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THE F-DEPTH OF AN IDEAL ON A MODULE

RENCAI LÜ AND ZHONGMING TANG

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Abstract. Let I be an ideal of a Noetherian local ring R and M a finitely generated R-module. The f-depth of I on M is the least integer r such that the local cohomology module $H_I^r(M)$ is not Artinian. This paper presents some part of the theory of f-depth including characterizations of f-depth and a relation between f-depth and f-modules.

1. Introduction

Let $(R, \mathfrak{m})$ be a (commutative) Noetherian local ring, I a proper ideal of R and M a finitely generated R-module. It is well known that the depth, $\operatorname{depth}(I, M)$, of I on M, i.e., the length of a maximal M-regular sequence in I, is the least integer r such that the local cohomology module $H_I^r(M) \neq 0$. Faltings [2] proved that the least integer r such that $H_I^r(M)$ is not finitely generated is

$$\min\{\operatorname{depth}(M_{\mathfrak{p}}) + \operatorname{ht}((I+\mathfrak{p})/\mathfrak{p}) | \mathfrak{p} \not\supseteq I\}.$$

Now we consider the problem of what is the least integer r such that $H_I^r(M)$ is not Artinian. In [4], Melkersson showed that, when $\operatorname{Supp}(M/IM) \not\subseteq \{\mathfrak{m}\}$, the least integer r such that $H_I^r(M)$ is not Artinian is

$$\min\{\operatorname{depth}(IR_{\mathfrak{p}}, M_{\mathfrak{p}}) | \mathfrak{p} \in \operatorname{Supp}(M/IM) \setminus \{\mathfrak{m}\}\},$$

and called this integer the f-depth (filter depth) of I on M. In this paper, for any proper ideal I, we define the f-depth of I on M as the length of a maximal M-filter regular sequence in I. Then it turns out that our f-depth of I on M is the least integer r such that $H_I^r(M)$ is not Artinian for any proper ideal I and, in the case $\operatorname{Supp}(M/IM) \not\subseteq \{\mathfrak{m}\}$, our f-depth coincides with the one of Melkersson.

After summarizing some results about filter regular sequences in section 2, we define f-depth in section 3. Then, we characterize f-depth by Ext, Koszul complexes and local cohomology modules. Section 4 contains an equivalent condition using f-depth for an R-module to be an f-module which is similar to a Cohen-Macaulay module.

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2. Preliminaries on filter regular sequences

Throughout the paper, let $(R, \mathfrak{m})$ be a (commutative) Noetherian local ring and M a finitely generated R-module. For any submodule N of M, we use $N :_M \langle \mathfrak{m} \rangle$ to denote the submodule $\{m \in M | \mathfrak{m}^n m \subseteq N \text{ for some } n > 0\}$.

Definition 2.1. Let $x_1, \dots, x_n \in \mathfrak{m}$. If, for $i = 1, \dots, n$,

$$(x_1, \dots, x_{i-1})M :_M x_i \subseteq (x_1, \dots, x_{i-1})M :_M \langle \mathfrak{m} \rangle,$$

then we say that $x_1, \dots, x_n$ is an M-filter regular sequence.

Notice that $x \in \mathfrak{m}$ is M-filter regular if and only if $x \notin \bigcup_{\mathfrak{p} \in \mathrm{Ass}_R(M) \setminus \{\mathfrak{m}\}} \mathfrak{p}$ and $x_1, x_2, \dots, x_n$ is an M-filter regular sequence if and only if x_1 is M-filter regular and $x_2, \dots, x_n$ is an M/x_1M-filter regular sequence. For a finitely generated R-module N, the length of N is denoted by $\ell(N)$. Then $\ell(N) < \infty$ if and only if $\dim(N) \leq 0$. Thus $x_1, \dots, x_n$ is an M-filter regular sequence is equivalent to

$$\ell((x_1, \dots, x_{i-1})M :_M x_i/(x_1, \dots, x_{i-1}M)) < \infty, i = 1, \dots, n.$$

On the other hand, we are reminded that $y_1, \dots, y_s \in R$ is a poor M-regular sequence if

$$(y_1, \dots, y_{i-1})M :_M y_i = (y_1, \dots, y_{i-1})M, i = 1, \dots, s,$$

and, if furthermore $(y_1, \dots, y_s)M \neq M$, we call $y_1, \dots, y_s$ an M-regular sequence. Then $x_1, \dots, x_n$ is an M-filter regular sequence if and only if, for any $\mathfrak{p} \in \mathrm{Supp}(M) \setminus \{\mathfrak{m}\}, x_1/1, \dots, x_n/1$ is a poor $M_\mathfrak{p}$-regular sequence, and, if $x_1, \dots, x_n$ is an M-filter regular sequence, then $x_1/1, \dots, x_n/1$ is an $M_\mathfrak{p}$-regular sequence for any $\mathfrak{p} \in \mathrm{Supp}(M) \setminus \{\mathfrak{m}\}$ with $x_1, \dots, x_n \in \mathfrak{p}$.

Since, for any poor M-regular sequence $y_1, \dots, y_s$ and any integers $i_j > 0, j = 1, \dots, s$, $y_1^{i_1}, \dots, y_s^{i_s}$ is also a poor M-regular sequence, it follows that if $x_1, \dots, x_n$ is an M-filter regular sequence, then, for any integers $i_j > 0$, $j = 1, \dots, n$, $x_1^{i_1}, \dots, x_n^{i_n}$ is also an M-filter regular sequence. However, a permutation of a filter regular sequence is not necessarily filter regular again. For example, let $K[x_1, x_2]$ be a polynomial ring and K a field. Set $R = K[x_1, x_2]_{(x_1, x_2)}$ and $M = R \oplus R/(x_2^2)$. Then, as $0 :_M x_1 = 0$ and $0 :_{M/x_1M} x_2 \subseteq R/(x_1, x_2^2)$ has finite length, it follows that x_1, x_2 is an M-filter regular sequence. But, as the prime ideal $(x_2) \in \mathrm{Ass}_R(M) \setminus \{(x_1, x_2)\}$ and $x_2 \in (x_2)$, we see that x_2 is not M-filter regular, hence, x_2, x_1 is not an M-filter regular sequence.

For any $x_1, \dots, x_n \in R$, let $H_i(x_1, \dots, x_n; M)$ be the i-th homology module of the Koszul complex $K_.(x_1, \dots, x_n; M)$ of M with respect to $x_1, \dots, x_n$. Then, we have the following:

Proposition 2.2. *If $x_1, \dots, x_n$ is an M-filter regular sequence, then*

$$\ell(H_i(x_1, \dots, x_n; M)) < \infty, \text{ for any } i > 0.$$

Proof. By induction on n. If $n = 1$, then $H_1(x_1; M) = 0 :_M x_1$ has finite length by definition. Now, assume that $n > 1$. From the long exact sequence

$$\cdots \to H_i(x_1, \dots, x_{n-1}; M) \to H_i(x_1, \dots, x_n; M) \to H_{i-1}(x_1, \dots, x_{n-1}; M)$$

$$\stackrel{(-1)^{i-1} x_n}{\to} H_{i-1}(x_1, \dots, x_{n-1}; M) \to \cdots \to H_0(x_1, \dots, x_n; M) \to 0,$$

we see that $\ell(H_i(x_1, \dots, x_n; M)) < \infty$ for all $i > 1$ from the induction assumption that $\ell(H_i(x_1, \dots, x_{n-1}; M)) < \infty$ and $\ell(H_{i-1}(x_1, \dots, x_{n-1}; M)) < \infty$. For the case $i = 1$, we have an exact sequence

$$\begin{aligned} H_1(x_1, \dots, x_{n-1}; M) &\to H_1(x_1, \dots, x_n; M) \to H_0(x_1, \dots, x_{n-1}; M) \\ &= M/(x_1, \dots, x_{n-1})M \xrightarrow{x_n} M/(x_1, \dots, x_{n-1})M. \end{aligned}$$

As $H_1(x_1, \dots, x_{n-1}; M)$ and $0 :_{M/(x_1, \dots, x_{n-1})M} x_n$ have finite length, we see that $\ell(H_1(x_1, \dots, x_n; M)) < \infty$. Hence, for all $i > 0$, $\ell(H_i(x_1, \dots, x_n; M)) < \infty$. □

Notice that the converse of Proposition 2.2 is not true because the conditions $\ell(H_i(x_1, \dots, x_n; M)) < \infty$ for all $i > 0$ do not depend on the order of $x_1, \dots, x_n$, but the filter regularity does.

3. Characterizations of f-depth

In order to show that f-depth is well-defined, we need the following:

Lemma 3.1. *Let $I \subseteq \mathfrak{m}$ be an ideal. If $\ell(Hom_R(R/I, M)) < \infty$, then there exists $x \in I$ which is M-filter regular.*

Proof. Assume the contrary. Then $I \subseteq \bigcup_{\mathfrak{p} \in \mathrm{Ass}_R(M) \setminus \{\mathfrak{m}\}} \mathfrak{p}$, so that $I \subseteq \mathfrak{p}$ for some $\mathfrak{p} \in \mathrm{Ass}_R(M) \setminus \{\mathfrak{m}\}$. Hence $\mathfrak{p} = \mathrm{ann}_R(m)$ for some $m \in M$, so $\dim(Rm) > 0$. But $Im = 0$, so $Rm \subseteq 0 :_M I \cong \mathrm{Hom}_R(R/I, M)$. Thus $\dim(\mathrm{Hom}_R(R/I, M)) > 0$, a contradiction. □

Proposition 3.2. *Let $I \subseteq \mathfrak{m}$ be an ideal and $n > 0$ an integer. Then the following are equivalent:*

(1) *$\ell(Ext^i_R(R/I, M)) < \infty$, for all $i < n$;*
(2) *I contains an M-filter regular sequence of length n.*

When $x_1, \dots, x_n \in I$ is an M-filter regular sequence,

$$Ext^n_R(R/I, M)_\mathfrak{p} \cong Hom_R(R/I, M/(x_1, \dots, x_n)M)_\mathfrak{p}, \text{ for any } \mathfrak{p} \in Spec(R) \setminus \{\mathfrak{m}\}.$$

Proof. Assume that $\ell(\mathrm{Ext}^i_R(R/I, M)) < \infty$, for all $i < n$. We use induction on n to show that I contains an M-filter regular sequence of length n. If $n = 1$, then I contains an M-filter regular element by Lemma 3.1. Now assume that $n > 1$ and the result is true for $n - 1$. Then, by Lemma 3.1 again, there is $x_1 \in I$ which is M-filter regular. From the short exact sequences

$$0 \to 0 :_M x_1 \to M \xrightarrow{x_1} x_1M \to 0,$$
$$0 \to x_1M \to M \to M/x_1M \to 0,$$

we get the long exact sequences

$$\begin{aligned} 0 \to \mathrm{Hom}_R(R/I, 0 :_M x_1) &\to \mathrm{Hom}_R(R/I, M) \xrightarrow{x_1} \mathrm{Hom}_R(R/I, x_1M) \\ &\to \mathrm{Ext}^1_R(R/I, 0 :_M x_1) \to \cdots, \\ 0 \to \mathrm{Hom}_R(R/I, x_1M) &\to \mathrm{Hom}_R(R/I, M) \to \mathrm{Hom}_R(R/I, M/x_1M) \\ &\to \mathrm{Ext}^1_R(R/I, x_1M) \to \cdots. \end{aligned}$$

As $\ell(0 :_M x_1) < \infty$, we have that $\ell(\mathrm{Ext}^i_R(R/I, 0 :_M x_1)) < \infty$ for any $i \geq 0$. Then we see that $\ell(\mathrm{Ext}^i_R(R/I, x_1M)) < \infty$ for all $i < n$ from the first long exact sequence. From the second long exact sequence, we get that $\ell(\mathrm{Ext}^i_R(R/I, M/x_1M)) < \infty$ for all $i < n - 1$. Then, by the induction assumption, there exist $x_2, \dots, x_n \in I$ which is an M/x_1M-filter regular sequence. Hence $x_1, x_2, \dots, x_n$ is an M-filter regular sequence.

Conversely, suppose that I contains an M-filter regular sequence of length n. Let $x_1, \dots, x_n \in I$ be an M-filter regular sequence. For any $\mathfrak{p} \in \operatorname{Supp}(M/IM) \setminus \{\mathfrak{m}\}$, $x_1/1, \dots, x_n/1$ is an $M_\mathfrak{p}$-regular sequence. Then, by a well-known property of $M_\mathfrak{p}$-regular sequences, we have

$$\operatorname{Ext}^i_{R_\mathfrak{p}}(R_\mathfrak{p}/IR_\mathfrak{p}, M_\mathfrak{p}) = 0, \text{ for all } i < n,$$

and

$$\operatorname{Ext}^n_{R_\mathfrak{p}}(R_\mathfrak{p}/IR_\mathfrak{p}, M_\mathfrak{p}) \cong \operatorname{Hom}_{R_\mathfrak{p}}(R_\mathfrak{p}/IR_\mathfrak{p}, M_\mathfrak{p}/(x_1/1, \dots, x_n/1)M_\mathfrak{p}).$$

Thus

$$\operatorname{Ext}^i_R(R/I, M)_\mathfrak{p} = 0, \text{ for all } i < n,$$
$$\operatorname{Ext}^n_R(R/I, M)_\mathfrak{p} \cong \operatorname{Hom}_R(R/I, M/(x_1, \dots, x_n)M)_\mathfrak{p}.$$

But, for any $\mathfrak{p} \notin \operatorname{Supp}(M/IM)$ and any $i \geq 0$, it is clear that

$$\operatorname{Ext}^i_R(R/I, M)_\mathfrak{p} = 0 \text{ and } \operatorname{Hom}_R(R/I, M/(x_1, \dots, x_n)M)_\mathfrak{p} = 0.$$

Hence $\ell(\operatorname{Ext}^i_R(R/I, M)) < \infty$ for all $i < n$ and for any $\mathfrak{p} \in \operatorname{Spec}(R) \setminus \{\mathfrak{m}\}$,

$$\operatorname{Ext}^n_R(R/I, M)_\mathfrak{p} \cong \operatorname{Hom}_R(R/I, M/(x_1, \dots, x_n)M)_\mathfrak{p}.$$

This completes the proof. □

If $x_1, \dots, x_n$ is a maximal M-filter regular sequence in I, then, by Lemma 3.1, $\dim(\operatorname{Hom}_R(R/I, M/(x_1, \dots, x_n)M)) > 0$. It follows from Proposition 3.2 that $\dim(\operatorname{Ext}^n_R(R/I, M)) > 0$. Hence any two maximal M-filter regular sequences in I (if any exist) have the same length.

Definition 3.3. Let I be a proper ideal of R. The f-depth (filter depth) of I on M is defined as the length of any maximal M-filter regular sequence in I, denoted by f-depth(I, M). Here, when the maximal M-filter regular sequence in I does not exist, we understand that the length is ∞.

Notice that f-depth$(I, M) = 0$ if and only if $I \subseteq \mathfrak{p}$ for some $\mathfrak{p} \in \operatorname{Ass}_R(M) \setminus \{\mathfrak{m}\}$ and if $x \in I$ is M-filter regular, then

$$\text{f-depth}(I, M) = \text{f-depth}(I, M/xM) + 1.$$

Furthermore, by Proposition 3.2, we have that

$$\text{f-depth}(I, M) = \min\{n | \dim(\operatorname{Ext}^n_R(R/I, M)) > 0\},$$

where, when $\dim(\operatorname{Ext}^n_R(R/I, M)) \leq 0$ for all $n \geq 0$, we understand the right side of the above equality to be ∞.

Proposition 3.4. *Let I, J be proper ideals of R. If $\sqrt{I} = \sqrt{J}$, then*

$$\textit{f-depth}(I, M) = \textit{f-depth}(J, M).$$

Proof. Let $x_1, \dots, x_n \in I$ be an M-filter regular sequence. Then, as $\sqrt{I} = \sqrt{J}$, there exists an integer $\alpha > 0$ such that $x_1^\alpha, \dots, x_n^\alpha \in J$. But since $x_1^\alpha, \dots, x_n^\alpha$ is also M-filter regular, we see that

$$\text{f-depth}(I, M) \leq \text{f-depth}(J, M).$$

Similarly,

$$\text{f-depth}(J, M) \leq \text{f-depth}(I, M).$$

Thus f-depth$(I, M) =$ f-depth(J, M). □

Suppose that $\dim(M/IM) > 0$. If $x \in I$ is M-filter regular, then $x \notin \mathfrak{p}$, for any $\mathfrak{p} \in \mathrm{Ass}_R(M)$ with $\dim(R/\mathfrak{p}) = \dim(M)$. Thus $\dim(M/xM) = \dim(M) - 1$. It follows that every M-filter regular sequence in I is a subsystem of parameters for M. Furthermore, we have the following:

Proposition 3.5. *If* $\dim(M/IM) > 0$*, then*

$$\text{depth}(I, M) \leq \text{f-depth}(I, M) \leq \text{ht}_M I,$$

where $\text{ht}_M I$ *is the infimum of lengths of strictly decreasing chains of prime ideals in* $\text{Supp}(M)$ *starting from a prime ideal containing* I*.*

Proof. As any M-regular sequence is an M-filter regular sequence, we see that $\text{depth}(I, M) \leq \text{f-depth}(I, M)$. So it remains to show that $\text{f-depth}(I, M) \leq \text{ht}_M(I)$.

By assumption, $\text{Supp}(M/IM) \not\subseteq \{\mathfrak{m}\}$. Let $x_1, \ldots, x_n \in I$ be any M-filter regular sequence. It is enough to show that $n \leq \text{ht}_M(\mathfrak{p})$ for any $\mathfrak{p} \in \text{Supp}(M/IM) \setminus \{\mathfrak{m}\}$. As $x_1, \ldots, x_n \in \mathfrak{p}$, $x_1/1, \ldots, x_n/1 \in R_\mathfrak{p}$ is an $M_\mathfrak{p}$-regular sequence. Hence $n \leq \dim(M_\mathfrak{p}) = \text{ht}_M(\mathfrak{p})$, as required. □

Proposition 3.6. *f-depth*$(I, M) = \infty$ *if and only if* I *contains a system of parameters for* M*.*

Proof. Notice that I containing a system of parameters for M is equivalent to $\dim(M/IM) = 0$. Thus, by Proposition 3.5, we only need to show that, when $\dim(M/IM) = 0$, for any integer $n > 0$ we can find an M-filter regular sequence of length n in I. But, in this case, we have that $\ell(\text{Ext}^i_R(R/I, M)) < \infty$ for all $i \geq 0$. Then the result follows from Proposition 3.2. □

Proposition 3.7. *Let* $V(I)$ *be the set of prime ideals containing* I*. Then*

$$\text{f-depth}(I, M) = \min\{\text{f-depth}(\mathfrak{p}, M) | \mathfrak{p} \in V(I)\}.$$

Proof. If $\dim(M/IM) = 0$, then $\text{f-depth}(I, M) = \infty$. But since the prime ideal containing I is just $\mathfrak{m}$ and $\text{f-depth}(\mathfrak{m}, M) = \infty$, the equality holds. Now assume that $\dim(M/IM) > 0$. Set $r = \min\{\text{f-depth}(\mathfrak{p}, M) | \mathfrak{p} \in V(I)\}$. As there is some $\mathfrak{p} \in V(I)$ such that $\dim(M/\mathfrak{p}M) > 0$, we have that $\text{f-depth}(\mathfrak{p}, M) < \infty$ hence, $r < \infty$. We use induction on r to show that $\text{f-depth}(I, M) = r$. If $r = 0$, then there exists a prime ideal $\mathfrak{p} \supseteq I$ such that $\text{f-depth}(\mathfrak{p}, M) = 0$. Thus, as $I \subseteq \mathfrak{p}$, $\text{f-depth}(I, M) = 0$. Suppose that $r > 0$. Then, for any $\mathfrak{p} \in \text{Ass}_R(M) \setminus \{\mathfrak{m}\}$, as $\text{f-depth}(\mathfrak{p}, M) = 0$ and $r > 0$, we have $I \not\subseteq \mathfrak{p}$. Hence $I \not\subseteq \bigcup_{\mathfrak{p} \in \text{Ass}_R(M) \setminus \{\mathfrak{m}\}} \mathfrak{p}$. Thus there exists $x_1 \in I$ which is M-filter regular. Set $M_1 = M/x_1M$. Then

$$\min\{\text{f-depth}(\mathfrak{p}, M_1) | \mathfrak{p} \in V(I)\} = \min\{\text{f-depth}(\mathfrak{p}, M) - 1 | \mathfrak{p} \in V(I)\} = r - 1.$$

Hence, by the induction assumption, $\text{f-depth}(I, M_1) = r - 1$, so $\text{f-depth}(I, M) = \text{f-depth}(I, M_1) + 1 = r$, as required. □

For any finitely generated R-module N, its $\mathfrak{m}$-adic completion is denoted by $\widehat{N}$. The following proposition states that f-depth does not change after passing to completion.

Proposition 3.8. *f-depth*$(I, M) =$ *f-depth*$(\widehat{I}, \widehat{M})$.

Proof. This is because, for any $i \geq 0$,

$$\dim(\mathrm{Ext}^i_R(R/I, M)) = \dim(\widehat{\mathrm{Ext}^i_R(R/I, M)}) = \dim(\mathrm{Ext}^i_{\widehat{R}}(\widehat{R}/\widehat{I}, \widehat{M})).$$

□

The following theorems give two characterizations of f-depth.

Theorem 3.9. *Let $y_1, \ldots, y_n \in I$ such that $I = (y_1, \ldots, y_n)$. Then*

$$\text{f-depth}(I, M) = n - \sup\{i | \dim(H_i(y_1, \ldots, y_n; M)) > 0\},$$

where, if there is no integer i with $\dim H_i(y_1, \ldots, y_n; M) > 0$, we understand that the right side of the above equality is ∞.

Proof. If $\dim(M/IM) = 0$, then f-depth$(I, M) = \infty$ and $\dim(H_i(y_1, \ldots, y_n; M)) \leq 0$ for any i (since $I \cdot H_i(y_1, \ldots, y_n; M) = 0$), so the theorem is true in this case. Now assume that $\dim(M/IM) > 0$. Let $r =$ f-depth(I, M). We use induction on r. If $r = 0$, then $I \subseteq \mathfrak{p}$ for some $\mathfrak{p} \in \mathrm{Ass}_R(M) \setminus \{\mathfrak{m}\}$. Thus $\mathfrak{p} = \mathrm{ann}_R(m)$ for some $m \in M$. As $Im = 0$, we see that $m \in 0 :_M I = H_n(y_1, \ldots, y_n; M)$. Then $\mathfrak{p} \in \mathrm{Ass}_R(H_n(y_1, \ldots, y_n; M))$. But since $\mathfrak{p} \neq \mathfrak{m}$, we have that $\dim(H_n(y_1, \ldots, y_n; M)) > 0$, and the equality holds. Suppose that $r > 0$. Let $x \in I$ be an M-filter regular element and $M_1 = M/xM$. Then, as f-depth$(I, M_1) = r - 1$, we have, by the induction assumption, that

$$\sup\{i | \dim(H_i(y_1, \ldots, y_n; M_1) > 0\} = n - r + 1.$$

Note that, as $\mathrm{Supp}(H_i(y_1, \ldots, y_n; M_1)) \subseteq \mathrm{Supp}(M/IM)$, the above equality is equivalent to $H_i(y_1, \ldots, y_n; M_1)_\mathfrak{p} = 0$ for all $i > n-r+1$ and any $\mathfrak{p} \in \mathrm{Supp}(M/IM) \setminus \{\mathfrak{m}\}$, and $H_{n-r+1}(y_1, \ldots, y_n; M_1)_\mathfrak{p} \neq 0$ for some $\mathfrak{p} \in \mathrm{Supp}(M/IM) \setminus \{\mathfrak{m}\}$.

For any $\mathfrak{p} \in \mathrm{Supp}(M/IM) \setminus \{\mathfrak{m}\}$, as $x \in \mathfrak{p}$, we see that $x/1$ is $M_\mathfrak{p}$-regular. From the short exact sequence

$$0 \to M_\mathfrak{p} \xrightarrow{x/1} M_\mathfrak{p} \to (M_1)_\mathfrak{p} \to 0,$$

we have a long exact sequence

$$\cdots \to H_i(y_1/1, \ldots, y_n/1; M_\mathfrak{p}) \xrightarrow{x/1} H_i(y_1/1, \ldots, y_n/1; M_\mathfrak{p})$$

$$\to H_i(y_1/1, \ldots, y_n/1; (M_1)_\mathfrak{p}) \to H_{i-1}(y_1/1, \ldots, y_n/1; M_\mathfrak{p}) \xrightarrow{x/1} \cdots.$$

As $H_i(y_1/1, \ldots, y_n/1; M_\mathfrak{p})$ is annihilated by $x/1$, the above long exact sequence is split into short exact sequences

$$0 \to H_i(y_1/1, \ldots, y_n/1; M_\mathfrak{p}) \to H_i(y_1/1, \ldots, y_n/1; (M_1)_\mathfrak{p})$$

$$\to H_{i-1}(y_1/1, \ldots, y_n/1; M_\mathfrak{p}) \to 0,$$

i.e.,

$$0 \to H_i(y_1, \ldots, y_n; M)_\mathfrak{p} \to H_i(y_1, \ldots, y_n; M_1)_\mathfrak{p} \to H_{i-1}(y_1, \ldots, y_n; M)_\mathfrak{p} \to 0.$$

Then $H_i(y_1, \ldots, y_n; M)_\mathfrak{p} = 0$ for any $i > n - r$ and any $\mathfrak{p} \in \mathrm{Supp}(M/IM) \setminus \{\mathfrak{m}\}$, and $H_{n-r}(y_1, \ldots, y_n; M)_\mathfrak{p} \neq 0$ for some $\mathfrak{p} \in \mathrm{Supp}(M/IM) \setminus \{\mathfrak{m}\}$. Hence $\sup\{i | \dim(H_i(y_1, \ldots, y_n; M)) > 0\} = n - r$. The theorem follows. □

Theorem 3.10 ([4, Theorem 3.1])**.** *For any proper ideal I of R,*

$$\text{f-depth}(I, M) = \min\{r | H^r_I(M) \text{ is not Artinian}\}.$$

Proof. If $\dim(M/IM) = 0$, then $\sqrt{I + \operatorname{ann}_R(M)} = \mathfrak{m}$, hence, $H^r_I(M) \cong H^r_{\mathfrak{m}}(M)$ is Artinian for any $r \geq 0$. Thus $\min\{r | H^r_I(M) \text{ is not Artinian}\} = \infty$. In this case, f-depth$(I, M) = \infty$ and the result is true.

Now we assume that $\dim(M/IM) > 0$. Let $n =$ f-depth(I, M). Then $n = \min\{i | \dim(\operatorname{Ext}^i_R(R/I, M)) > 0\}$. Note that $\dim(\operatorname{Ext}^i_R(R/I, M)) \leq 0$ for all $i < n$ and $\dim(\operatorname{Ext}^n_R(R/I, M)) > 0$ is equivalent to $\operatorname{Ext}^i_{R_\mathfrak{p}}(R_\mathfrak{p}/IR_\mathfrak{p}, M_\mathfrak{p}) = 0$ for any $i < n$ and any $\mathfrak{p} \in \operatorname{Supp}(M/IM) \setminus \{\mathfrak{m}\}$, but, $\operatorname{Ext}^n_{R_\mathfrak{p}}(R_\mathfrak{p}/IR_\mathfrak{p}, M_\mathfrak{p}) \neq 0$ for some $\mathfrak{p} \in \operatorname{Supp}(M/IM) \setminus \{\mathfrak{m}\}$; i.e., $\operatorname{depth}(IR_\mathfrak{p}, M_\mathfrak{p}) \geq n$ for any $\mathfrak{p} \in \operatorname{Supp}(M/IM) \setminus \{\mathfrak{m}\}$ and $\operatorname{depth}(IR_\mathfrak{p}, M_\mathfrak{p}) = n$ for some $\mathfrak{p} \in \operatorname{Supp}(M/IM) \setminus \{\mathfrak{m}\}$. It follows that

$$n = \min\{\operatorname{depth}(IR_\mathfrak{p}, M_\mathfrak{p}) | \mathfrak{p} \in \operatorname{Supp}(M/IM) \setminus \{\mathfrak{m}\}\}.$$

Then, by [4, Theorem 3.1], $n = \min\{r | H^r_I(M) \text{ is not Artinian}\}$. □

4. F-depth and f-modules

A finitely generated R-module M is called an f-module if every system of parameters for M is an M-filter regular sequence. F-modules were introduced in [1] as a generalization of Cohen-Macaulay modules.

The following theorem gives a characterization of f-modules by f-depth.

Theorem 4.1. *M is an f-module if and only if, for any $\mathfrak{p} \in$ Supp$(M) \setminus \{\mathfrak{m}\}$,*

$$\text{f-depth}(\mathfrak{p}, M) = \dim(M) - \dim(R/\mathfrak{p}).$$

Proof. Suppose that M is an f-module. Let $\mathfrak{p} \in \operatorname{Supp}(M) \setminus \{\mathfrak{m}\}$ and $x_1, \ldots, x_r$ be a maximal M-filter regular sequence in $\mathfrak{p}$. Then

$$\mathfrak{p} \subseteq \bigcup_{\mathfrak{q} \in \operatorname{Ass}_R(M/(x_1,\ldots,x_r)M) \setminus \{\mathfrak{m}\}} \mathfrak{q};$$

hence, $\mathfrak{p} \subseteq \mathfrak{q}$ for some $\mathfrak{q} \in \operatorname{Ass}_R(M/(x_1, \ldots, x_r)M) \setminus \{\mathfrak{m}\}$. As M is an f-module and $x_1, \ldots, x_r$ is a subsystem of parameters for M, we have, by [1, (2.5)], that $\dim(R/\mathfrak{q}) = \dim(M/(x_1, \ldots, x_r)M)$. But since $\mathfrak{p} \in \operatorname{Supp}(M/(x_1, \ldots, x_r)M)$, we see that $\mathfrak{p} = \mathfrak{q}$, hence

$$\text{f-depth}(\mathfrak{p}, M) = r = \dim(M) - \dim(R/\mathfrak{p}).$$

Conversely, suppose that

$$\text{f-depth}(\mathfrak{p}, M) = \dim(M) - \dim(R/\mathfrak{p})$$

for all $\mathfrak{p} \in \operatorname{Supp}(M) \setminus \{\mathfrak{m}\}$. We use induction on $d := \dim(M)$ to show that M is an f-module. The case $d = 0$ is trivial. Suppose that $d > 0$. Let $x_1, x_2, \ldots, x_d$ be a system of parameters for M. Then x_1 is M-filter regular, otherwise $x_1 \in \mathfrak{p}$ for some $\mathfrak{p} \in \operatorname{Ass}_R(M) \setminus \{\mathfrak{m}\}$, hence

$$\text{f-depth}(\mathfrak{p}, M) + \dim(R/\mathfrak{p}) = \dim(R/\mathfrak{p}) \leq \dim(M/x_1M) = \dim(M) - 1,$$

a contradiction. Set $M_1 = M/x_1M$. Then $\dim(M_1) = d - 1$ and $x_2, \ldots, x_d$ is a system of parameters for M_1 and, for any $\mathfrak{p} \in \operatorname{Supp}(M_1) \setminus \{\mathfrak{m}\}$, as $x_1 \in \mathfrak{p}$, we have that

$$\begin{aligned} \text{f-depth}(\mathfrak{p}, M_1) &= \text{f-depth}(\mathfrak{p}, M) - 1 = \dim(M) - \dim(R/\mathfrak{p}) - 1 \\ &= \dim(M_1) - \dim(R/\mathfrak{p}). \end{aligned}$$

Thus, by induction assumption, $x_2, \ldots, x_d$ is an M_1-filter regular sequence. Then $x_1, x_2, \ldots, x_d$ is an M-filter regular sequence and M is an f-module. □

Remark 4.2. Suppose that M is an f-module. Let I be a proper ideal of R such that $I \supseteq \mathrm{ann}_R(M)$ and $\sqrt{I} \neq \mathfrak{m}$. Then, by Proposition 3.7 and Theorem 4.1, we have that

$$\begin{aligned} \text{f-depth}(I, M) &= \min\{\text{f-depth}(\mathfrak{p}, M) | \mathfrak{p} \in \mathrm{Supp}(M/IM) \setminus \{\mathfrak{m}\}\} \\ &= \dim(M) - \max\{\dim(R/\mathfrak{p}) | \mathfrak{p} \in \mathrm{Supp}(M/IM) \setminus \{\mathfrak{m}\}\} \\ &= \dim(M) - \dim(R/I). \end{aligned}$$

Note that $\mathrm{ht}_M(I) \leq \dim(M) - \dim(R/I)$ and it follows from Proposition 3.5 that

$$\text{f-depth}(I, M) = \mathrm{ht}_M(I) = \dim(M) - \dim(R/I).$$

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Department of mathematics, Suzhou University, Suzhou 215006, People's Republic of China

Department of mathematics, Suzhou University, Suzhou 215006, People's Republic of China

E-mail address: zmtang@suda.edu.cn