PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 130, Number 7, Pages 1983–1992
S 0002-9939(01)06247-5
Article electronically published on November 21, 2001

ON THE PREDICTABILITY OF DISCRETE DYNAMICAL SYSTEMS

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(Communicated by Alan Dow)

Abstract. Let X be a metric space. A function $f : X \to X$ is said to be non-sensitive at a point $a \in X$ if for every $\epsilon > 0$ there is a $\delta > 0$ such that for any choice of points $a_0 \in B(a; \delta)$, $a_1 \in B(f(a_0); \delta)$, $a_2 \in B(f(a_1); \delta), \dots$, we have that $d(a_m, f^m(a)) < \epsilon$ for every $m \geq 0$. Let $H(X)$ be the set of all homeomorphisms from X onto X endowed with the topology of uniform convergence. The main goal of the present paper is to prove that for certain spaces X, "most" functions in $H(X)$ are non-sensitive at "most" points of X.

1. Introduction

Consider a discrete dynamical system (X, f), where X is a metric space with metric d. If $a \in X$, the sequence a, $f(a)$, $f^2(a), \dots$ can be thought of as the actual behaviour of the system (X, f) at a. However, in concrete situations, we are often unable to compute the initial condition a exactly. We just compute a value a_0 close to a. It may also be the case that we cannot compute $f(a_0)$ exactly, but just a value a_1 close to $f(a_0)$. Then we compute a value a_2 close to $f(a_1)$ and so on. In this way, we obtain a sequence a_0, a_1, $a_2, \dots$ that can be thought of as the predicted behaviour of the system (X, f) at a. It is natural to ask whether or not this predicted behaviour is close to the actual behaviour of the system. This leads to the following definition:

Definition 1. We say that f is non-sensitive at a if for every $\epsilon > 0$ there is a $\delta > 0$ such that for any choice of points

$$a_0 \in B(a; \delta),\ a_1 \in B(f(a_0); \delta),\ a_2 \in B(f(a_1); \delta), \dots ,$$

we have that

$$d(a_m, f^m(a)) < \epsilon \quad \text{for every } m \geq 0.$$

If f is non-sensitive at a, then the discrete dynamical system (X, f) is "predictable at a", in the sense that we can predict the future evolution of a in the system forever as accurately as we want provided we can compute the initial condition and the values of f precisely enough.

The above definition may remind the reader of the well-known notion of "shadowing" which, in the case X is compact, may be defined as follows [1]: f is said

Received by the editors August 10, 1999 and, in revised form, January 16, 2001.
2000 *Mathematics Subject Classification.* Primary 37B25, 37B20, 54H20.
Key words and phrases. Homeomorphisms, predictability, recurrence, Baire category, Lebesgue measure.

to have the shadowing property (also called the pseudo-orbit tracing property) if for every $\epsilon > 0$ there is a $\delta > 0$ such that for every sequence $a_0, a_1, a_2, \dots$ in X satisfying $d(f(a_i), a_{i+1}) < \delta$ for all $i \geq 0$, there is a point $x \in X$ such that

$$d(a_m, f^m(x)) < \epsilon \quad \text{for all } m \geq 0.$$

There is an important difference between "shadowing" and "non-sensitivity" despite the obvious fact that the former is a global notion whereas the latter is a pointwise one. To understand this difference, let $a \in X$ and choose points $a_0 \in B(a; \delta)$, $a_1 \in B(f(a_0); \delta)$, $a_2 \in B(f(a_1); \delta)$, etc. The shadowing property gives us the existence of a point $x \in X$ (possibly different from a) such that $d(a_m, f^m(x)) < \epsilon$ for all $m \geq 0$. On the other hand, non-sensitivity at a guarantees that this holds with $x = a$. This is a major difference if we are interested in the problem of predictability: the shadowing property guarantees only that the predicted behaviour of the system (X, f) at a is close to some actual behaviour of the system (but this actual behaviour may be different from the actual behaviour at the initial condition a!). For results and further references on "shadowing", see the book [1] by Aoki and Hiraide.

Let $H(X)$ be the set of all homeomorphisms from X onto X. If X is compact, we consider $H(X)$ endowed with the supremum metric: $d(f, g) = \sup_{x \in X} d(f(x), g(x))$; $H(X)$ is then a Baire space.

If M is a Baire space, we say that "most elements of M" satisfy a certain property P if the set of all $x \in M$ that do not satisfy property P is of the first category in M. The terms "typical" and "generic" are often used instead of "most".

A result related to the notion of predictability was obtained by the author in [2]. Let B^n denote the closed unit ball of $\mathbf{R}^n$. Then, Theorem 20 of [2] can be stated as follows:

> *Let $n \geq 2$. For most functions $f \in H(B^n)$, the family $\{f^m; m \geq 1\}$ is equicontinuous at most points of B^n.*

The equicontinuity of $\{f^m; m \geq 1\}$ at a point a can be thought of as a kind of "predictability with respect to the initial condition a". This property is much weaker than the notion of non-sensitivity (which implies that the family $\{f^m; m \geq 1\}$ is uniformly equicontinuous on the orbit $\{a, f(a), f^2(a), \dots\}$ of a under f). Nevertheless, it is indeed true that we can assure the non-sensitivity of f at most points of B^n, for most $f \in H(B^n)$. In fact, this is true in a much more general setting:

Theorem 2. *Fix $n \geq 1$. Let X be a metrizable compact topological n-manifold with (or without) boundary* [5] *and fix a metric d compatible with the topology of X. Then, most functions in $H(X)$ are non-sensitive at most points of X.*

We shall prove Theorem 2 in Section 4.

In the case X is a metrizable compact smooth manifold without boundary, Odani [6] proved that most functions in $CLD(X)$ (the closure in $H(X)$ of the set of all diffeomorphisms of X) have the shadowing property. His arguments were based on tools from Differentiable Dynamics. This work was preceeded by Yano [9], who proved that most functions in $H(S^1)$ have the shadowing property (where S^1 is the unit circle). By Munkres [4] and Whitehead [8], $CLD(X) = H(X)$ whenever $\dim X \leq 3$; but Munkres [4] also showed that this is false if $\dim X > 3$. The problem of whether most functions in $H(X)$ have the shadowing property was recently solved in the affirmative for $\dim X$ arbitrary by Pilyugin and Plamenevskaya [7]. However, this problem seems to remain open if X is an arbitrary metrizable compact topological manifold with (or without) boundary.

An interesting problem is to change the notion of "most" on B^n from category sense to measure-theoretic (probability) sense, that is, to consider "most" as meaning "full measure". In this direction we have the following result:

Theorem 3. *Fix $n \geq 2$. Most functions in $H(B^n)$ are non-sensitive at every point of a full (Lebesgue) measure subset of B^n.*

As an immediate consequence of this theorem we have the following:

Corollary 4. *Let $n \geq 2$. For most functions $f \in H(B^n)$, if f is sensitive to initial conditions on a subset Y of B^n, then $\overline{Y}$ has Lebesgue measure zero.*

Recall that f is sensitive to initial conditions on $Y \subset B^n$ if there is an $\epsilon > 0$ such that for any $y \in Y$ and any $\delta > 0$, there is an $x \in B^n$ with $\|x - y\| < \delta$ and there is an $m \geq 1$ so that $\|f^m(x) - f^m(y)\| \geq \epsilon$.

We shall prove Theorem 3 in Section 2. Its proof will establish the following result at the same time:

Theorem 5. *Let $n \geq 2$. For most functions $f \in H(B^n)$, the set Ω_f of all non-wandering points of f has Lebesgue measure zero.*

Theorem 5 improves Theorem 17 of [2], which asserts that for most functions $f \in H(B^n)$, the set P_f of all periodic points of f has Lebesgue measure zero. In this direction, it is interesting to observe the following fact:

Proposition 6. *Fix $n \geq 2$. For most functions $f \in H(B^n)$, most points of Ω_f are recurrent and non-periodic.*

We shall prove Proposition 6 in Section 3.

2. Proof of Theorems 3 and 5

Consider $\mathbf{R}^n$ endowed with the metric given by its euclidean norm $\|\cdot\|$. Given $A \subset \mathbf{R}^n$, $\overline{A}$, Int$A$, BdA and diam$A$ denote the closure, the interior, the boundary and the diameter of A in $\mathbf{R}^n$, respectively, and we define

$$N_\delta(A) = \bigcup_{a \in A} \{x \in \mathbf{R}^n; \|x - a\| < \delta\} \quad \text{for } \delta > 0.$$

Throughout the present section, X denotes the closed unit ball of $\mathbf{R}^n$ and μ denotes Lebesgue measure on $\mathbf{R}^n$. By an open box we mean a set of the form

$$\{(x_1, \dots, x_n) \in \mathbf{R}^n; \alpha_i < x_i < \alpha_i + \delta \text{ for } 1 \leq i \leq n\},$$

where $\alpha_1, \dots, \alpha_n \in \mathbf{R}$ and $\delta > 0$. A closed box is just the closure of an open box. By a tree we mean a finite rooted tree [3]. If T is a tree, $V(T)$ denotes the set of all vertices of T. Moreover, if $v_1, v_2 \in V(T)$, we write "$v_1 > v_2$" or "$v_2 < v_1$" to mean that v_1 and v_2 are adjacent [3] and that the unique path connecting v_2 to the root of T passes through v_1. A B-tree is a pair (T, φ), where T is a tree and φ is a bijective correspondence between $V(T)$ and a collection of pairwise disjoint closed boxes contained in Int$X$. If (T, φ) is a B-tree, we usually omit the correspondence φ and speak just of the B-tree T; moreover, we make no distinction between a vertex of T and its corresponding closed box.

For each $k \geq 1$, let A_k be the set of all functions $f \in H(X)$ for which there are finitely many B-trees $T_1, \dots, T_s$ so that the following properties are satisfied:

(i) $C \cap F = \emptyset$ whenever $C \in V(T_i)$, $F \in V(T_j)$ and $i \neq j$.

(ii) $\mathrm{diam}C < 1/k$ whenever $C \in V(T_i)$ for some $1 \le i \le s$.
(iii) If $C, F \in V(T_i)$ and $C > F$, then $f(F) \subset \mathrm{Int}C$.
(iv) For each i, there is a chain $C_{i,1} > C_{i,2} > \cdots > C_{i,t_i}$ of successive boxes in $V(T_i)$, beginning with the root $C_{i,1}$ of T_i, so that $f(C_{i,1}) \subset \mathrm{Int}C_{i,t_i}$.
(v) The set

$$\bigcup\{C; C \in V(T_i) - \{C_{i,1}, \ldots, C_{i,t_i}\} \text{ for some } 1 \le i \le s\}$$

together with the set

$$\bigcup_{i=1}^{s}[(C_{i,t_i} - f(C_{i,1})) \cup (C_{i,t_i-1} - f^2(C_{i,1})) \cup \ldots \cup (C_{i,1} - f^{t_i}(C_{i,1}))]$$

form a set of Lebesgue measure $> \mu(X) - 1/k$.

It is immediate to check that each A_k is open in $H(X)$. Moreover, $\mu(\Omega_f) < 1/k$ for every $f \in A_k$, and so $\mu(\Omega_f) = 0$ whenever $f \in \bigcap_{k=1}^{\infty} A_k$. Now, suppose $f \in \bigcap_{k=1}^{\infty} A_k$. Then, for each $k \ge 1$, there are B-trees $T_{k,1}, \ldots, T_{k,s_k}$ so that (i) through (v) hold with $T_{k,1}, \ldots, T_{k,s_k}$ in place of $T_1, \ldots, T_s$. Put

$$M_k = \bigcup\{C; C \in V(T_{k,j}) \text{ for some } 1 \le j \le s_k\} \quad (k \ge 1) \quad \text{and} \quad M = \bigcap_{r=1}^{\infty}\bigcup_{k=r}^{\infty} M_k.$$

Since $\mu(M_k) > \mu(X) - 1/k$ for every $k \ge 1$, it follows that $\mu(M) = \mu(X)$. Moreover, for each $k \ge 1$, there is a $0 < \delta_k < 1/k$ such that

$$f(N_{\delta_k}(F)) \subset \mathrm{Int}C \quad \text{whenever } C > F,$$

where $C, F \in V(T_{k,j})$ for some $1 \le j \le s_k$, and

$$f(N_{\delta_k}(C_{k,j,1})) \subset \mathrm{Int}C_{k,j,t_{k,j}},$$

where $C_{k,j,1} > \cdots > C_{k,j,t_{k,j}}$ is related to $T_{k,j}$ $(1 \le j \le s_k)$ as $C_{i,1} > \cdots > C_{i,t_i}$ is related to T_i in property (iv). Hence, if $a \in M_k$ and we choose $a_0 \in B(a;\delta_k)$, $a_1 \in B(f(a_0);\delta_k)$, $a_2 \in B(f(a_1);\delta_k), \ldots$, then $\|a_m - f^m(a)\| < 2/k$ for every $m \ge 0$. This implies that f is non-sensitive at every point of M.

It remains to show that each A_k is dense in $H(X)$. Fix $k \ge 1$, $f \in H(X)$ and $\epsilon > 0$. Let $0 < \delta < 1/k$ be such that

$$\mathrm{diam}f(Y) < \frac{\epsilon}{2} \quad \text{whenever } Y \subset X \text{ and } \mathrm{diam}Y < \delta.$$

Let $\mathcal{C}$ be a finite collection of pairwise disjoint closed boxes contained in $\mathrm{Int}X$ such that

$$\mu(X - \bigcup_{C \in \mathcal{C}} C) < \frac{1}{k} \;\text{ and }\; \mathrm{diam}C < \delta \text{ for every } C \in \mathcal{C}.$$

Let r be the cardinality of $\mathcal{C}$ and choose $\eta > 0$ such that

$$\mathrm{diam}f(Y) < \frac{\epsilon}{2^{r+1}} \quad \text{whenever } Y \subset X \text{ and } \mathrm{diam}Y < \eta.$$

Our next goal is to construct finitely many B-trees $T_1, \ldots, T_s$ satisfying (i) and the following properties:

(a) $\mathrm{diam}C < \delta$ whenever $C \in V(T_i)$ for some $1 \le i \le s$.
(b) $\mathcal{C} \subset V(T_1) \cup \ldots \cup V(T_s)$.
(c) If $C, F \in V(T_i)$ and $C > F$, then we have two possibilities:
- either $C \in \mathcal{C}$ and $f(F) \subset C$,
- or $C \notin \mathcal{C}$ and $(\mathrm{Int}C) \cap f(\mathrm{Bd}F) \ne \emptyset$.

(d) For each i, there is a chain $C_{i,1} > C_{i,2} > \cdots > C_{i,t_i}$ of successive boxes in $V(T_i)$, beginning with the root $C_{i,1}$ of T_i, with the following property: $f(C_{i,1}) \subset C_{i,t_i}$ and/or there is a $C_{i,t_i+1} \in V(T_i)$ with $C_{i,t_i} > C_{i,t_i+1}$ such that $C_{i,1}, \ldots, C_{i,t_i+1} \notin \mathcal{C}$ and both $C_{i,1}$ and C_{i,t_i+1} are contained in an open box W_i of diameter $< \eta$.

In order to explain how to construct the trees $T_1, \ldots, T_s$, we shall need a variable $\mathcal{B}$, which will denote the set of all closed boxes that have already been used in the construction up to the current step (at the beginning we have $\mathcal{B} = \emptyset$). It is also important to observe that during the construction the trees will be seen as variables. Even when the construction of a tree T_i is apparently done, it may be necessary to change it later on in the process.

We begin by choosing a closed box $C_1 \in \mathcal{C}$ and by putting it as a vertex of T_1 (note that now $\mathcal{B} = \{C_1\}$). Suppose that in a certain moment T_1 consists of the vertices $C_1 < C_2 < \cdots < C_j$. We look at the set $f(C_j)$. There are three possibilities:

Case 1. $f(C_j) \subset C$ for some $C \in \mathcal{B}$.

We stop the construction of T_1 (for the time being). So, C_j is the root of T_1.

Case 2. $f(C_j) \subset C$ for some $C \in \mathcal{C} - \mathcal{B}$.

Let C_{j+1} be such a C. We put C_{j+1} as a vertex of T_1 adjacent to C_j and satisfying $C_j < C_{j+1}$.

Case 3. $f(C_j) \not\subset \bigcup\{C; C \in \mathcal{C} \text{ or } C \in \mathcal{B}\}$.

Then we also have that $f(\mathrm{Bd}C_j) \not\subset \bigcup\{C; C \in \mathcal{C} \text{ or } C \in \mathcal{B}\}$. Hence, we can choose a closed box $C_{j+1} \subset \mathrm{Int}X$ disjoint from each box in $\mathcal{C} \cup \mathcal{B}$ such that $\mathrm{diam}C_{j+1} < \delta$ and $(\mathrm{Int}C_{j+1}) \cap f(\mathrm{Bd}C_j) \neq \emptyset$. We put C_{j+1} as a vertex of T_1 adjacent to C_j and satisfying $C_j < C_{j+1}$.

If Case 1 never happens, then the construction can go on forever. In this case we will stop the construction of T_1 as soon as we obtain a chain $C_{1,1} > C_{1,2} > \cdots > C_{1,t_1} > C_{1,t_1+1}$ beginning with the root $C_{1,1}$ of T_1 so that $C_{1,1}, \ldots, C_{1,t_1+1} \notin \mathcal{C}$ and both $C_{1,1}$ and C_{1,t_1+1} are contained in an open box W_1 of diameter $< \eta$. Since $\mathcal{C}$ is finite and X is bounded, we will obtain such a chain in a finite number of steps.

Suppose that we have already constructed $T_1, \ldots, T_{i-1}$. If $\mathcal{B} \supset \mathcal{C}$, we are done. If this is not the case, we choose a $C'_1 \in \mathcal{C} - \mathcal{B}$ and put it as a vertex of T_i. If in a certain moment T_i consists of the vertices $C'_1 < C'_2 < \cdots < C'_j$, we then look at $f(C'_j)$. Cases 2 and 3 are treated as before. However, Case 1 should be divided in two possibilities:

Case 1a. $f(C'_j) \subset C$ for some $C \in V(T_i)$.

We stop the construction of T_i (for the time being). So, C'_j is the root of T_i.

Case 1b. $f(C'_j) \subset C$ for some $C \in \mathcal{B} - V(T_i)$.

Let $\tilde{C}$ be such a C. Then $\tilde{C}$ is a vertex of a previous tree, say $\tilde{C} \in V(T_{i_0})$, where $1 \leq i_0 < i$. In this case we will have no tree T_i for the time being. We will just enlarge T_{i_0} by putting the chain $C'_1 < C'_2 < \cdots < C'_j$ as a new branch of it, satisfying the relation $C'_j < \tilde{C}$.

If Cases 1a and 1b never happen, we will stop the construction of T_i when we obtain a chain $C_{i,1} > C_{i,2} > \cdots > C_{i,t_i} > C_{i,t_i+1}$ beginning with the root $C_{i,1}$ of T_i so that $C_{i,1}, \ldots, C_{i,t_i+1} \notin \mathcal{C}$ and both $C_{i,1}$ and C_{i,t_i+1} are contained in an open box W_i of diameter $< \eta$.

By the way the trees $T_1, \ldots, T_s$ were constructed, it is immediate to check that they have all the desired properties. Note also that $s \leq r$.

Let $I = \{i \in \{1, \ldots, s\}; f(C_{i,1}) \subset C_{i,t_i}\}$ and $J = \{1, \ldots, s\} - I$. For each $C \in V(T_1) \cup \ldots \cup V(T_s)$, choose an open box V_C with

$$\text{diam} V_C < \delta \quad \text{and} \quad C \subset V_C \subset \overline{V_C} \subset \text{Int} X,$$

so that the family $\{\overline{V_C}\}_{C \in V(T_1) \cup \ldots \cup V(T_s)}$ is pairwise disjoint. We may also assume that

$$\overline{V_{C_{i,1}}} \subset W_i \ \text{ and } \ \overline{V_{C_{i,t_i+1}}} \subset W_i \quad \text{for every } i \in J.$$

Now, we are going to define a function $g_0 \in H(X)$ as follows: Suppose that $C, F \in V(T_i)$ for some i, and $C > F$. By (c), there is a closed box $B \subset \text{Int} F$ such that $f(B) \subset \text{Int} C$. If $F \notin \{C_{i,t_i+1}; i \in J\}$, choose a $\varphi \in H(\overline{V_F})$ such that $\varphi(F) \subset B$ and $\varphi(x) = x$ for all $x \in \text{Bd} V_F$, and define

$$g_0(x) = f(\varphi(x)) \quad \text{for all } x \in \overline{V_F}.$$

If $F \in \{C_{i,t_i+1}; i \in J\}$, then $C \notin \mathcal{C}$ and so $(\text{Int} C) \cap f(\text{Bd} F) \neq \emptyset$. Choose an open box Z_F with $F \subset Z_F \subset \overline{Z_F} \subset V_F$ such that there is a closed box $D_F \subset V_F - \overline{Z_F}$ with $f(D_F) \subset \text{Int} C$, choose a $\varphi \in H(\overline{Z_F})$ such that $\varphi(F) \subset B$ and $\varphi(x) = x$ for all $x \in \text{Bd} Z_F$, and define

$$g_0(x) = f(\varphi(x)) \quad \text{for all } x \in \overline{Z_F}.$$

We make this definition for all $C, F \in V(T_i)$ with $C > F$ $(1 \leq i \leq s)$. For $i \in I$, there also exists a closed box $B \subset \text{Int} C_{i,1}$ such that $f(B) \subset \text{Int} C_{i,t_i}$. So, choose a $\varphi \in H(\overline{V_{C_{i,1}}})$ such that $\varphi(C_{i,1}) \subset B$ and $\varphi(x) = x$ for all $x \in \text{Bd} V_{C_{i,1}}$, and put

$$g_0(x) = f(\varphi(x)) \quad \text{for all } x \in \overline{V_{C_{i,1}}}.$$

Let K be the union of all these closed boxes where we have already defined g_0. We finally put $g_0(x) = f(x)$ for all $x \in X - K$. Then $g_0 \in H(X)$, $d(g_0, f) < \epsilon/2$, and (iii) holds for every $i \in \{1, \ldots, s\}$ and (iv) holds for every $i \in I$ with g_0 in place of f.

Now, we need to change g_0 a little bit in order to obtain (iv) also for $i \in J$. If $i \in J$ and $1 \leq j \leq t_i + 1$, we denote the open box $V_{C_{i,j}}$ simply by $V_{i,j}$. Moreover, for $i \in J$, we write Z_i and D_i in place of $Z_{C_{i,t_i+1}}$ and $D_{C_{i,t_i+1}}$, respectively. Recall that

$$D_i \subset V_{i,t_i+1} - \overline{Z_i} \quad \text{and} \quad f(D_i) \subset \text{Int} C_{i,t_i} \quad (i \in J).$$

Write $J = \{i_1, \ldots, i_w\}$ and put

$$K_1 = K \cup \overline{V_{i_2,1}} \cup \ldots \cup \overline{V_{i_w,1}} \cup D_{i_2} \cup \ldots \cup D_{i_w}.$$

Choose

$$a_1 \in \text{Int} C_{i_1,1} \subset \overline{V_{i_1,1}} \subset W_{i_1} - K_1 \quad \text{and} \quad b_1 \in \text{Int} D_{i_1} \subset W_{i_1} - K_1.$$

Since K_1 is the union of a finite collection of pairwise disjoint closed boxes contained in $\text{Int} X$, $W_{i_1} - K_1$ is connected. So, there is a continuous path $\alpha : [0,1] \to W_{i_1} - K_1$ from a_1 to b_1. Moreover, we may assume that $\alpha([0,1]) \subset \text{Int} X$. Cover

$\alpha([0,1])$ by finitely many open balls $B_1, \ldots, B_\ell$ whose closures are contained in $(W_{i_1} - K_1) \cap \mathrm{Int} X$ so that

$$\overline{B_1} \subset \mathrm{Int} C_{i_1,1}, \quad \overline{B_\ell} \subset \mathrm{Int} D_{i_1} \quad \text{and} \quad B_i \cap B_{i+1} \neq \emptyset \ \text{ for every } 1 \leq i < \ell.$$

By working on $\overline{V_{i_1,1}} \cup \overline{B_1} \cup \ldots \cup \overline{B_\ell}$, we see that it is possible to construct a $\varphi \in H(X)$ such that

$$\varphi(C_{i_1,1}) \subset \mathrm{Int} D_{i_1} \quad \text{and} \quad \varphi(x) = x \ \text{ if } x \notin \overline{V_{i_1,1}} \cup \overline{B_1} \cup \ldots \cup \overline{B_\ell}.$$

Put $g_1 = g_0 \circ \varphi$. Then $g_1 \in H(X)$, $g_1 = g_0$ on $X - (\overline{V_{i_1,1}} \cup \overline{B_1} \cup \ldots \cup \overline{B_\ell})$ and

$$g_1(C_{i_1,1}) \subset g_0(D_{i_1}) = f(D_{i_1}) \subset \mathrm{Int} C_{i_1,t_{i_1}}.$$

Moreover, since $\mathrm{diam}(\overline{V_{i_1,1}} \cup \overline{B_1} \cup \ldots \cup \overline{B_\ell}) < \eta$, we have $d(g_1, g_0) < \epsilon/2^{r+1}$ and

$$\mathrm{diam} g_1(Y) < \frac{\epsilon}{2^r} \quad \text{whenever } Y \subset X - K \text{ and } \mathrm{diam} Y < \eta.$$

Put

$$K_2 = K \cup \overline{V_{i_1,1}} \cup \overline{V_{i_3,1}} \cup \ldots \cup \overline{V_{i_w,1}} \cup D_{i_3} \cup \ldots \cup D_{i_w}.$$

Choose

$$a_2 \in \mathrm{Int} C_{i_2,1} \subset \overline{V_{i_2,1}} \subset W_{i_2} - K_2 \quad \text{and} \quad b_2 \in \mathrm{Int} D_{i_2} \subset W_{i_2} - K_2,$$

and argue as before. We then obtain a $g_2 \in H(X)$ with $g_2(C_{i_2,1}) \subset \mathrm{Int} C_{i_2,t_{i_2}}$. Moreover, since g_2 differs from g_1 only on a set of diameter $< \eta$, we have $d(g_2, g_1) < \epsilon/2^r$ and

$$\mathrm{diam} g_2(Y) < \frac{\epsilon}{2^{r-1}} \quad \text{if } Y \subset X - K \text{ and } \mathrm{diam} Y < \eta.$$

By continuing this process, we will obtain a function $g_w \in H(X)$ such that

$$d(g_w, g_0) < \frac{\epsilon}{2^{r+1-(w-1)}} + \frac{\epsilon}{2^{r+1-(w-2)}} + \cdots + \frac{\epsilon}{2^{r+1}} < \frac{\epsilon}{2},$$

and so $d(g_w, f) < \epsilon$. Moreover, properties (iii) and (iv) hold for every $i \in \{1, \ldots, s\}$ if we replace f by g_w.

Finally, it remains to deal with property (v). Recall that $\mathcal{C} \subset V(T_1) \cup \ldots \cup V(T_s)$ and that $\mu(\bigcup_{C \in \mathcal{C}} C) > \mu(X) - 1/k$. If we had every box of $\mathcal{C}$ appearing in the first union in property (v) we would be done. However, this may not be the case, since for $i \in I$ there may exist boxes of $\mathcal{C}$ in the chain $C_{i,1} > C_{i,2} > \cdots > C_{i,t_i}$. For each $i \in I$, let U_i be an open box such that

$$C_{i,1} \subset U_i \subset \overline{U_i} \subset V_{C_{i,1}} \quad \text{and} \quad g_w(\overline{U_i}) \subset \mathrm{Int} C_{i,t_i}.$$

Let $\varphi_i \in H(\overline{U_i})$ be such that $\varphi_i(x) = x$ for all $x \in \mathrm{Bd} U_i$ and $\mathrm{diam} \varphi_i(C_{i,1})$ is very small. Put $g = g_w \circ \varphi_i$ on $\overline{U_i}$ for each $i \in I$ and $g = g_w$ on $X - \bigcup_{i \in I} \overline{U_i}$. Then $g \in H(X)$ and $d(g, f) < \epsilon$, because $g_w(\overline{U_i}) \subset f(\overline{V_{C_{i,1}}})$ for each $i \in I$, by the way the g_j's were constructed. Moreover, (iii) and (iv) still hold with g in place of f. By choosing φ_i so that $\mathrm{diam} \varphi_i(C_{i,1})$ is small enough, we will have that the measure of

$$(C_{i,t_i} - g(C_{i,1})) \cup (C_{i,t_i-1} - g^2(C_{i,1})) \cup \ldots \cup (C_{i,1} - g^{t_i}(C_{i,1}))$$

is so close to the measure of $C_{i,t_i} \cup C_{i,t_i-1} \cup \ldots \cup C_{i,1}$ that (v) will also hold with g in place of f. This completes the proof.

3. Proof of Proposition 6

Put $X = B^n$. We know that for most functions $f \in H(X)$, $\Omega_f = \overline{P_f}$ and the set of all periodic points of f with period m is dense in the set of all periodic points of f with period q whenever q divides m [2]. Fix an $f \in H(X)$ which has these two properties. Let R_f denote the set of all recurrent points of f. For each $m \geq 1$ and each $r \geq 1$, let $V_{m,r} = \{x \in X; \|x - f^t(x)\| < 1/m \text{ for some } t \geq r\}$. Then, each $V_{m,r}$ is open and

$$R_f - P_f = \bigcap_{m,r,k} (V_{m,r} - F_{f^k})$$

(where F_{f^k} denotes the set of all fixed points of f^k). Thus, it remains to show that each set $(V_{m,r} - F_{f^k}) \cap \Omega_f$ is dense in Ω_f. For this purpose, let U be an open set that meets Ω_f. Choose a $y \in P_f \cap U$. Let p be the period of y and choose an integer t of the form sp (for some $s \geq 1$) which is greater than k. Now, choose a periodic point z of f with period t which lies in U. Then, $z \in (V_{m,r} - F_{f^k}) \cap \Omega_f \cap U$, which completes the proof.

4. Proof of Theorem 2

Let $i(X)$ denote the interior of the manifold X. If $A \subset X$, then $\overline{A}$ and $\mathrm{Int}A$ denote the closure and the interior of A in X, respectively, and we define $N_\delta(A) = \bigcup_{a \in A} B(a;\delta)$ for $\delta > 0$. Moreover, if $f : X \to X$ is a mapping, we define $(f \circ N_\delta)^0(A) = A$, $(f \circ N_\delta)^1(A) = f(N_\delta(A))$, $(f \circ N_\delta)^2(A) = f(N_\delta(f(N_\delta(A))))$, and so on.

Fix a sequence $z_1, z_2, \ldots$ in $i(X)$ which is dense in X. For each $r \geq 1$ and $k \geq 1$, let $\mathcal{O}_{r,k}$ be the set of all functions $f \in H(X)$ for which there is a closed set $V \subset i(X)$ and there are integers $q \geq 0$ and $m \geq 1$ so that $f^q(z_k) \in \mathrm{Int}V$, $f^m(V) \subset \mathrm{Int}V$ and $\mathrm{diam} f^i(V) < 1/r$ for $0 \leq i \leq m-1$.

Clearly, each $\mathcal{O}_{r,k}$ is open. Let $f \in \bigcap_{r,k} \mathcal{O}_{r,k}$. By definition, for each $r \geq 1$ and each $k \geq 1$, there are a closed set $V_{r,k} \subset i(X)$ and integers $q_{r,k} \geq 0$ and $m_{r,k} \geq 1$ such that $f^{q_{r,k}}(z_k) \in \mathrm{Int}V_{r,k}$, $f^{m_{r,k}}(V_{r,k}) \subset \mathrm{Int}V_{r,k}$ and $\mathrm{diam} f^i(V_{r,k}) < 1/r$ for $0 \leq i \leq m_{r,k} - 1$. Let $W_{r,k}$ be an open ball centered at z_k such that

$$f^{q_{r,k}}(\overline{W_{r,k}}) \subset \mathrm{Int}V_{r,k} \quad \text{and} \quad \mathrm{diam} f^i(\overline{W_{r,k}}) < 1/r \;\; \text{for } 0 \leq i \leq q_{r,k} - 1.$$

Choose $0 < \delta_{r,k} < 1/r$ such that

$$(f \circ N_{\delta_{r,k}})^{q_{r,k}}(\overline{W_{r,k}}) \subset \mathrm{Int}V_{r,k}, \quad (f \circ N_{\delta_{r,k}})^{m_{r,k}}(V_{r,k}) \subset \mathrm{Int}V_{r,k},$$

$$\mathrm{diam}(f \circ N_{\delta_{r,k}})^i(\overline{W_{r,k}}) < \frac{1}{r} - \delta_{r,k} \quad \text{for } 0 \leq i \leq q_{r,k} - 1$$

and

$$\mathrm{diam}(f \circ N_{\delta_{r,k}})^i(V_{r,k}) < \frac{1}{r} - \delta_{r,k} \quad \text{for } 0 \leq i \leq m_{r,k} - 1.$$

Put $D_r = \bigcup_{k=1}^{\infty} W_{r,k}$. Then D_r is open and dense in X. Moreover, if $a \in D_r$, then $a \in W_{r,k}$ for some $k \geq 1$, and so for any choice of points $a_0 \in B(a;\delta_{r,k})$, $a_1 \in B(f(a_0);\delta_{r,k})$, $a_2 \in B(f(a_1);\delta_{r,k}), \ldots$, we have that $d(a_m, f^m(a)) < 1/r$ for every $m \geq 0$. Therefore, $D = \bigcap_{r=1}^{\infty} D_r$ is a dense G_δ subset of X and f is non-sensitive at every point of D. Thus, it remains to show that each $\mathcal{O}_{r,k}$ is dense in $H(X)$. Fix $r \geq 1$, $k \geq 1$, $f \in H(X)$ and $\epsilon > 0$. Let a be a cluster point of the sequence $(f^j(z_k))_{j\geq 0}$ and choose a neighborhood W of a in X for which there is a homeomorphism $\psi : W \to B^n$ with

$$\psi(i(X) \cap \mathrm{Int}W) = \{x \in \mathbf{R}^n; \|x\| < 1\}.$$

Let $q \geq 0$ be the smallest integer such that $f^q(z_k) \in \mathrm{Int}W$. Let $s \geq 1$ be such that $f^{q+s}(z_k) \in \mathrm{Int}W$. We have two possibilities:

Case 1. $n \geq 2$.

We choose a point $b \in (i(X) \cap \mathrm{Int}W) - \{f^j(z_k); j \in \mathbf{Z}\}$ so close to $f^q(z_k)$ that we have $f^s(b) \in \mathrm{Int}W$, and we let $m \geq 1$ be the smallest integer such that $f^m(b) \in \mathrm{Int}W$.

Case 2. $n = 1$.

We may think of W as being $[-1, 1]$. So, we may define the sets

$$L = \{x \in W; -1 < x < f^q(z_k)\} \quad \text{and} \quad R = \{x \in W; f^q(z_k) < x < 1\}.$$

If $f^{q+s}(z_k) = f^q(z_k)$, choose $b \in R - \{f^j(z_k); j \in \mathbf{Z}\}$ so close to $f^q(z_k)$ that we have $f^s(b)$ and $f^{2s}(b)$ in $\mathrm{Int}W$; then $f^s(b)$ or $f^{2s}(b)$ belong to R. If $f^{q+s}(z_k) \neq f^q(z_k)$, then either $f^{q+s}(z_k) \in L$ or $f^{q+s}(z_k) \in R$; say $f^{q+s}(z_k) \in R$. In this case, choose $b \in R - \{f^j(z_k); j \in \mathbf{Z}\}$ so that $f^s(b) \in R$. Let $m \geq 1$ be the smallest integer such that $f^m(b) \in R \subset \mathrm{Int}W$.

Let $\phi \in H(X)$ be such that

$$\phi(f^m(b)) = b \quad \text{and} \quad \phi(x) = x \text{ for all } x \in (X - \mathrm{Int}W) \cup \{f^q(z_k)\}.$$

If $n = 1$, we also assume that $\phi(x) = x$ for all $x \in L$. Define $g = \phi \circ f \in H(X)$. Then

$$g(x) = f(x) \quad \text{whenever } f(x) \in (X - \mathrm{Int}W) \cup \{f^q(z_k)\}$$

and b is a periodic point of g with period m. Moreover,

$$g^q(z_k) = f^q(z_k) \in \mathrm{Int}W \quad \text{and} \quad g^j(z_k) = f^j(z_k) \notin \mathrm{Int}W \text{ for } 0 \leq j \leq q-1.$$

Let Z and V be closed neighborhoods of b such that

$$g^q(z_k) \in \mathrm{Int}V \subset V \subset \mathrm{Int}Z \subset Z \subset \mathrm{Int}W$$

and $\psi(Z)$ is a closed ball contained in $\{x \in \mathbf{R}^n; \|x\| < 1\}$. If $n = 1$, we also assume that

$$g(b), \ldots, g^{m-1}(b) \notin Z.$$

Since $g(b), \ldots, g^{m-1}(b) \notin Z$, there is a closed neighborhood V' of b such that $V' \subset \mathrm{Int}Z$, $Z, g(V'), \ldots, g^{m-1}(V')$ are pairwise disjoint, $g^m(V') \subset \mathrm{Int}V$ and $\mathrm{diam} g^i(V') < 1/r$ for $1 \leq i \leq m-1$. Now, let $\varphi \in H(X)$ be such that

$$\varphi(V) \subset V' \quad \text{and} \quad \varphi(x) = x \text{ for all } x \in (X - \mathrm{Int}Z) \cup \{b\}.$$

Let $h = g \circ \varphi \in H(X)$. Then

$$h^q(z_k) = g^q(z_k) \in \mathrm{Int}V,$$
$$h^m(V) = h^{m-1}(g(\varphi(V))) \subset g^m(V') \subset \mathrm{Int}V$$

and

$$\mathrm{diam} h^i(V) \leq \mathrm{diam} g^i(V') < \frac{1}{r} \quad \text{for } 1 \leq i \leq m-1.$$

Moreover, by choosing W small enough we can also guarantee that $\mathrm{diam}V < 1/r$ (hence, $h \in \mathcal{O}_{r,k}$) and $d(h, f) < \epsilon$. This completes the proof.

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