomorphisms. A map $f \in E(\Omega)$ is injective if and only if

$$\forall (c' \in C(\Omega))\ \forall (c'' \in C(\Omega)), \quad ((f \circ c' = f \circ c'') \Rightarrow (c' = c'')).$$

We denote the class of injective endomorphisms of Ω by $E_i(\Omega)$. For every $f \in E_i(\Omega)$, $f_i(\Omega)$ is open [2]. The family $\{f(\Omega),\ f \in E_i(\Omega)\}$ of subsets of Ω forms a base of topology, because every $z \in \Omega$ has a neighborhood $f(\Omega)$, where $f(\zeta) = z + \lambda(\zeta - z)$, f belongs to $E_i(\Omega)$ for every λ such that $|\lambda|$ is small.

To summarize, we described subsets of Ω and the topology on it using only the semigroup structure of $E(\Omega)$. Since this is so, the semigroup structure also defines the notions of an open set, closed set, compact set, closure of a set.

Now we can easily prove continuity of the map ψ constructed above. Indeed, let $g(\Omega_2)$, $g \in E_i(\Omega_2)$, be a set from the base of topology of Ω_2. We take $f = \varphi^{-1} g$. Then $f \in E_i(\Omega_1)$ and $\psi^{-1}(g(\Omega_2)) = f(\Omega_1)$, which proves that ψ is continuous. Since φ is an isomorphism, the same argument works to prove that ψ^{-1} is also continuous, and thus ψ is a homeomorphism.

Therefore the domains Ω_1, Ω_2 are homeomorphic, and hence [8] they have the same dimension, i. e. $n = m$.

3. Localization

We need the following lemma.

Lemma 1. *Suppose H is a semigroup with identity, and f an element of H with the following two properties:*

(i) $hf = fh$, for every h in H;

(ii) $h_1 f = h_2 f$ implies $h_1 = h_2$, for every h_1 and h_2 in H.

Then there exist a semigroup S_f and a monomorphism $i : H \to S_f$ such that $i(f)$ is invertible in S_f and commutes with all elements of S_f. Moreover, the semigroup S_f satisfies the following universal property: for every semigroup S_1 with a monomorphism $i_1 : H \to S_1$ such that $i_1(f)$ is invertible in S_1 and commutes with all elements of S_1, there exists a unique monomorphism $\hat{i}_1 : S_f \to S_1$ such that $i_1 = \hat{i}_1 \circ i$.

Remark 1. Uniqueness of $\hat{i}_1$ implies that the semigroup S_f with the universal property is unique up to an isomorphism.

Proof. We construct S_f as follows. First we consider formal expressions of the form hf^k, where $h \in H$ and k is an integer (it may be positive, negative or zero). Then we define a multiplication on this set: $h_1 f^{k_1} * h_2 f^{k_2} = h_1 h_2 f^{k_1+k_2}$. Next we consider a relation on the set of formal expressions: $h_1 f^{k_1} \sim h_2 f^{k_2}$ if $k_1 \le k_2$ and $h_1 = h_2 f^{k_2 - k_1}$ in H, or $k_2 \le k_1$ and $h_2 = h_1 f^{k_1 - k_2}$ in H. It is easy to verify that this is an equivalence relation and it is compatible with the operation $*$; that is, $x \sim y$, $u \sim v$ implies $x * u \sim y * v$.

Lastly, let S_f be the set of equivalence classes with the binary operation induced by $*$. For S_f to be a semigroup, we need to show that the binary operation $*$ is associative. Let $h_1 f^{k_1} \sim h_1' f^{k_1'}$, $h_2 f^{k_2} \sim h_2' f^{k_2'}$ and $h_3 f^{k_3} \sim h_3' f^{k_3'}$. We need to show that $(h_1 f^{k_1} * h_2 f^{k_2}) * h_3 f^{k_3} \sim h_1' f^{k_1'} * (h_2' f^{k_2'} * h_3' f^{k_3'})$. By the definition of the operation $*$, the last equivalence is the same as $h_1 h_2 h_3 f^{k_1+k_2+k_3} \sim h_1' h_2' h_3' f^{k_1'+k_2'+k_3'}$. Assuming that $k_1 + k_2 + k_3 \le k_1' + k_2' + k_3'$, we have essentially one possibility to consider (the others are either similar or trivial): $k_1 \le k_1'$, $k_2 \le k_2'$, $k_3' \le k_3$. In this case $h_1h_2h_3f^{k_3-k_3'} = h_1'h_2'h_3'f^{k_1'-k_1+k_2'-k_2}$. Now we can use the cancellation property (ii) to get the desired equivalence.

The semigroup H is embedded into S_f via $i:\ h \mapsto [hf^0]$. The element $i(f) = [\mathrm{id}f]$, where id is the identity in H, is invertible in S_f with the inverse $[\mathrm{id}f^{-1}]$. Clearly, $[\mathrm{id}f]$ commutes with all elements of S_f.

Now, suppose that S_1, $i_1:\ H \to S_1$ is a semigroup and a monomorphism, such that $i_1(f)$ is invertible in S_1 and commutes with all elements of S_1. Then we define

$$\hat{i}_1([hf^k]) = i_1(h)(i_1(f))^k.$$

This definition does not depend on a representative of $[hf^k]$. Indeed, suppose $h_1f^{k_1} \sim h_2f^{k_2}$ and assume $k_1 \le k_2$. Then $h_1 = h_2f^{k_2-k_1}$, and thus $i_1(h_1) = i_1(h_2)i_1(f)^{k_2-k_1}$. Hence $i_1(h_1)i_1(f)^{k_1} = i_1(h_2)i_1(f)^{k_2}$.

So defined, $\hat{i}_1$ is a homomorphism:

$$\begin{aligned}\hat{i}_1([h_1f^{k_1}][h_2f^{k_2}]) &= \hat{i}_1([h_1h_2f^{k_1+k_2}])\\ &= i_1(h_1h_2)i_1(f)^{k_1+k_2} = i_1(h_1)i_1(h_2)i_1(f)^{k_1}i_1(f)^{k_2}\\ &= i_1(h_1)i_1(f)^{k_1}i_1(h_2)i_1(f)^{k_2} = \hat{i}_1([h_1f^{k_1}])\hat{i}_1([h_2f^{k_2}]).\end{aligned}$$

The relation $\hat{i}_1 \circ i = i_1$ holds, since $\hat{i}_1([hf^0]) = i_1(h)$ for all $h \in H$.

Uniqueness of $\hat{i}_1$ is clear. Lemma 1 is proved. □

4. Extension of φ

Following [4], we say that for a bounded domain Ω an element $f \in E(\Omega)$ is *good at* $z \in \Omega$, denoted by $f \in G_z(\Omega)$, if

1. z is a unique fixed point of f;
2. $f(\Omega)$ has compact closure in Ω;
3. f is injective in Ω.

Property 3 of a good element was already stated in terms of the semigroup structure of Ω. Since the topology on Ω was described using only the semigroup structure, Property 2 can also be stated in these terms. Property 1 can be expressed in terms of the semigroup structure as

$$(f \circ c_z = c_z) \wedge ((f \circ c_\zeta = c_\zeta) \Rightarrow (c_\zeta = c_z)).$$

Since f is an endomorphism of a domain, all eigenvalues λ of its linear part at z satisfy $|\lambda| \le 1$ [10]. Moreover, $|\lambda| < 1$ because the closure of $f(\Omega)$ is a compact set in Ω. The injectivity of f implies [2] that it is biholomorphic onto $f(\Omega)$ and the Jacobian determinant of f does not vanish at any point of Ω.

It is clear that for every $z \in \Omega$ a good element f at z exists. For example, we can take $f(\zeta) = z + \lambda(\zeta - z)$ with sufficiently small $|\lambda|$.

Consider a good element $f \in G_z(\Omega)$ and its commutant $H_f(\Omega)$ in $E(\Omega)$:

$$H_f(\Omega) = \{h \in E(\Omega):\quad hf = fh\}.$$

Clearly $H_f(\Omega)$ is a subsemigroup of $E(\Omega)$. The element f, being good (hence injective), satisfies the cancellation property (ii) of Lemma 1 in $H_f(\Omega)$. Thus, by Lemma 1, we have the extension S_f of $H_f(\Omega)$ in which f is invertible and commutes with all elements of S_f. In the case of analytic endomorphisms we can embed $H_f(\Omega)$ into the subsemigroup of A_z, the semigroup of germs of analytic mappings at z under composition, consisting of elements that commute with the germ of f

and containing the germ of f^{-1}. We use the universal property of Lemma 1 to conclude that S_f is isomorphic to a subsemigroup of A_z. We identify S_f with this semigroup, i. e. we consider elements of S_f as germs of analytic mappings at z.

In proving that ψ is (anti-) biholomorphic we need to show that it is so in a neighborhood of every point of Ω_1. Since an (anti-) biholomorphic type of a domain is preserved by translations in $\mathbb{C}^n$, it is enough to show that ψ is (anti-) biholomorphic in a neighborhood of $0 \in \mathbb{C}^n$, assuming that Ω_1 and Ω_2 contain 0 and $\psi(0) = 0$.

Let $\varphi: \ E(\Omega_1) \to E(\Omega_2)$ be an isomorphism of the semigroups, f a good element, $f \in G_0(\Omega_1)$, and $H_f(\Omega_1)$ the commutant of f. Then clearly $H_g(\Omega_2) = \varphi(H_f(\Omega_1))$ is the commutant of $g = \varphi f$. By Lemma 1, we have the extensions $S_f, \ S_g$ of $H_f(\Omega_1)$ and $H_g(\Omega_2)$ respectively, and by the universal property of this lemma the isomorphism φ extends to an isomorphism

$$\Phi: \quad S_f \to S_g.$$

5. System of projections and linearization

Let Ω be a bounded domain in $\mathbb{C}^n$. We say that a good element $f \in G_0(\Omega)$ is *very good at* 0, and write $f \in VG_0(\Omega)$, if the corresponding semigroup $S_f \subset A_0$ constructed in Section 4 contains a system of elements, which we call a system of projections, $\{p_i\}_{i=1}^n$ with the following properties:

(a) $\forall \ (i = 1, \dots, n), \quad (p_i \neq 0)$;
(b) $\forall \ (i = 1, \dots, n), \quad (p_i^2 = p_i)$;
(c) $\forall \ (i, \ j = 1, \dots, n, \ i \neq j), \quad (p_i p_j = 0)$.

There does exist a very good element, since we can take f to be a homothetic transformation at 0 with sufficiently small coefficient, p_i a projection on the i'th coordinate of the standard coordinate system. Clearly, $p_i f = f p_i$ and there exists k such that $p_i f^k \in E(\Omega)$, and hence $p_i \in S_f$. From now on, we fix a very good element $f \in VG_0(\Omega)$, associated semigroups $H_f(\Omega), \ S_f$ and a system of projections $\{p_i\}$.

We introduce another subsemigroup of $E(\Omega)$:

$$P_f(\Omega) = \{h \in G_0(\Omega) \cap H_f(\Omega), \quad hp_i = p_i h, \quad i = 1, \dots, n\},$$

where the commutativity relations are in $S_f \subset A_0$. Notice that $P_f(\Omega) \neq \emptyset$ since f belongs to it.

Lemma 2. *For every $h \in P_f(\Omega)$ there exists a biholomorphic germ θ_h at $0 \in \mathbb{C}^n$ such that $\theta_h h = \Lambda \theta_h$, where $\Lambda = \operatorname{diag}(\lambda_1, \dots, \lambda_n)$ is an invertible diagonal matrix which is similar to $dh(0)$ in $GL(n, \mathbb{C})$.*

Proof. The relations $p_i \neq 0, \ p_i^2 = p_i, \ p_i p_j = 0, \ i \neq j$, imply that for $P_i = dp_i(0)$, the linear part of p_i at 0, we have $P_i \neq 0, \ P_i^2 = P_i, \ P_i P_j = 0, \ i \neq j$. Since the matrices P_i commute, there exists [7] a matrix $A \in GL(n, \mathbb{C})$ such that $P_i' = AP_iA^{-1} = \Delta_i = \operatorname{diag}(0, \dots, 1, \dots, 0)$, where the only non-zero entry appears in the i'th place.

Since $p_i^2 = p_i, \ i = 1, \dots, n$, we can use the argument given in [10] to linearize p_i, i.e. there exists a biholomorphic germ ξ_i at 0 such that $\xi_i p_i = P_i \xi_i, \ d\xi_i(0) = \mathrm{id}, \ i = 1, \dots, n$. The map ξ_i is constructed in [10] as follows:

$$\xi_i = \mathrm{id} + (2P_i - \mathrm{id})(p_i - P_i), \quad i = 1, \dots, n.$$

If we take $\xi_i' = A\xi_i$, we have $\xi_i' p_i = P_i' \xi_i'$. For simplicity of notations, we assume that ξ_i itself conjugates p_i to a diagonal matrix, that is, $P_i = P_i'$ (in this case P_i is not necessarily $dp_i(0)$, but rather $Adp_i(0)A^{-1}$; $d\xi_i(0) = A$). For every $i = 1, \ldots, n$ we have $h_i P_i = P_i h_i$, where $h_i = \xi_i h \xi_i^{-1}$. Let $H_i = dh_i(0)$. Then $H_i P_i = P_i H_i$, and hence in the i'th row and the i'th column the matrix H_i has only one non-zero entry, λ_i, which is located at their intersection. Thus λ_i has to be an eigenvalue of H_i, and hence of the linear part of h. In particular, $0 < |\lambda_i| < 1$.

Let $I_i : \mathbb{C} \to \mathbb{C}^n$ be the embedding $z \mapsto (0, \ldots, z, \ldots, 0)$, where the only non-zero entry is z, which is in the i'th place; and $\pi_i : \mathbb{C}^n \to \mathbb{C}$, a projection $(z_1, \ldots, z_n) \mapsto z_i$, corresponding to the i'th axis. For every $i = 1, \ldots, n$, the map $\pi_i h_i I_i$ sends a neighborhood of 0 in $\mathbb{C}$ into $\mathbb{C}$, and its derivative at 0, λ_i, is an eigenvalue of h. Hence ([3], p. 31) $\pi_i h_i I_i$ is linearized by the unique solution $\eta_{h,i}$ of the Schröder equation

$$\eta(\pi_i h_i I_i) = \lambda_i \eta, \quad \eta(0) = 0, \ \eta'(0) = 1. \tag{6}$$

Since $P_i I_i = I_i$, $\pi_i P_i I_i = \mathrm{id}_{\mathbb{C}}$, we can rewrite (6) as

$$\eta_{h,i}\pi_i h_i P_i I_i = \lambda_i \eta_{h,i}\pi_i P_i I_i, \quad \text{or} \quad \eta_{h,i}\pi_i h_i P_i = \lambda_i \eta_{h,i}\pi_i P_i.$$

But $h_i P_i = P_i h_i$, and so

$$\eta_{h,i}\pi_i P_i h_i = \lambda_i \eta_{h,i}\pi_i P_i. \tag{7}$$

The equation (7), in its turn, is equivalent to

$$\eta_{h,i}\pi_i \xi_i p_i h = \lambda_i \eta_{h,i}\pi_i \xi_i p_i. \tag{8}$$

We denote

$$\theta_{h,i} = \eta_{h,i}\pi_i \xi_i p_i, \tag{9}$$

a map from a neighborhood of $0 \in \mathbb{C}^n$ into $\mathbb{C}$. Then (8) becomes $\theta_{h,i} h = \lambda_i \theta_{h,i}$. Now we define

$$\theta_h = (\theta_{h,1}, \ldots, \theta_{h,n}),$$

which is a germ of an analytic map at 0. This germ linearizes h:

$$\theta_h h = (\theta_{h,1} h, \ldots, \theta_{h,n} h) = (\lambda_1 \theta_{h,1}, \ldots, \lambda_n \theta_{h,n}) = \Lambda \theta_h,$$

where $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$ is an invertible diagonal matrix, which has eigenvalues of $dh(0)$ on its diagonal.

The germ θ_h is biholomorphic. Indeed,

$$\theta_{h,i} = \eta_{h,i}\pi_i \xi_i p_i = \eta_{h,i}\pi_i P_i \xi_i, \quad i = 1, \ldots, n.$$

Using the chain rule, we see that $d\theta_h(0) = A$, where A is an invertible diagonal matrix that diagonalizes P_i. We conclude that θ_h is biholomorphic. Lemma 2 is proved. □

6. Simultaneous linearization

Using Lemma 2, we can linearize elements of $P_f(\Omega)$. Namely, for every $h \in P_f(\Omega)$ there exists θ_h (constructed in Section 5) such that $\theta_h h = \Lambda_h \theta_h$, where Λ_h is an invertible diagonal matrix. In particular, we can linearize f:

$$\theta_f f = \Lambda_f \theta_f,$$

where the germ θ_f is biholomorphic at 0, and Λ_f is an invertible diagonal matrix.

Lemma 3. *For every $h \in P_f(\Omega)$ we have $\theta_h = \theta_f$.*

Proof. Let us consider the germ

$$\theta = \Lambda_f^{-1}\theta_h f, \tag{10}$$

which is clearly biholomorphic. We have

$$\theta h = \Lambda_f^{-1}\theta_h f h = \Lambda_f^{-1}\theta_h h f = \Lambda_f^{-1}\Lambda_h\theta_h f = \Lambda_h\Lambda_f^{-1}\theta_h f = \Lambda_h\theta.$$

Using (10), we write the equation $\theta h = \Lambda_h\theta$ in the coordinate form:

$$(1/\lambda_{f,i})\theta_{h,i} f h = (\lambda_{h,i}/\lambda_{f,i})\theta_{h,i} f, \quad i = 1, \dots, n.$$

By (9) and the definition of ξ_i,

$$(1/\lambda_{f,i})\eta_{h,i}\pi_i P_i f_i h_i = (\lambda_{h,i}/\lambda_{f,i})\eta_{h,i}\pi_i P_i f_i, \quad i = 1, \dots, n,$$

where $f_i = \xi_i f \xi_i^{-1}$. Using the commutativity relations $f_i P_i = P_i f_i,\ h_i P_i = P_i h_i$, which hold since $\{p_i\} \subset S_f,\ h \in P_f(\Omega)$, we get

$$(1/\lambda_{f,i})\eta_{h,i}\pi_i f_i h_i P_i = (\lambda_{h,i}/\lambda_{f,i})\eta_{h,i}\pi_i f_i P_i, \quad \text{or}$$
$$(1/\lambda_{f,i})\eta_{h,i}\pi_i f_i h_i I_i = (\lambda_{h,i}/\lambda_{f,i})\eta_{h,i}\pi_i f_i I_i, \quad i = 1, \dots, n.$$

This is the same as

$$((1/\lambda_{f,i})\eta_{h,i}\pi_i f_i I_i)(\pi_i h_i I_i) = \lambda_{h,i}((1/\lambda_{f,i})\eta_{h,i}\pi_i f_i I_i), \quad i = 1, \dots, n,$$

since h_i locally preserves the i'th coordinate axis ($h_i P_i = P_i h_i$). It is easily seen that

$$((1/\lambda_{f,i})\eta_{h,i}\pi_i f_i I_i)(0) = 0,$$
$$((1/\lambda_{f,i})\eta_{h,i}\pi_i f_i I_i)'(0) = 1.$$

A normalized solution to a Schröder equation is unique, though; thus we have

$$\eta_{h,i}(\pi_i f_i I_i) = \lambda_{f,i}\eta_{h,i}, \quad \eta_{h,i}(0) = 0,\ \eta'_{h,i}(0) = 1.$$

Using the uniqueness argument again, we obtain $\eta_{h,i} = \eta_{f,i}$, and hence $\theta_h = \theta_f$. The lemma is proved. □

According to Lemma 3, the single biholomorphic germ θ_f conjugates the subsemigroup $P_f(\Omega)$ to some subsemigroup D_f of invertible diagonal matrices in D_n, the set of all $n \times n$ diagonal matrices with entries in $\mathbb{C}$. We show that D_f contains all invertible diagonal matrices with sufficiently small entries. To do this, first we extend θ_f to an analytic map on the whole domain Ω using the formula

$$\theta_f = \Lambda_f^{-l}\theta_f f^l,$$

where l is chosen so large that $\mathrm{Cl}\{f^l(\Omega)\}$ is contained in a neighborhood of 0 where θ_f is originally defined and biholomorphic; the symbol Cl denotes closure. From the procedure of extending θ_f to Ω we see that it is one-to-one and bounded in the whole domain.

Now, let $\Lambda = \mathrm{diag}(\lambda_1, \dots, \lambda_n)$ be a matrix such that $\mathrm{Cl}\{\Lambda\theta_f(\Omega)\} \subset W$, where W is a neighborhood of $0 \in \mathbb{C}^n$ for which $\mathrm{Cl}\{\theta_f^{-1}W\} \subset \Omega$. Such a matrix Λ exists since θ_f is bounded in Ω. Consider $h = \theta_f^{-1}\Lambda\theta_f$, which belongs to $G_0(\Omega)$. The map h commutes with f and all p_i's. Indeed, using the formula $\theta_f f \theta_f^{-1} = \Lambda_f$, we conclude that $hf = fh$ is equivalent to $\Lambda\Lambda_f = \Lambda_f\Lambda$, which is a true relation since both matrices Λ and Λ_f are diagonal. The relations $hp_i = p_i h,\ i = 1, \dots, n$, are verified similarly, using the formula $\theta_f p_i \theta_f^{-1} = P_i$, which follows from the definition of θ_f.

7. Solving a matrix equation

We proved that for an element $f \in VG_0(\Omega)$ there exists a biholomorphic germ θ_f conjugating the semigroup $P_f(\Omega)$ to a subsemigroup $D_f \subset D_n$, which contains all invertible diagonal matrices with sufficiently small entries.

Let $f \in VG_0(\Omega_1)$ and $g = \varphi f$. Then $g \in VG_0(\Omega_2)$, and there is an isomorphism

$$\Phi: \quad S_f \to S_g.$$

For the mappings f and g we have

$$\theta_f f = \Lambda_f \theta_f, \quad \theta_g g = M_g \theta_g,$$

where $\Lambda_f,\ M_g$ are invertible diagonal matrices.

Let us consider the germ $L = \theta_g \psi \theta_f^{-1}$. This germ conjugates the semigroups $D_f,\ D_g$:

$$\begin{aligned} L\Lambda L^{-1} &= \theta_g \psi \theta_f^{-1} \Lambda \theta_f \psi^{-1} \theta_g^{-1} \\ &= \theta_g \psi h \psi^{-1} \theta_g^{-1} = \theta_g j \theta_g^{-1} = M, \end{aligned}$$

where $h \in P_f,\ \theta_f h = \Lambda \theta_f;\ j = \varphi h,\ \theta_g j = M\theta_g$.

Define $R(\Lambda) = L\Lambda L^{-1}$. Then $R:\ D_f \to D_g$,

$$R(\Lambda_1\Lambda_2) = R(\Lambda_1)R(\Lambda_2), \quad \Lambda_1,\ \Lambda_2 \in D_f.$$

In what follows, we will identify D_n with the multiplicative semigroup $\mathbb{C}^n$ ($D_n \cong \mathbb{C}^n$) in the obvious way and consider a topology on D_n induced by the standard topology on $\mathbb{C}^n$.

We are going to extend R to an isomorphism of D_n. First, we denote by $\overline{D}_f,\ \overline{D}_g$ the closures of $D_f,\ D_g$ in D_n, and for $\Lambda \in \overline{D}_f$ we set

$$R(\Lambda) = \lim R(\Lambda_k), \quad \Lambda_k \to \Lambda,\ \Lambda_k \in D_f.$$

This limit exists and does not depend on the sequence $\{\Lambda_k\}$, which follows from the fact that $\psi^{\pm 1}, \theta_f^{\pm 1}, \theta_g^{\pm 1}$ are continuous. The map R is an isomorphism of topological semigroups $\overline{D}_f$ and $\overline{D}_g$ (the inverse of R has a similar representation).

Next, we extend the map R to D_n as

$$R(\Gamma) = R(\Gamma\Lambda)R(\Lambda)^{-1}, \quad \Gamma \in D_n,$$

where $\Lambda \in D_f$ is chosen so that $\Gamma\Lambda \in \overline{D}_f$. This definition does not depend on the choice of Λ. Indeed, since all matrices in question are diagonal (hence commute), the relation $R(\Gamma\Lambda_1)R(\Lambda_1)^{-1} = R(\Gamma\Lambda_2)R(\Lambda_2)^{-1}$ is equivalent to $R(\Gamma\Lambda_1)R(\Lambda_2) = R(\Gamma\Lambda_2)R(\Lambda_1)$, which holds.

The extended map R is clearly an isomorphism of D_n onto itself. Thus we have

$$R(\Lambda'\Lambda'') = R(\Lambda')R(\Lambda''), \quad \Lambda',\ \Lambda'' \in D_n. \tag{11}$$

Injectivity of R and (11) imply that $R(\Delta_i) = \Delta_j$ for all i, where $j = j(i)$ depends on i; $j(i)$ is a permutation on $\{1, \ldots, n\}$ (we recall that $\Delta_i = \operatorname{diag}(0, \ldots, 1, \ldots, 0)$). This is because $\{\Delta_i\}_{i=1}^n$ is the only system in D_n with the following relations: $\Delta_i \neq 0,\ \Delta_i^2 = \Delta_i,\ \Delta_i\Delta_j = 0,\ i \neq j$.

Since all matrices Λ and their images $R(\Lambda)$ are diagonal, we can consider the matrix equation (11) as n scalar equations:

$$r_j(\lambda_1'\lambda_1'', \ldots, \lambda_n'\lambda_n'') = r_j(\lambda_1', \ldots, \lambda_n')r_j(\lambda_1'', \ldots, \lambda_n''), \quad j = 1, \ldots, n, \tag{12}$$

where r_j are components of R. If we rewrite the equation $R(\Delta_i\Lambda) = \Delta_j R(\Lambda)$ in the coordinate form, we see that $r_j(\lambda_1, \ldots, \lambda_n) = r_j(0, \ldots, \lambda_i, \ldots, 0) = q_j(\lambda_i)$; that is, each r_j depends on only one of the λ_i's. For each j the corresponding equation in (12) in terms of the q_j's becomes

$$q_j(\lambda_i'\lambda_i'') = q_j(\lambda_i')q_j(\lambda_i'').$$

This equation has ([4], p. 130) either the constant solution $q_j(\lambda_i) = 1$, or

$$q_j(\lambda_i) = \lambda_i^{\alpha_{ij}}\overline{\lambda}_i^{\beta_{ij}}, \quad \alpha_{ij},\ \beta_{ij} \in \mathbb{C}, \quad \alpha_{ij} - \beta_{ij} = \pm 1.$$

Going back to the function L, we have

$$\begin{aligned} L\mathrm{diag}(\lambda_1, \ldots, \lambda_n) = \mathrm{diag}(\lambda_{i(1)}^{\alpha_1}\overline{\lambda}_{i(1)}^{\beta_1}, \ldots, \lambda_{i(n)}^{\alpha_n}\overline{\lambda}_{i(n)}^{\beta_n})L,& \\ \alpha_i - \beta_i = \pm 1, \quad i = 1, \ldots, n,& \end{aligned} \tag{13}$$

where $i(j)$ is the inverse permutation to $j(i)$.

Let us choose and fix $(\mu_1, \ldots, \mu_n)$ such that $(1/\mu_1, \ldots, 1/\mu_n)$ belongs to a neighborhood W_0 of $0 \in \mathbb{C}^n$ where L is defined, and let W_1 be a neighborhood of $0 \in \mathbb{C}^n$ such that $(\mu_1 z_1, \ldots, \mu_n z_n) \in W_0$, whenever $(z_1, \ldots, z_n) \in W_1$. Then from (13) we have

$$\begin{aligned} L(z_1, \ldots, z_n) &= L\mathrm{diag}(\mu_1 z_1, \ldots, \mu_n z_n)(1/\mu_1, \ldots, 1/\mu_n) \\ &= \mathrm{diag}((\mu_{i(1)}z_{i(1)})^{\alpha_1}(\overline{\mu_{i(1)}z_{i(1)}})^{\beta_1}, \ldots, (\mu_{i(n)}z_{i(n)})^{\alpha_n}(\overline{\mu_{i(n)}z_{i(n)}})^{\beta_n}) \\ &\times L(1/\mu_1, \ldots, 1/\mu_n) = B(z_1^{\alpha_1}\overline{z}_1^{\beta_1}, \ldots, z_n^{\alpha_n}\overline{z}_n^{\beta_n}), \end{aligned}$$

where B is a constant matrix. The last formula is the explicit expression for L.

8. Proving that ψ is (anti-) biholomorphic

To prove that ψ is (anti-) biholomorphic is the same as to prove that L is (anti-) biholomorphic, because the relation $L = \theta_g \circ \psi \circ \theta_f^{-1}$ holds. We showed that

$$L(z_1, \ldots, z_n) = B(z_1^{\alpha_1}\overline{z}_1^{\beta_1}, \ldots, z_n^{\alpha_n}\overline{z}_n^{\beta_n}), \quad \alpha_i - \beta_i = \pm 1, \quad i = 1, \ldots, n, \tag{14}$$

in a neighborhood W_1 of 0. From the representation (14) we see that L is $\mathbb{R}$-differentiable and non-degenerate in $W_1 \setminus \bigcup_{k=1}^n \{(z_1, \ldots, z_n) : \ z_k = 0\}$. Since this is true for every point in the domain Ω_1, the map ψ is $\mathbb{R}$-differentiable and non-degenerate everywhere, with the possible exception of an analytic set. Let us remove this set from Ω_1, as well as its image under ψ from Ω_2. We call the domains obtained in this way Ω', Ω''. Now the map $\psi : \ \Omega' \to \Omega''$ is $\mathbb{R}$-differentiable and non-degenerate everywhere. It is clear that if we prove that ψ is (anti-) biholomorphic between Ω', Ω'', then it is (anti-) biholomorphic between Ω_1, Ω_2 due to a standard continuation argument [11]. So we can think that ψ is $\mathbb{R}$-differentiable and non-degenerate in Ω_1 itself. The map L thus has to be $\mathbb{R}$-differentiable and non-degenerate at 0. However, this is the case if and only if $\alpha_i + \beta_i = 1,\ i = 1, \ldots, n$. Together with the equation $\alpha_i - \beta_i = \pm 1$ it gives us that either $\alpha_i = 1,\ \beta_i = 0$, or $\alpha_i = 0,\ \beta_i = 1$.

It remains to show that either $\alpha_i = 1$ and $\beta_i = 0$, or $\alpha_i = 0$ and $\beta_i = 1$, simultaneously for all i. Suppose, by way of contradiction, that we have $L(z_1, \dots, z_n) = B(\dots, z_i, \dots, \overline{z}_j, \dots)$. Then

$$L^{-1}(w_1, \dots, w_n) = (\dots, l_i(w_1, \dots, w_n), \dots, l_j(\overline{w}_1, \dots, \overline{w}_n), \dots),$$

where $l_i,\ l_j$ are linear analytic functions. Let us look at an endomorphism f_0 of Ω_1 in the form

$$f_0 = \theta_f^{-1} \lambda(\dots, \theta_{f,i}\theta_{f,j}, \dots, \theta_{f,j}, \dots)\theta_f,$$

where $\theta_{f,i}\theta_{f,j}$ is in the i'th place and $\theta_{f,j}$ in the j'th; $|\lambda|$ is sufficiently small. Using (1) and the definition of L, we have

$$\theta_g \varphi f_0 \theta_g^{-1} = \theta_g \psi f_0 \psi^{-1} \theta_g^{-1} = L\theta_f f_0 \theta_f^{-1} L^{-1}.$$

So,

$$\begin{aligned} &\theta_g \varphi f_0 \theta_g^{-1}(w_1, \dots, w_n) \\ &\qquad = B'(\dots, l_i(w_1, \dots, w_n) l_j(\overline{w}_1, \dots, \overline{w}_n), \dots, \bar{l}_j(w_1, \dots, w_n), \dots) \end{aligned}$$

for some constant matrix B'. This map, and hence φf_0, is not analytic, though, in a neighborhood of 0, which is a contradiction. Thus L, and hence ψ, is either analytic or antianalytic in a neighborhood of 0.

Theorem 1 is proved completely.

9. Proof of Theorem 2

Since φ is an epimorphism, it takes constant endomorphisms of Ω_1 to constant endomorphisms of Ω_2, which follows from (2). Thus we can define a map $\psi :\ \Omega_1 \to \Omega_2$ as in (3). Following the same steps as in verifying (1), we obtain

$$\varphi f \circ \psi = \psi \circ f, \quad \text{for all } f \in E(\Omega_1). \tag{15}$$

We will show that (15) implies bijectivity of ψ. The map ψ is surjective. Indeed, let $w \in \Omega_2$, and let c_w be the corresponding constant endomorphism. Since φ is an epimorphism, there exists $f \in E(\Omega_1)$ such that $\varphi f = c_w$. If we plug this f into (15), we get

$$\psi f(z) = w$$

for all $z \in \Omega_1$. Thus ψ is surjective.

To prove that ψ is injective, we show that for every $w \in \Omega_2$, the full preimage of w under ψ, $\psi^{-1}(w)$, consists of one point.

Assume for contradiction that $S_w = \psi^{-1}(w)$ consists of more than one point for some $w \in \Omega_2$. The set S_w cannot be all of Ω_1, since ψ is surjective. For $z_0 \in \partial S_w \cap \Omega_1$ we can find $z_1 \in S_w$ and $\zeta \notin S_w$ which are arbitrarily close to z_0. Let z_2 be a fixed point of S_w different from z_1. Consider a homothetic transformation h such that $h(z_1) = z_1,\ h(z_2) = \zeta$. Since the domain Ω_1 is bounded, we can choose points z_1 and ζ sufficiently close to each other so that h belongs to $E(\Omega_1)$. Applying (15) to h, we obtain

$$\begin{aligned} \varphi h(w) = \varphi h \circ \psi(z_1) = \psi \circ h(z_1) = \psi(z_1) = w; \\ \varphi h(w) = \varphi h \circ \psi(z_2) = \psi \circ h(z_2) = \psi(\zeta) \neq w. \end{aligned}$$

The contradiction shows injectivity of ψ. Thus we have proved that ψ is bijective.

According to (15) we have

$$\varphi f = \psi \circ f \circ \psi^{-1}, \quad \text{for all } f \in E(\Omega_1),$$

which implies that φ is an isomorphism.

Theorem 2 is proved.

References

[1] V. I. Arnold, *Geometrical Methods in the Theory of Ordinary Differential Equations,* Springer–Verlag, 1988. MR **89h:**58049

[2] S. Bochner, W. Martin, *Several Complex Variables,* Princeton University Press, 1948. MR **10:**366a

[3] L. Carleson, T. Gamelin, *Complex Dynamics,* Springer–Verlag, 1993. MR **94h:**30033

[4] A. Eremenko, *On the Characterization of a Riemann Surface by its Semigroup of Endomorphisms,* Trans. of AMS 338 No. 1, 1993, 123–131. MR **93j:**30023

[5] M. Heins, *Complex Function Theory,* Acad. Press, NY, 1968. MR **39:**413

[6] A. Hinkkanen, *Functions Conjugating Entire Functions to Entire Functions and Semigroups of Analytic Endomorphisms,* Complex Variables Theory Appl. 18, 1992, 149–154. MR **93m:**30037

[7] K. Hoffman, R. Kunze, *Linear Algebra,* Prentice-Hall, Inc, 1971. MR **43:**1998

[8] W. Hurewicz, H. Wallman, *Dimension Theory,* Princeton University Press, 1948. MR **3:**312b

[9] H. Iss'sa, *On the Meromorphic Function Field of a Stein Variety,* Ann. of Math. (2) 83, 1966, 34–46. MR **32:**2613

[10] S. Kobayashi, *Hyperbolic Complex Spaces,* Springer–Verlag, 1998. MR **99m:**32026

[11] S. Krantz, *Function Theory of Several Complex Variables,* John Wiley & Sons, Inc, 1982. MR **84c:**32001

[12] K. D. Magill, *A Survey of Semigroups of Continuous Selfmaps,* Semigroup Forum 11 (1975/76), 189–282. MR **52:**14140

Department of Mathematics, Purdue University, West Lafayette, Indiana 47907

E-mail address: `smerenko@math.purdue.edu`