PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 130, Number 7, Pages 2001–2005
S 0002-9939(01)06279-7
Article electronically published on November 15, 2001

ALMOST PERIODIC SOLUTIONS FOR UNDAMPED NONHOMOGENEOUS DELAY-DIFFERENTIAL EQUATIONS

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(Communicated by Carmen C. Chicone)

ABSTRACT. We first establish a result giving conditions that certain undamped delay differential equations with almost periodic time dependence have unique almost periodic solutions. Using this result we obtain conditions that a second order scalar nonlinear delay differential equation with almost periodic forcing will have a unique almost periodic solution having saddle-type stability properties. These results use the method of averaging.

In a previous paper [1], the author gave conditions that a second order scalar differential equation of the form

$$x'' + x - \varepsilon\nu x + \varepsilon^3 x^3 = f(t), \tag{1}$$

where ε and ν are positive constants and f is almost periodic (a.p. for short), will have an a.p. solution; cf. also [2]. It was found that this a.p. solution $x(t)$ is for ε and ν sufficiently small unique and has saddle point type stability, and in fact $\varepsilon x(t)$ tends to a linear combination of $\cos t$ and $\sin t$ as $\varepsilon \to 0$ uniformly for $t \in R$, the set of reals. This result was shown to be a special case of a result for a more general first order equation in R^n of the form

$$x' = (A + \varepsilon C(t))x + \varepsilon g(x, \varepsilon) + \varepsilon p(t), \tag{2}$$

where A and $C(t)$ are real $n \times n$ matrices with A similar to a diagonal matrix with pure imaginary entries, the entries of $C(t)$ are a.p., g and its first partial derivatives with respect to the components of x are continuous in (x, ε), and $p(t)$ is an R^n-valued a.p. function. For any $x \in R^n$, $|x|$ will denote some norm in R^n.

Our purpose in this paper is to show that if a time-delay term is introduced into (1), the same result will hold provided the magnitude of this term is sufficiently small, and will follow from a more general n-dimensional equation of the form

$$x' = (A + \varepsilon C(t))x + \varepsilon g(x, \varepsilon) + \varepsilon\alpha h(x_t) + \varepsilon p(t), \tag{3}$$

where $A, C(t), g$, and p are as before, ε and α are positive constants, and $h : C_r \to R^n$ is a function on C_r, the set of functions $\phi(\theta)$ continuous on $[-r, \theta]$ to R^n. For fixed t, and an R^n-valued function $x(t)$ continuous on R, x_t is the element of C_r given by $x(t + \theta), -r \le \theta \le 0$. We use the norm $\|\phi\|_r = \sup\{|\phi(\theta)| : -r \le \theta \le 0\}$ for $\phi \in C_r$. We assume h is uniformly continuous on bounded subsets of C_r.

Received by the editors February 14, 2000 and, in revised form, January 22, 2001.
2000 *Mathematics Subject Classification.* Primary 34K14.

As in [1], we use the method of averaging, but in a way somewhat simpler than the general method of averaging for delay equations discussed in [3] and also [4], in which conditions are required on the spectra of linear operators on C_r.

As in [1], we make the change of variables $e^{tA}y = x$ in (3) and easily arrive at the equation

$$y' = \varepsilon e^{-tA}[C(t)e^{tA}y + g(e^{tA}y, \varepsilon) + \alpha h(e^{A_t}y_t) + p(t)] \tag{4}$$

where $e^{A_t}y_t = e^{(t+\theta)A}y(t+\theta), -r \leq \theta \leq 0$, is a member of C_r for each $t \in R$.

The following lemmas will be proved using essentially the same arguments as were used in the proof of Theorem 1 in [1]. We use the notation

$$m_t(F(t)) = \lim_{T\to\infty} T^{-1} \int_0^T F(t)dt$$

for any function $F : R \to R^n$ which is a.p.

Lemma 1. *Let* $C_0 = m_t(e^{-tA}C(t)e^{tA})$,

$$g_0(y) = m_t(e^{-tA}g(e^{tA}y, 0)),$$
$$b_0(\alpha, z) = m_t(e^{-tA}p(t) + \alpha h(e^{A_t}z_t)),$$

where $y \in R^n$, *and* z *is an a.p.* R^n *valued function. Suppose*

$$C_0 y + g_0(y) + b_0(0, z) = 0 \tag{4.1}$$

has a solution $\bar{y}$ *such that all the eigenvalues of* $C_0 + \partial g_0/\partial y(\bar{y})$ *have nonzero real parts; we call such matrices noncritical. Fix* $b_1 > |\bar{y}|$. *Then if* $z(t)$ *is any* R^n*-valued a.p. function such that* $|z(t)| \leq b_1$ *for* $t \in R$, *there exist* $\varepsilon_0 > 0$ *and* $\alpha_0 > 0$ *such that if* $0 < \varepsilon \leq \varepsilon_0,\ 0 < \alpha \leq \alpha_0$, *the equation*

$$y' = \varepsilon e^{-tA}[C(t)e^{tA}y + g(e^{tA}y, \varepsilon) + \alpha h(e^{A_t}z_t) + p(t)] \tag{4z}$$

will, for such a $z(t)$, *have a unique a.p. solution* $\bar{y}(t, \varepsilon, \alpha, z)$ *such that*

$$\bar{y}(t, \varepsilon, \alpha, z) \to \bar{y} \text{ as } (\varepsilon, \alpha) \to (0, 0) \text{ uniformly for } t \in R. \tag{5}$$

Proof. We sketch a proof since it is quite similar to the proof of Theorem 1 in [1]. First, it can be shown that $C_0, g_0(y)$, and $b_0(\alpha, z)$ are well defined since the arguments of m_t which define them are a.p., provided $z(t)$ is a.p.; we omit the details. We note that $b_0(0, z)$ is independent of z so that $\bar{y}$ is also independent. From the conditions on $C_0 + \partial g/\partial y(\bar{y})$, it follows that by the Implicit Function Theorem, there exists a $\alpha_0 > 0$ such that if $0 \leq \alpha \leq \alpha_0$,

$$C_0 y + g_0(y) + b_0(\alpha, z) = 0 \tag{4.1α}$$

has a solution $\bar{y}(\alpha, z)$ such that $\bar{y}(\alpha, z) \to \bar{y}$ as $\alpha \to 0$ uniformly for $z(t)$ such that $|z(t)| \leq b_1$ where $b_1 > 0$ is as above. For such $z(t)$, it also follows that for α_0 sufficiently small,

$$C_0 + \frac{\partial g_0}{\partial y}(\bar{y}\ \alpha, z) \tag{4.2α}$$

is noncritical for $0 \leq \alpha \leq \alpha_0$.

By the classical (first order) method of averaging (cf. [3]), our lemma will follow as in the proof of Theorem 1 in [1]. We will, however, include some of the details which will be used in the proof of Theorem 1 in this paper. First, there exists an R^n-valued function $U(t, u, \varepsilon, \alpha, z)$ a.p. in t uniformly for (u, a, α) in compact sets of $R^n \times (0,\infty) \times [0,\infty]$ and fixed $z(t)$ a.p. in t with $|z(t)| \le b_1, t \in R$, such that the change of variable defined by

$$y = u + \varepsilon U(t,u,\varepsilon,\alpha,z) \tag{6}$$

takes (4z) into

$$u' = \varepsilon f_0(u,\varepsilon,\alpha,z) + \varepsilon F(t,u,\varepsilon,\alpha,z) \tag{7}$$

where $f_0(u,\varepsilon,\alpha,z) = C_0 u + g_0(u) + b_0(\alpha,z)$ and F satisfies a local Lipschitz condition in u, and $F(t,u,\alpha,0,z) = 0$. Since (4.1α) holds for $y = \bar{y}(\alpha,z), z$ as indicated and $0 \le \alpha \le \alpha_0$, then if $u = v + \bar{y}(\alpha,z)$, (7) goes into

$$v' = \varepsilon A_0 v + \varepsilon q(t,v,\varepsilon,\alpha,z) \tag{8}$$

where $A_0 = C_0 + \partial g_0/\partial y(\bar{y}(\alpha,z))$, and for ε, z, and α as suitably restricted. Since A_0 is then noncritical, (8) has a unique a.p. solution $v(t,\varepsilon,\alpha,z) \to 0$ as $(\varepsilon,\alpha) \to (0,0)$ uniformly for $t \in R$. □

If we denote by AP the set of all a.p. R^n-valued functions $z(t)$ and use the norm $\|z\| = \sup\{|z(t)| : t \in R\}, AP$ is a Banach space over the reals. Let $AP(b_1)$ be the subset of AP consisting of functions $z(t)$ with $\|z\| \le b_1$. We shall show that if h is Lipchitzian, the mapping $\Phi : AP(b_1) \to AP(b_1)$ defined by the a.p. solution $y(t,\varepsilon,\alpha,z)$ of (4z) is a contraction for ε and α sufficiently small and this will prove

Theorem 1. *Assume that the hypotheses of Lemma* 1 *hold and also that for each bounded set $B \subset C_r$ there exists a constant $L(B) > 0$ such that for $\phi \in B, \psi \in B, |h(\phi) - h(\psi)| \le L(B)\|\phi - \psi\|_r$. Then there exist $\varepsilon_0 > 0, \alpha_0 > 0$ such that for $0 < \varepsilon \le \varepsilon_0, 0 < \alpha \le \alpha_0$, (3) has a unique a.p. solution $\bar{x}(t,\varepsilon,\alpha)$ such that $\bar{x}(t,\varepsilon,\alpha) \to e^{tA}\bar{y}$ as $(\varepsilon,\alpha) \to (0,0)$ uniformly for $t \in R$; here $\bar{y}$ is as defined in Lemma* 1.

Proof. Because of (5) and the condition $b_1 > |\bar{y}|$, we may assume α_0 and ε_0 so small that $\Phi(AP(b_1)) \subset AP(b_1), \Phi$ as defined above. Let $z_i(t) \in AP(b_1), i = 1,2$; let $\bar{u}(t,\varepsilon,\alpha,z_i)$ be the unique a.p. function corresponding to $\bar{y}(t,\varepsilon,\alpha,z_i)$, the unique a.p. solution of $(4z_i)$ given by Lemma 1, $i = 1,2$. Let $\bar{v}(t,\varepsilon,\alpha,z_i)$ be the solution of (8) given by $\bar{v}(t,\varepsilon,\alpha,z_i) = \bar{u}(t,\varepsilon,\alpha,z_i) - \bar{y}(\alpha,z_i)$. Then for ε_0 and α_0 sufficiently small, there exists a $\rho, 0 < \rho < 1$, and a $K > 0$ such that

$$\|\bar{v}_1 - \bar{v}_2\| \le \rho\|z_1 - z_2\|, \tag{8.1}$$

$$\|\bar{u}_1 - \bar{u}_2\| \le \|\bar{v}_1 - \bar{v}_2\|, \quad \text{and} \tag{8.2}$$

$$\|\bar{y}_1 - \bar{y}_2\| \le \|\bar{u}_1 - \bar{u}_2\| + \varepsilon K\|\bar{u}_1 - \bar{u}_2\|; \tag{8.3}$$

cf. (6). Here we have used the simpler notation $\bar{v}_i = \bar{v}(t,\varepsilon,\alpha,z_i)$, etc. So from (8.1)-(8.3) we get

$$\|\bar{y}_1 - \bar{y}_2\| \le \rho(1 + \varepsilon_0 K)\|z_1 - z_2\|,$$

which for ε_0 sufficiently small clearly shows that Φ is a contraction, the unique fixed point of which is an a.p. solution $\bar{y}(t,\varepsilon,\alpha)$ of (4). By our conditions on $A, x = \bar{x}(t,\varepsilon,\alpha) = e^{tA}\bar{y}(t,\varepsilon,\alpha)$ is an a.p. solution of (3). □

We now consider the second order scalar equation

$$x'' + x - \varepsilon\nu x + \varepsilon^3 x^3 - \varepsilon\alpha\ell(x_t) = f(t) \tag{1.1}$$

where ε, ν, α are positive constants, f is a.p., and let $\ell(\phi) : C_r \to R$ be linear on C_r. Multiplying (1.1) by ε and putting $\varepsilon x = y$, we get

$$y'' + y - \varepsilon\nu y + \varepsilon y^3 - \alpha\varepsilon\ell(y_t) = \varepsilon f(t). \tag{1.2}$$

Since these equations are perturbations of the corresponding scalar equations in [1], the method to apply Theorem 1 to (1.2) is quite similar to the one used in [1], but we include most of the details here for the sake of clarity.

A system in R^2 equivalent to (1.2) is

(1.3)
$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \varepsilon\begin{pmatrix} 0 & 0 \\ \nu & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} - \varepsilon\begin{pmatrix} 0 \\ x_1^3 \end{pmatrix} - \varepsilon\alpha\begin{pmatrix} 0 \\ \ell(x_t) \end{pmatrix} + \varepsilon\begin{pmatrix} 0 \\ f(t) \end{pmatrix}$$

where $x_1 = y,\ x_2 = y'$.

If we put

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad g = \begin{pmatrix} 0 \\ x_1^3 \end{pmatrix},$$
$$h(\phi) = \begin{pmatrix} 0 \\ \ell(\phi) \end{pmatrix}, \quad p(t) = \begin{pmatrix} 0 \\ f(t) \end{pmatrix}, \quad C(t) = \begin{pmatrix} 0 & 0 \\ \nu & 0 \end{pmatrix},$$

then (1.3) becomes (3) with $n = 2$.

It follows by direct calculation that

$$C_0 = m_t(e^{-tA}C(t)e^{tA}) = \frac{\nu}{2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$
$$g_0(x) = m_t(e^{-tA}g(e^{tA}x)) = \frac{3}{8}\begin{pmatrix} (x_1^2 + x_x^2)x_2^2 \\ -(x_1^2 + x_2^2)x_1 \end{pmatrix},$$
$$b_0(\alpha, z) = m_t\left(e^{-tA}\begin{pmatrix} 0 \\ f(t) + \alpha\ell(z_t) \end{pmatrix}\right) = \begin{pmatrix} -b_1 - \alpha d_1(z) \\ b_2 + \alpha d_2(z) \end{pmatrix},$$
where $b_1 = m_t(f(t)\sin t), b_2 = m_t(f(t)\cos t)$,
$d_1 = m_t(\ell(z_t)\sin t, d_2 = m_t(\ell(z_t)\cos t)$,

and $z(t)$ is any a.p. function such that $|z(t)| \leq b_1$ for $t \in R, b_1$ as in Lemma 1.
We now use Theorem 1. Condition (4.1) becomes

$$\begin{aligned} &-4\nu x_2 + 3(x_1^2 + x_2^2)x_2 - 4b_1 = 0, \\ &4\nu x_1 - 3(x_1^2 + x_2^2)x_1 + 4b_2 = 0. \end{aligned} \tag{4.11}$$

If $b_1 = b_2 = 0$, the solution $(x_1 x_2) = (0, 0)$ of (4.11) is not of interest since there $\partial g_0/\partial x$ is the zero matrix, and since $C_0 = \frac{\nu}{2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ is clearly critical, our theorem does not apply.

Suppose $b_1 = 0, b_2 > 0$ and $\nu > (9b_2/2)^{\frac{2}{3}}$. Then by routine analysis (4.11) has a real solution $(\bar{x}_1, 0), \bar{x}_1 < 0, 4\nu/9 < \bar{x}_1^2 < 4\nu/3$. By direct calculation, it then can be verified that the matrix

$$C_0 + \partial g_0/\partial y(\bar{x})$$

with $\bar{x} = (\bar{x}_1, 0)$ has 2 nonzero eigenvalues of opposite sign (cf. [1]) and so is noncritical. So Theorem 1 applies and there exist $\alpha_0 > 0, \varepsilon_0 > 0$ such that (1.3) has an a.p. solution $(x_1(t, \varepsilon, \alpha), x_2(t, \varepsilon, \alpha))$ for $0 < \alpha \leq \alpha_0, 0 < \varepsilon \leq \varepsilon_0$, such that $x_1(t, \varepsilon, \alpha) \to \bar{x}_1 \cos t$ and $x_2(t, \varepsilon, \alpha) \to x_1 \sin t$ as $(\varepsilon, \alpha) \to (0, 0)$ uniformly for $t \in R$.

Clearly the cases $b_2 < 0, b_1 = 0$, and $b_2 = 0, |b_1| > 0$ can be handled similarly; cf. [1], [2]. We thus have the following

Theorem 2. *Let* $b_1 b_2 = 0, |b_1 + b_2| = \mu > 0$. *Then if* $\nu > (9\mu/2)^{\frac{2}{3}}$ *there exist* ε_0 *and* α_0 *such that for* $0 < \varepsilon \leq \varepsilon_0, 0 < \alpha \leq \alpha_0$, (1.1) *has a unique a.p. solution* $x(t, \varepsilon, \alpha)$ *such that*

$$\varepsilon x(t, \varepsilon, \alpha) \to (\bar{x}_1 \cos t + \bar{x}_2 \sin t)$$

as $(\varepsilon, \alpha) \to (0, 0)$ *uniformly for* $t \in R$. *Here* $\bar{x}_1 \neq 0, \bar{x}_2 = 0$ *if* $b_1 = 0, b_2 \neq 0$, *and* $\bar{x}_1 = 0, \bar{x}_2 \neq 0$ *if* $b_1 \neq 0, b_2 = 0$, *and* $(\bar{x}_1, \bar{x}_2)$ *solves* (4.11).

In terms of the real Fourier series for f,

$$A_0 + \sum_{i=1}^{\infty} A_i \cos \lambda_i t + B_i \sin \lambda_i t, \tag{1.4}$$

we can easily obtain the following:

Corollary. *If* $\lambda_1 = 1, B_1 \neq 0$, *then for* α *and* ε *sufficiently small and* $\nu > (9|B_1|/4)^{2/3}$, (1.1) *will have a unique a.p. solution* $\bar{x}(t, \varepsilon, \alpha) \to \bar{x}_1 \cos t$ *as* $(\varepsilon, \alpha) \to (0, 0)$ *uniformly for* $t \in R$.

Other results are also easily obtained. For example, if $\nu = \varepsilon^{-1}$, and α and ε are sufficiently small, we can infer the existence of a unique a.p. solution of the special case of (1.1):

$$x'' + \varepsilon^3 x^3 + \varepsilon \alpha \ell(x_t) = f(t). \tag{1.5}$$

We may also assume even in the general case of Theorem 1, that $r = \infty$; i.e., $h(\phi)$ is a function on C_∞, the space of functions continuous and bounded on $(-\infty, 0]$ with the usual supremum norm; i.e., the infinite delay case.

An open question is whether α can be deleted from (2) and (1.1) and replaced by some other condition on h in (2) or ℓ in (1.1), or in general, whether such results as Theorem 1 can be obtained in cases where the delay term is not small.

Finally, it may be observed that equations such as (1.1) can model a kind of oscillator with a control that depends linearly on the state y not only at time t, but at a previous time $t = r$, the delay being a consequence of the fact that the control is situated at a distance from the oscillator. If the forcing is the sum of periodic functions with incommensurate periods, such as, for example, $a \cos t + b \cos \pi t$, it might be of interest to see if for sufficiently small linearities and controls, the oscillator will display an almost periodic time dependence.

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