

## ON COVERING MAPPINGS ON SOLENOIDS

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ABSTRACT. Covering mappings on the dyadic solenoid are studied. Some results stated by Zhou Youcheng (2000) are discussed in a more detailed way by indicating certain inaccuracies in their proofs. These are either corrected or supplemented, or else suitable counterexamples are constructed. Some open questions are asked and connections with related results are considered.

In this paper a *space* always means a metric space, and a *continuum* is defined as a compact connected space. A *mapping* means a continuous function. We denote by  $\mathbb{N}$  the set of all positive integers, by  $\mathbb{C}$  the space of all complex numbers (equipped with the natural topology), and by  $\mathbb{S}^1$  the unit circle, i.e.,  $\mathbb{S}^1 = \{z \in \mathbb{C} : |z| = 1\}$ . As usual,  $i$  means the imaginary unit, i.e.,  $i^2 = -1$ , and  $\exp$  stands for the exponential function, i.e.,  $\exp(i\alpha) = \cos \alpha + i \sin \alpha$ .

A surjective mapping  $f : X \rightarrow Y$  between continua  $X$  and  $Y$  is called:

- *open* provided that open subsets of the domain are mapped onto open subsets of the range;
- *k-to-1* (where  $k \in \mathbb{N}$ ) provided that  $\text{card } f^{-1}(y) = k$  for each  $y \in Y$ ;
- a *local homeomorphism* provided that for each  $x \in X$  there exists a neighborhood  $U \subset X$  of  $x$  such that  $f(x) \in \text{int } f(U)$  and  $f|U : U \rightarrow f(U)$  is a homeomorphism;
- a *covering mapping* provided that for each  $y \in Y$  there exists a neighborhood  $V \subset Y$  of  $y$  such that  $f^{-1}(V)$  is the disjoint union of open subsets of  $X$  each of which is mapped homeomorphically onto  $V$  under  $f$ .

The following proposition is known (see [9, p. 199]; compare [10, Proposition 4, p. 1067]).

**Proposition 1.** *For any mapping  $f : X \rightarrow Y$  between continua the conditions below are equivalent:*

- 1°.  *$f$  is a  $k$ -to-1 open mapping for some  $k \in \mathbb{N}$ ;*
- 2°.  *$f$  is a local homeomorphism;*
- 3°.  *$f$  is a covering mapping.*

Recall that the number  $k$  in the above proposition is called a *degree* of the covering mapping (see e.g. [9, p. 199]).

The following notation will be used. Given an inverse sequence  $\{X_n, f_n\}$  of compact spaces  $X_n$  with bonding mappings  $f_n : X_{n+1} \rightarrow X_n$ , where the set of positive integers  $\mathbb{N}$  is taken as the directed set of indices, we denote by  $X_\infty = \varprojlim \{X_n, f_n\}$  its inverse limit.

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Let two inverse sequences  $\mathbf{S} = \{X_n, f_n\}$  and  $\mathbf{T} = \{Y_n, g_n\}$  be given such that there is a sequence  $\mathbf{h} = \{h_n : n \in \mathbb{N}\}$  called a *mapping* between  $\mathbf{S}$  and  $\mathbf{T}$  (so  $\mathbf{h} : \mathbf{S} \rightarrow \mathbf{T}$ ) satisfying

$$(2) \quad h_n \circ f_n = g_n \circ h_{n+1}.$$

Then there exists a mapping  $h_\infty : X_\infty \rightarrow Y_\infty$  defined by  $h_\infty(\langle x_n \rangle) = \langle h_n(x_n) \rangle$ , and it is called the *limit mapping* induced by  $\mathbf{h}$  (compare e.g. [7, Exercise 2.22, p. 26]).

Recall that a diagram

$$(3) \quad \begin{array}{ccc} X' & \xleftarrow{f} & X \\ h' \downarrow & & \downarrow h \\ Y' & \xleftarrow{g} & Y \end{array}$$

is said to be *exact* (or *bi-commutative* in [5, §3, IV, p. 193]) if it *commutes*, i.e.,  $h' \circ f = g \circ h$ , and if

$$(4) \quad h'(x') = g(y) \text{ implies } h^{-1}(y) \cap f^{-1}(x') \neq \emptyset \text{ for every } x' \in X' \text{ and } y \in Y.$$

A mapping  $\mathbf{h} : \mathbf{S} \rightarrow \mathbf{T}$  is said to be *exact* provided that the diagram

$$(5) \quad \begin{array}{ccc} X_n & \xleftarrow{f_n} & X_{n+1} \\ h_n \downarrow & & \downarrow h_{n+1} \\ Y_n & \xleftarrow{g_n} & Y_{n+1} \end{array}$$

is exact for each  $n \in \mathbb{N}$  (see [8, p. 58]). The next quoted result is shown in [3, Theorem 3, p. 58].

**Proposition 6.** *Let an exact mapping  $\mathbf{h} : \mathbf{S} \rightarrow \mathbf{T}$  between inverse sequences  $\mathbf{S}$  and  $\mathbf{T}$  be given. If each  $h_n : X_n \rightarrow Y_n$  is open, then the limit mapping  $h_\infty : X_\infty \rightarrow Y_\infty$  is open as well.*

A *solenoid* means the inverse limit of an inverse sequence of circles  $X_n = \mathbb{S}^1$  with open mappings  $f_n$  for each  $n \in \mathbb{N}$ . Further, if  $f_n(z) = z^2$  for each  $z \in \mathbb{S}^1$  and each  $n \in \mathbb{N}$ , then the solenoid is said to be *dyadic* and is denoted by  $\Sigma$ .

For any  $k \in \mathbb{N}$  let  $p^k : \mathbb{S}^1 \rightarrow \mathbb{S}^1$  denote the  $k$ -th potency mapping, i.e.,  $p^k(z) = z^k$  for each  $z \in \mathbb{S}^1$ . Note that  $p^k$  is a covering mapping for any  $k$ .

For each  $n \in \mathbb{N}$  and for some fixed  $k \in \mathbb{N}$  put

$$X_n = Y_n = \mathbb{S}^1, \quad f_n = g_n = p^2, \quad h_n = p^k,$$

and consider the following diagram:

$$(7) \quad \begin{array}{ccccccc} \mathbb{S}^1 & \xleftarrow{p^2} & \mathbb{S}^1 & \xleftarrow{p^2} & \mathbb{S}^1 & \xleftarrow{p^2} & \dots & \Sigma \\ p^k \downarrow & & \downarrow p^k & & \downarrow p^k & & \dots & \downarrow h_\infty^k \\ \mathbb{S}^1 & \xleftarrow{p^2} & \mathbb{S}^1 & \xleftarrow{p^2} & \mathbb{S}^1 & \xleftarrow{p^2} & \dots & \Sigma \end{array}$$

The next result is formulated as Lemma 1 of [10, p. 1067].

**Proposition 8.** *For each  $k \in \mathbb{N}$  the limit mapping  $h_\infty^k : \Sigma \rightarrow \Sigma$  in diagram (7) is a covering mapping.*

The proof of Proposition 8, as given in [10, p. 1067], is not correct. Namely it is said that any homeomorphism  $h$  of the unit circle  $\mathbb{S}^1$  may be lifted under the mapping  $h_n = p^k$ , i.e., there exists a mapping  $h' : \mathbb{S}^1 \rightarrow \mathbb{S}^1$  such that  $h = h_n \circ h'$ . However, this is possible in the trivial case  $k = 1$  only. Indeed, for any point  $z \in \mathbb{S}^1$  we have  $h_n^{-1}(z) = h'(h^{-1}(z))$ , whence the cardinalities of both members of the equality are the same. But  $\text{card } h_n^{-1}(z) = k$ , while  $\text{card } h'(h^{-1}(z)) = 1$ .

To present another proof of this proposition we need several statements. Consider the basic subdiagram of diagram (7):

$$(9) \quad \begin{array}{ccc} \mathbb{S}^1 & \xleftarrow{p^2} & \mathbb{S}^1 \\ p^k \downarrow & & \downarrow p^k \\ \mathbb{S}^1 & \xleftarrow{p^2} & \mathbb{S}^1 \end{array}$$

**Statement 10.** *Diagram (9) is exact if and only if  $k$  is odd.*

*Proof.* Assume  $k$  is odd. Let  $z_1, z_2 \in \mathbb{S}^1$  satisfy  $p^k(z_1) = p^2(z_2)$ . We have to show, according to (4), that  $(p^2)^{-1}(z_1) \cap (p^k)^{-1}(z_2) \neq \emptyset$ . Define  $z_0 = z_1^k = z_2^2$  and observe that the  $2k$ -element set  $Z_0 = (p^2)^{-1}((p^k)^{-1}(z_0)) = (p^k)^{-1}((p^2)^{-1}(z_0))$  divides  $\mathbb{S}^1$  into  $2k$  equal parts. Since  $z_1 \in (p^k)^{-1}(z_0)$ , the set  $Z_1 = (p^2)^{-1}(z_1)$  is a 2-point subset of  $Z_0$  consisting of two opposite points. Similarly, since  $z_2 \in (p^2)^{-1}(z_0)$ , the set  $Z_2 = (p^k)^{-1}(z_2)$  is a  $k$ -point subset of  $Z_0$  that divides  $\mathbb{S}^1$  into  $k$  equal parts. Note that  $Z_2$  does not contain any pair of opposite points because  $k$  is odd. Thus  $Z_1 \cap Z_2 \neq \emptyset$ , as needed.

Assume now that diagram (9) is exact. Let

$$z_0 = -1, \quad z_1 = \exp\left(\frac{3\pi}{k}i\right), \quad z_2 = i, \quad z_3 = \exp\left(\frac{3\pi}{2k}i\right), \quad z_4 = \exp\left(\frac{\pi}{2k}i\right), \quad \xi = \exp\left(\frac{2\pi}{k}i\right).$$

Then  $z_1^k = z_2^2 = z_0$ . Further,

$$\begin{aligned} (p^2)^{-1}(z_1) &= \{z_3, -z_3\}, \quad \text{and} \\ (p^k)^{-1}(z_2) &= \{z_4\xi^j : j \in \{0, 1, \dots, k-1\}\}. \end{aligned}$$

Since  $(p^2)^{-1}(z_1) \cap (p^k)^{-1}(z_2) \neq \emptyset$  by assumption (4), it follows that for some  $j \in \{0, 1, \dots, k-1\}$  either  $\arg z_3 = \arg(z_4\xi^j)$  or  $\arg(-z_3) = \pi + \arg z_3 = \arg(z_4\xi^j)$ , i.e.,

$$\frac{3\pi}{2k} = \begin{cases} \frac{\pi}{2k} + j \cdot \frac{2\pi}{k} & \text{in the first case,} \\ -\pi + \frac{\pi}{2k} + j \cdot \frac{2\pi}{k} & \text{in the second case.} \end{cases}$$

The first case leads to  $3 = 1 + 4j$ , i.e.,  $j = \frac{1}{2}$ , which is impossible. In the second case we have  $3 = -2k + 1 + 4j$ , i.e.,  $k = 2j - 1$ , so  $k$  is odd as needed. The proof is complete. □

Accept the following notation. Let

$$(11) \quad \gamma_j^k = \exp\left(\frac{2j\pi}{k}i\right) \quad \text{for } k \in \mathbb{N} \text{ and } j \in \{0, 1, \dots, k-1\}.$$

In other words,  $\gamma_j^k$  means the  $k$ -th root of 1 satisfying  $\arg \gamma_j^k = j \cdot \frac{2\pi}{k}$ . Further, put

$$(12) \quad \Gamma^k(1) = \{\gamma_j^k : j \in \{0, 1, \dots, k-1\}\},$$

where  $\gamma_j^k$  are as in (11).

If  $k$  is odd, define a function  $\varphi : \Gamma^k(1) \rightarrow \Gamma^k(1)$  by

$$(13) \quad \varphi(\gamma_j^k) = \begin{cases} \exp(\frac{j}{2k} \cdot 2\pi i) & \text{if } j \in \{0, 1, \dots, k-1\} \text{ and } j \text{ is even,} \\ \exp(\frac{k+j}{2k} \cdot 2\pi i) & \text{if } j \in \{0, 1, \dots, k-1\} \text{ and } j \text{ is odd.} \end{cases}$$

It is evident that  $\varphi$  is well defined and that

$$(14) \quad \text{for each odd } k \in \mathbb{N} \text{ and for each } \gamma \in \Gamma^k(1), \text{ the value } \varphi(\gamma) \text{ is the only element of } \Gamma^k(1) \text{ such that } (\varphi(\gamma))^2 = \gamma.$$

**Statement 15.** *If  $k$  is odd, then the limit mapping  $h_\infty^k$  in diagram (7) is open and  $k$ -to-1, and thus it is a covering mapping.*

*Proof.* The openness is a consequence of Statement 10 and Proposition 6. We will show that  $h_\infty^k$  is  $k$ -to-1. To this aim note that since all mappings  $h_n = p^k$  are surjections, the limit mapping  $h_\infty^k$  it is a surjection, too (see [2, Theorem 3.2.14, p. 142]). Let  $w = (w_1, w_2, \dots) \in \Sigma$ . Thus there exists a point  $v = (v_1, v_2, \dots) \in \Sigma$  such that  $h_\infty^k(v) = w$ . Let  $\Gamma^k(1)$  be defined as in (12). Then, for each  $n \in \mathbb{N}$  the set of  $k$ -th roots of  $w_n \in \mathbb{S}^1$  is  $\Gamma^k(w_n) = \{v_n \cdot \gamma_j^k : j \in \{0, 1, \dots, k-1\}\}$ .

Let  $\varphi : \Gamma^k(1) \rightarrow \Gamma^k(1)$  be defined by (13). Then (14) holds. For a nonnegative integer  $m$  denote by  $\varphi^m$  the  $m$ -th iteration of  $\varphi$ , with  $\varphi^0$  as the identity.

Let  $x = (x_1, x_2, \dots) \in (h_\infty^k)^{-1}(w)$ .

CLAIM. There exist  $j \in \{0, 1, \dots, k-1\}$  such that for each  $n \in \mathbb{N}$  we have  $x_n = v_n \cdot \varphi^{n-1}(\gamma_j^k)$ .

We proceed by induction. If  $n = 1$ , then  $x_1 \in (p^k)^{-1}(w_1) = \Gamma^k(w_1)$ , thus  $x_1 = v_1 \cdot \gamma_j^k = v_1 \cdot \varphi^0(\gamma_j^k)$  for some  $j \in \{0, 1, \dots, k-1\}$ .

Assume that the equality in the claim holds for a certain  $j \in \{0, 1, \dots, k-1\}$  and for some  $n \in \mathbb{N}$ . By the definition  $x_{n+1} \in \Gamma^k(w_{n+1})$ , hence  $x_{n+1} = v_{n+1} \cdot \gamma_l^k$  for some  $l \in \{0, 1, \dots, k-1\}$ . Then

$$v_{n+1}^2 \cdot (\gamma_l^k)^2 = x_{n+1}^2 = x_n = v_n \cdot \varphi^{n-1}(\gamma_j^k)$$

by the inductive hypothesis. Since  $v_{n+1}^2 = v_n$ , it follows that  $(\gamma_l^k)^2 = \varphi^{n-1}(\gamma_j^k)$ . By the definition of  $\varphi$  we conclude that  $\gamma_l^k = \varphi(\varphi^{n-1}(\gamma_j^k)) = \varphi^n(\gamma_j^k)$ . Consequently,  $x_{n+1} = v_{n+1} \varphi^n(\gamma_j^k)$ , and thus the claim is proved.

It follows from the claim that if  $x \in (h_\infty^k)^{-1}(w)$ , then

$$x = (v_1 \gamma_j^k, v_2 \varphi(\gamma_j^k), v_3 \varphi^2(\gamma_j^k), \dots) \quad \text{for some } j \in \{0, 1, \dots, k-1\}.$$

Since  $\gamma_j^k \in \Gamma^k(1)$  and  $\text{card } \Gamma^k(1) = k$ , the set  $(h_\infty^k)^{-1}(w)$  therefore has exactly  $k$  elements. Since  $w \in \Sigma$  was chosen as an arbitrary element of  $\Sigma$ , we see that  $h_\infty^k$  is a  $k$ -to-1 mapping, as needed.

Therefore  $h_\infty^k$  is a covering mapping according to Proposition 1. □

The following fact is an easy observation.

**Fact 16.** *If  $k = r \cdot s$  for some  $r, s \in \mathbb{N}$ , then for the limit mapping  $h_\infty^k : \Sigma \rightarrow \Sigma$  in diagram (7) we have*

$$h_\infty^k = h_\infty^r \circ h_\infty^s.$$

*Proof.* Indeed, let  $x = (x_1, x_2, x_3, \dots) \in \Sigma$ . Then

$$\begin{aligned} (h_\infty^r \circ h_\infty^s)(x) &= h_\infty^r(h_\infty^s((x_1, x_2, x_3, \dots))) = h_\infty^r((p^s(x_1), p^s(x_2), p^s(x_3), \dots)) \\ &= h_\infty^r(((x_1)^s, (x_2)^s, (x_3)^s, \dots)) = (((x_1)^s)^r, ((x_2)^s)^r, ((x_3)^s)^r, \dots) \\ &= ((x_1)^k, (x_2)^k, (x_3)^k, \dots) = h_\infty^k(x). \end{aligned}$$

□

**Statement 17.** For  $k = 2$  the limit mapping  $h_\infty^2 : \Sigma \rightarrow \Sigma$  in diagram (7) is a homeomorphism.

*Proof.* Let  $x = (x_1, x_2, x_3, \dots) \in \Sigma$ . Then  $x_{n+1}^2 = x_n$  for each  $n \in \mathbb{N}$ , whence  $h_\infty^2(x) = (x_1^2, x_1, x_2, x_3, \dots)$ . Define an auxiliary mapping  $h : \Sigma \rightarrow \Sigma$  by  $h(x) = (x_2, x_3, x_4, \dots)$ . Then  $h$  is well defined, and it is the inverse of  $h_\infty^2$ . Thus  $h_\infty^2$  is a bijection, and being continuous and defined on a compact space, it is a homeomorphism. □

Thus, as a consequence, we infer the following.

**Statement 18.** If  $k = 2^m$  for some  $m \in \mathbb{N}$ , then the limit mapping  $h_\infty^k : \Sigma \rightarrow \Sigma$  in diagram (7) is a homeomorphism.

*Proof of Proposition 8.* If  $k$  is odd, then the limit mapping  $h_\infty^k$  is a covering by Statement 15.

If  $k$  is even, then  $k = l \cdot 2^m$  for some  $l, m \in \mathbb{N}$  with  $l$  being odd. Hence, by Fact 16, the limit mapping  $h_\infty^k$  is the composition of  $h_\infty^l$ , which is a covering mapping by Statement 15, and of  $h_\infty^{2^m}$ , which is a homeomorphism by Statement 18. Thus the conclusion follows. □

Proposition 8 has been proved to be applied as a lemma (viz. [10, Lemma 1, p. 1067]) to show the next quoted result; see [10, Theorem 1, p. 1068]. The mentioned theorem should, however, be formulated in the following form, which is less general than that in [10]. This form corresponds to the proof of the theorem as given in that paper.

**Theorem 19.** The dyadic solenoid admits any odd degree  $k$  covering mapping of the form  $h_\infty^k$ , while it does not admit any covering mapping  $h_\infty^k$  of an even degree  $k$ .

Observe however, that Proposition 8 is not mentioned in the proof of the theorem as it is written in [10, p. 1068]. Instead, the author of [10] writes: “Since  $\Sigma$  is a homogeneous space, hence any fiber has a cardinal  $k$ .” This argument is not correct: the homogeneity of a space does not imply that all fibers of a mapping have the same cardinality. Indeed, if  $f : \mathbb{S}^1 \rightarrow \mathbb{S}^1$  is a monotone mapping which shrinks a fixed (nondegenerate) arc of the domain to a singleton of the range, then both domain and range are homogeneous, while fibers do not have the same cardinality. An argument which is proper in this place is just Proposition 8.

Theorem 19 is related to [1, Corollary 2, p. 282] which states that there does not exist any exactly 2-to-1 mapping defined on the dyadic solenoid. In connection with these two results the following questions seem to be natural and interesting.

**Question 20.** Let  $k > 2$  be an even integer. Is it true that there does not exist any exactly  $k$ -to-1 mapping defined on the dyadic solenoid?

Note that we do not assume openness of the mapping in the above question.

By a *graph* we mean a 1-dimensional polyhedron. Let  $f, h : G \rightarrow G$  be open surjective mappings of a graph  $G$  onto itself such that  $h$  is a covering mapping of a degree  $k \in \mathbb{N}$ , and that the diagram

$$\begin{array}{ccc} G & \xleftarrow{f} & G \\ h \downarrow & & \downarrow h \\ G & \xleftarrow{f} & G \end{array}$$

is exact. Further, let  $\mathbf{S} = \{X_n, f_n\}$  be an inverse sequence such that  $X_n = G$  and  $f_n = f$  for each  $n \in \mathbb{N}$ , and let  $X_\infty = \varprojlim \mathbf{S}$ , and  $\mathbf{h} : \mathbf{S} \rightarrow \mathbf{S}$  be the mapping between the inverse sequences. Denote by  $h_\infty : X_\infty \rightarrow X_\infty$  the limit mapping induced by  $\mathbf{h}$ .

**Questions 21.** a) For what graphs  $G$  and numbers  $k$  the limit mapping  $h_\infty$  is a covering mapping of degree  $k$ ?

b) For what graphs  $G$  and numbers  $k$  the inverse limit space  $X_\infty$  admits a covering mapping onto itself?

Note that the above results (Proposition 8, Statement 15 and Theorem 19) give partial answers to these questions (for  $G = \mathbb{S}^1$  and for odd  $k$ ). Observe also that the graphs  $G$  considered in Questions 21 have to be cyclic, because if they are acyclic (i.e. trees), then  $X_\infty$  is a tree-like continuum, and any local homeomorphism (so any covering mapping) on such continuum is a homeomorphism (see [6, Corollary, p. 67]; for an even stronger result see [4, Theorem, p. 2572]).

Two mappings  $f : X \rightarrow Y$  and  $g : X \rightarrow Y$  are said to be *homotopic* provided that there is a mapping  $H : X \times [0, 1] \rightarrow Y$  (called a *homotopy that joins  $f$  and  $g$* ) such that

$$H(x, 0) = f(x) \text{ and } H(x, 1) = g(x) \text{ for each } x \in X.$$

We will write  $f \sim g$  if  $f$  is homotopic to  $g$ .

The following situation is discussed in Theorem 2 of [10, p. 1068]. Let  $\mathbf{S} = \{X_n, f_n\}$  be an inverse sequence such that  $X_n = \mathbb{S}^1$  and  $f_n = p^2$  for each  $n \in \mathbb{N}$ . Let mappings  $\mathbf{h} : \mathbf{S} \rightarrow \mathbf{S}$  and  $\mathbf{g} : \mathbf{S} \rightarrow \mathbf{S}$  be such that for some odd positive integer  $k$

$$h_n = p^k \text{ and } g_n \sim h_n \text{ for each } n \in \mathbb{N}.$$

In other words, we consider again diagram (7), but now, besides the vertical down arrows that correspond to mappings  $h_n = p^k$  we have other ones (also down) that correspond to  $g_n : \mathbb{S}^1 \rightarrow \mathbb{S}^1$  such that

$$g_n \text{ is a covering mapping, and } g_n \sim p^k \text{ for each } n.$$

Under these assumptions it is said in Theorem 2 of [10, p. 1068] that the limit mapping  $g_\infty$  induced by  $\mathbf{g}$  is homotopic to the limit mapping  $h_\infty^k$  induced by  $\mathbf{h}$ , and it is also a covering mapping of degree  $k$ . More precisely, the conclusions of the quoted theorem are the following:

- (a) the mapping  $\mathbf{g} : \mathbf{S} \rightarrow \mathbf{S}$  induces a limit mapping  $g_\infty : \Sigma \rightarrow \Sigma$ ;
- (b) the limit mapping  $g_\infty$  is homotopic to  $h_\infty^k$ ;
- (c)  $g_\infty$  is a covering mapping of degree  $k$ .

Below we give an example showing that, contrary to (a), the sequence  $\{g_n : n \in \mathbb{N}\}$  need not induce any limit mapping.

**Example 22.** Let  $\mathbf{S}$  be the inverse sequence described above. There is a sequence of covering mappings  $g_n : \mathbb{S}^1 \rightarrow \mathbb{S}^1$  such that  $g_n \sim \text{id}|_{\mathbb{S}^1}$  for all  $n \in \mathbb{N}$ , and that no limit mapping is induced by this sequence.

*Proof.* Let  $k = 1$ . Thus  $h_n = p^1 = \text{id}|_{\mathbb{S}^1}$  and  $h_\infty^1 = \text{id}|_{\Sigma}$ . Define  $g_n : \mathbb{S}^1 \rightarrow \mathbb{S}^1$  and  $H : \mathbb{S}^1 \times [0, 1] \rightarrow \mathbb{S}^1$  by

$$g_n(z) = iz \quad \text{for each } n \in \mathbb{N},$$

$$H(z, t) = z \exp\left(\frac{\pi t}{2}i\right) \quad \text{for each } (z, t) \in \mathbb{S}^1 \times [0, 1].$$

Thus  $g_n$  is a rotation of  $\mathbb{S}^1$  by  $\frac{\pi}{2}$  (hence it is a covering mapping) and  $g_n \sim h_n$  for each  $n \in \mathbb{N}$ , but the diagram

$$(23) \quad \begin{array}{ccc} \mathbb{S}^1 & \xleftarrow{p^2} & \mathbb{S}^1 \\ g_n \downarrow & & \downarrow g_{n+1} \\ \mathbb{S}^1 & \xleftarrow{p^2} & \mathbb{S}^1 \end{array}$$

does not commute. The value of the point  $(1, 1, 1, \dots) \in \Sigma$  under the sequence  $\{g_n : n \in \mathbb{N}\}$  is  $(i, i, i, \dots)$  which is not a point of  $\Sigma$ . Therefore the considered sequence does not induce any limit mapping.  $\square$

*Remark 24.* Example 22 shows that the assumptions made in [10, Theorem 2, p. 1068] do not suffice for the existence of the limit mapping induced by the sequence  $\{g_n : n \in \mathbb{N}\}$ . Therefore, the existence of  $g_\infty$  should be assumed in the theorem.

But even if the assertion (a) is included in the assumptions of Theorem 2 of [10], the proof of (b), as presented in [10], is not correct. Namely the needed homotopy between the limit mappings  $h_\infty^k$  and  $g_\infty$  is supposed to be constructed inductively: if one has a homotopy  $H_1 : \mathbb{S}^1 \times [0, 1] \rightarrow \mathbb{S}^1$  such that  $H_1(z, 0) = h_1(z)$  and  $H_1(z, 1) = g_1(z)$ , and if the existence of a suitable homotopy  $H_n : \mathbb{S}^1 \times [0, 1] \rightarrow \mathbb{S}^1$  is assumed for some  $n \in \mathbb{N}$ , it is claimed that one can define a homotopy  $H_{n+1} : \mathbb{S}^1 \times [0, 1] \rightarrow \mathbb{S}^1$  such that

$$(25) \quad (H_{n+1}(z, t))^2 = H_n(z^2, t) \quad \text{for each } (z, t) \in \mathbb{S}^1 \times [0, 1].$$

This is not true, as the following example illustrates.

**Example 26.** Let  $\mathbf{S}$  be the inverse sequence described above and, for an arbitrary odd positive integer  $k$ , let the mapping  $\mathbf{h} : \mathbf{S} \rightarrow \mathbf{S}$  be defined by  $h_n = p^k$  for each  $n \in \mathbb{N}$ . Then there is a mapping  $\mathbf{g} : \mathbf{S} \rightarrow \mathbf{S}$  such that, for each  $n \in \mathbb{N}$ :

- (27)  $g_n$  is a covering mapping;
- (28) there exists a homotopy  $H_n$  that joins  $g_n$  and  $h_n$  (so  $g_n \sim h_n$  for all  $n$ );
- (29) there is no homotopy  $H_{n+1}$  joining  $g_{n+1}$  and  $h_{n+1}$  for which equality (25) holds.

*Proof.* For each  $n \in \mathbb{N}$  define  $g_n : \mathbb{S}^1 \rightarrow \mathbb{S}^1$  by  $g_n(z) = z^k \exp(i\pi(2 - \frac{1}{2^n}))$ , and observe that each  $g_n$  is a covering mapping and that diagram (23) commutes.

For each  $n \in \mathbb{N}$  define  $H_n : \mathbb{S}^1 \times [0, 1] \rightarrow \mathbb{S}^1$  by

$$(30) \quad H_n(z, t) = z^k \exp(i\pi t(2 - \frac{1}{2^n})) \quad \text{for each } z \in \mathbb{S}^1 \text{ and } t \in [0, 1].$$

Then  $H_n(z, 0) = z^k = h_n(z)$ , and  $H_n(z, 1) = g_n(z)$  for each  $z \in \mathbb{S}^1$ . Therefore  $H_n$  is a homotopy between  $g_n$  and  $h_n$ . Thus  $g_n \sim h_n$ , and (28) holds.

To show (29) it is enough to verify that

(31) if  $G_{n+1} : \mathbb{S}^1 \times [0, 1] \rightarrow \mathbb{S}^1$  is an arbitrary homotopy that joins  $h_{n+1}$  and  $g_{n+1}$ , then there is a number  $t_0 \in [0, 1]$  such that  $(G_{n+1}(1, t_0))^2 \neq H_n(1, t_0)$ .

Indeed, note that, by (30),

$$H_n(1, t) \in H_n(\{1\} \times [0, 1]) = \{\exp(i\pi\theta) : \theta \in [0, 2 - \frac{1}{2^n}]\} \text{ for each } t \in [0, 1],$$

whence it follows that

$$g_{n+1}(1) = \exp(i\pi(2 - \frac{1}{2^{n+1}}))$$

does not belong to  $H_n(\{1\} \times [0, 1])$ , i.e.,

$$(32) \quad g_{n+1}(1) \neq H_n(1, t) \text{ for each } t \in [0, 1].$$

Further, let  $A$  and  $B$  be the two subarcs of the circumference  $\mathbb{S}^1$  such that  $A \cup B = \mathbb{S}^1$  and  $A \cap B = \{h_{n+1}(1) = 1, g_{n+1}(1)\}$ .

On the other hand, if  $\omega$  and  $-\omega$  denote the two square roots of  $g_{n+1}(1)$ , then

$$A \cap \{\omega, -\omega\} \neq \emptyset \neq B \cap \{\omega, -\omega\}.$$

Since  $G_{n+1}(\{1\} \times [0, 1])$  is arcwise connected and contains both points  $h_{n+1}(1) = 1$  and  $g_{n+1}(1)$ , either  $A$  or  $B$  is contained in  $G_{n+1}(\{1\} \times [0, 1])$ , and thus

$$G_{n+1}(\{1\} \times [0, 1]) \cap \{\omega, -\omega\} \neq \emptyset.$$

Therefore it follows that there is a number  $t_0 \in [0, 1]$  such that  $G_{n+1}(1, t_0) \in \{\omega, -\omega\}$ . Therefore  $(G_{n+1}(1, t_0))^2 = g_{n+1}(1)$ , whence  $(G_{n+1}(1, t_0))^2 \neq H_n(1, t_0)$  according to (32). So (31) follows and the proof is complete.  $\square$

*Remark 33.* Example 26 shows that the *methods* chosen in the proof of [10, Theorem 2, p. 1068] do not assure the existence of any homotopy that joins the two limit mappings between the dyadic solenoids  $\Sigma$ . Therefore a natural question arises of whether the existence of such a homotopy as it is claimed in (b) can be proved using some other methods. Our next example shows that this is not the case.

**Example 34.** Let  $\mathbf{S}$  be the inverse sequence described above and, for an arbitrary odd positive integer  $k$ , let the mapping  $\mathbf{h} : \mathbf{S} \rightarrow \mathbf{S}$  be defined by  $h_n = p^k$  for each  $n \in \mathbb{N}$ . Then there is a mapping  $\mathbf{g} : \mathbf{S} \rightarrow \mathbf{S}$  such that:

- (35)  $g_n$  is a covering mapping for each  $n \in \mathbb{N}$ ;
- (36) the mapping  $\mathbf{g} : \mathbf{S} \rightarrow \mathbf{S}$  induces a limit mapping  $g_\infty : \Sigma \rightarrow \Sigma$ ;
- (37)  $g_n \sim h_n$  for each  $n \in \mathbb{N}$ ;
- (38) the limit mapping  $g_\infty$  is not homotopic to  $h_\infty^k$ .

*Proof.* Let  $e = (1, 1, 1, \dots)$  be the identity in  $\Sigma$ . Fix a point  $x \in \Sigma$  such that  $x$  is not in the component of  $\Sigma$  containing the point  $e$ . Then  $x$  can be written in the form

$$x = (\exp(i\theta_1), \exp(i\theta_2), \exp(i\theta_3), \dots) \text{ with } \theta_n = 2\theta_{n+1} \text{ for each } n \in \mathbb{N}.$$

For each  $n \in \mathbb{N}$  define  $g_n : \mathbb{S}^1 \rightarrow \mathbb{S}^1$  by  $g_n(z) = z^k \exp(i\theta_n)$ . Since  $h_n = p^k$  is a covering mapping and  $g_n$  is the composition of  $h_n$  with a homeomorphism (in fact, a rotation)  $z \mapsto z \exp(i\theta_n)$ , it follows that  $g_n$  is a covering mapping, i.e., (35) holds.

Consider again diagram (23) and note that it commutes for each  $n \in \mathbb{N}$  (simply by the definition of  $g_n$ ). Thus (36) follows.

For each  $n \in \mathbb{N}$  a mapping  $H_n : \mathbb{S}^1 \times [0, 1] \rightarrow \mathbb{S}^1$  given by  $H_n(z, t) = z^k \exp(it\theta_n)$  is a homotopy between  $h_n$  and  $g_n$ , so (37) is true.

To verify (38) suppose on the contrary that there is a homotopy  $H : \Sigma \times [0, 1] \rightarrow \Sigma$  that joins  $h_\infty^k$  and  $g_\infty$ . Notice that

$$h_\infty^k(e) = e \text{ and } g_\infty(e) = (\exp(i\theta_1), \exp(i\theta_2), \exp(i\theta_3), \dots) = x.$$

Thus  $H(\{e\} \times [0, 1])$  is a locally connected subcontinuum of  $\Sigma$ , so it is an arc. Moreover, the points  $e$  and  $x$  belong to this arc, whence it follows that  $x$  is in the same component of  $\Sigma$  to which  $e$  belongs, a contradiction with the choice of  $x$ . Thus (38) is shown. The proof is complete.  $\square$

*Remark 39.* Examples 22 and 34 show that, to obtain (c), we have to join conditions (a) and (b) to the *assumptions* of Theorem 2 in [10].

Recall that, given a space  $X$ , a mapping  $f : X \rightarrow X$ , and an  $n \in \mathbb{N}$ , the symbol  $f^n : X \rightarrow X$  stands for the  $n$ -th iteration of  $f$ . A point  $x \in X$  is said to be a *periodic* point of  $f$  provided that there is  $n \in \mathbb{N}$  such that  $f^n(x) = x$ ; then  $n$  is called the *period* of  $x$  under  $f$ .

Proposition 9 of [10, p. 1070] states that the set of periodic points of  $h_\infty^k$  (see diagram (7) above) is dense in  $\Sigma$ . As a proof of the result it is said that “the proof is a direct checking from the definition of the mapping” and the definition of the bonding ones. Below we present a more complete proof of this result.

**Proposition 40** ([10, Proposition 9, p. 1070]). *Let  $k \in \mathbb{N}$  be given. If  $h_\infty^k : \Sigma \rightarrow \Sigma$  is the limit mapping induced by  $h_n = p^k$  for each  $n \in \mathbb{N}$  as in diagram (7), then the set of all periodic points of  $h_\infty^k$  is dense in  $\Sigma$ .*

*Proof.* Consider the inverse sequence

$$(41) \quad \mathbb{S}^1 \xleftarrow{p^2} \mathbb{S}^1 \xleftarrow{p^2} \mathbb{S}^1 \xleftarrow{p^2} \dots \Sigma$$

where each factor space  $X_n$  is  $\mathbb{S}^1$  and each bonding mapping  $f_n$  is  $p^2$ .

If  $k = 1$ , then  $h_n$  is the identity for each  $n$ , so  $h_\infty^k$  is the identity on  $\Sigma$ , and thus any point of  $\Sigma$  is a fixed point, so a periodic one, and the set of the periodic points of  $h_\infty^1$  is the whole  $\Sigma$ .

Let  $k > 1$ , and let  $\pi_n : \Sigma \rightarrow X_n = \mathbb{S}^1$  be the  $n$ -th projection mapping in the inverse sequence (41). Choose a basic open subset  $V$  of  $\Sigma$ . This means that there is  $n \in \mathbb{N}$  and an open set  $U_n$  in  $X_n = \mathbb{S}^1$  such that  $V = \pi_n^{-1}(U_n)$ . Let  $m \in \mathbb{N}$  be such that:

- 1)  $m$  is a prime;
- 2)  $m > k$ ;
- 3) there exists a point  $z_n \in U_n \subset X_n = \mathbb{S}^1$  such that  $z_n \in \Gamma^m(1)$ ,

where  $\Gamma^m(1)$  denotes the set of  $m$ -th roots of 1 as in (12). Note that  $m > k > 1$  and condition 1) imply that  $m$  is odd. So, if the function  $\varphi : \Gamma^m(1) \rightarrow \Gamma^m(1)$  is defined as in (13), we have  $(\varphi(z))^2 = z$  for each  $z \in \Gamma^m(1)$  according to (14).

Define

$$\zeta = (z_n^{2^{n-1}}, \dots, z_n^2, z_n, \varphi(z_n), \varphi^2(z_n), \dots).$$

Thus it follows from (14) that  $\zeta \in \Sigma$ . Furthermore, by the definition of  $z_n$  and by the definition (13) of  $\varphi$  (see condition 3) above), it follows that

$$(42) \quad \zeta^m = (1, 1, 1, \dots) = e$$

( $e$  is the identity of  $\Sigma$ ). Since  $z_n \in U_n \subset X_n = \mathbb{S}^1$  again by 3), we infer that

$$(43) \quad \zeta \in V.$$

Since  $m > k$  is a prime by 1) and 2), we can apply Fermat's Theorem (saying that if an integer  $q$  is not divisible by a prime  $m$ , then the number  $q^{m-1} - 1$  is divisible by  $m$ ) to get  $k^{m-1} \equiv 1 \pmod{m}$ , i.e.,  $m | (k^{m-1} - 1)$ . Since (42) implies  $\zeta^{k^{m-1}-1} = e$ , we obtain  $(h_\infty^k)^{m-1}(\zeta) = \zeta^{k^{m-1}} = \zeta$ . Consequently,  $\zeta$  is a point of period  $m - 1$  under  $h_\infty^k$ . Hence by (43) the conclusion follows, and thus the proof is complete.  $\square$

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