

## ON THE LOEWNER PROBLEM IN THE CLASS $\mathbf{N}_\kappa$

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ABSTRACT. Loewner's theorem on boundary interpolation of  $\mathbf{N}_\kappa$  functions is proved under rather general conditions. In particular, the hypothesis of Alpay and Rovnyak (1999) that the function  $f$ , which is to be extended to an  $\mathbf{N}_\kappa$  function, is defined and continuously differentiable on a nonempty open subset of the real line, is replaced by the hypothesis that the set on which  $f$  is defined contains an accumulation point at which  $f$  satisfies some kind of differentiability condition. The proof of the theorem in this note uses the representation of  $\mathbf{N}_\kappa$  functions in terms of selfadjoint relations in Pontryagin spaces and the extension theory of symmetric relations in Pontryagin spaces.

### 1. INTRODUCTION AND MAIN RESULT

Recall that the generalized Nevanlinna class  $\mathbf{N}_\kappa$  consists of all functions  $N$  which are meromorphic in the upper half plane  $\mathbb{C}^+$  and such that the kernel

$$(1) \quad \frac{N(z) - N(w)^*}{z - w^*}, \quad z, w \in \mathcal{D}(N),$$

has  $\kappa$  negative squares; here  $\mathcal{D}(N)$  denotes the domain of holomorphy of  $N$ . A function  $N \in \mathbf{N}_\kappa$  is always being extended to the lower half plane by symmetry:

$$(2) \quad N(z^*) = N(z)^*, \quad z \in \mathcal{D}(N),$$

and also into those points of the real axis where it can be continued analytically. For  $\kappa = 0$  the class  $\mathbf{N}_0$  consists of all functions  $N$  which are holomorphic in  $\mathbb{C}^+$  and such that  $\frac{\operatorname{Im} N(z)}{\operatorname{Im} z} \geq 0$ ,  $z \in \mathbb{C}^+$ . If  $N \in \mathbf{N}_\kappa$  by  $\mathcal{E}_N$  we denote the set of poles of  $N$  in  $\mathbb{C}^+$  and of generalized poles of  $N$  of nonpositive type on the real axis  $\mathbb{R}$ , the total multiplicity of all these poles or generalized poles of nonpositive type is  $\kappa$ ; see [KL4] and [L].

The function  $N$  belongs to the class  $\mathbf{N}_\kappa$  for some integer  $\kappa \geq 0$  if and only if it has a representation of the form

$$(3) \quad N(z) = N(\mu)^* + (z - \mu^*) \langle (I + (z - \mu)(A - z)^{-1})u, u \rangle, \quad z \in \rho(A),$$

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where  $A = A^*$  is a selfadjoint linear relation with nonempty resolvent set  $\rho(A)$  in a Pontryagin space  $(\Pi, \langle \cdot, \cdot \rangle)$ ,  $\mu \in \rho(A)$ , and  $u \in \Pi$ . The representation can be chosen  $u$ -minimal which means that

$$\overline{\text{span}}\{(I + (z - \mu)(A - z)^{-1})u \mid z \in \rho(A)\} = \Pi.$$

In that case,  $N$  belongs to the class  $\mathbf{N}_\kappa$  with  $\kappa = \text{ind}_-\Pi$ ,  $\mathcal{D}(N) = \rho(A)$ , the representation (3) is unique up to isomorphisms, and the set  $\mathcal{E}_N$  coincides with  $\sigma_0(A) \cap (\mathbb{C}^+ \cup \mathbb{R})$ , where  $\sigma_0(A)$  is the set of all eigenvalues of  $A$  with a nonpositive eigenvector; for these notions and results see, for example, [KL1], [DLS1], [DLS2], and [L].

In this note we study the Loewner problem in the class  $\mathbf{N}_\kappa$ : Given a real set  $\mathcal{A}$ , values  $f(s)$  and certain numbers  $\widehat{f}(s)$  for  $s \in \mathcal{A}$ , find conditions such that the function  $f$  can be extrapolated to a function of the class  $\mathbf{N}_\kappa$ . It turns out that the numbers  $\widehat{f}(s)$  are the derivatives  $f'(s)$  or at least bounds for them. In the classical case  $\kappa = 0$  this problem was studied in [Loe], [BS], [K], [SK] (see [D] for a nice account of these papers), and [RR], and the case for generalized Nevanlinna functions was considered in [AR]. Our method follows the method of [K], [SK], [KL2], [KL3], and [DL] which means that the problem is reduced to an extension problem for a symmetric linear relation in a Pontryagin space. In [SK] (where the case  $\kappa = 0$  is considered)  $\mathcal{A}$  is the open interval  $(-1, 1)$ ,  $f$  is continuous there and  $f$  is differentiable on a subset  $\mathcal{A}'$  of  $\mathcal{A}$  containing 0 and having Lebesgue measure 2. In [AR]  $\mathcal{A}$  is an open set, the function  $f$  is continuously differentiable there and  $\widehat{f} = f'$ , and the problem for  $\kappa > 0$  is reduced to the case  $\kappa = 0$ . We also mention the paper [ADL], where for  $\kappa = 0$  the case of a countable set  $\mathcal{A}$  is considered.

If  $\mathcal{A}$  is a subset of the real line by  $\mathcal{A}^i$  we denote the set of its inner points. Furthermore, for  $s \in \mathbb{R}$ , the notation  $z \widehat{\rightarrow} s$  means that  $z$  tends nontangentially to  $s$ . The main result of this note is the following theorem. In the abstract above we alluded to hypothesis 2.

**Theorem.** *Let  $\mathcal{A}$  be a subset of  $\mathbb{R}$  and let  $f, \widehat{f} : \mathcal{A} \rightarrow \mathbb{R}$  be two real functions such that:*

1. *For  $s, t \in \mathcal{A}$  the kernel*

$$K(s, t) := \begin{cases} \frac{f(t) - f(s)}{t - s}, & t \neq s, \\ \widehat{f}(t), & t = s, \end{cases}$$

*has  $\kappa$  negative squares.*

2. *The set  $\mathcal{A}$  contains an accumulation point  $t_0$  such that  $f$  and  $\widehat{f}$  are continuous in  $t_0$  and*

$$\lim_{s \rightarrow t_0, s \in \mathcal{A}} \frac{f(s) - f(t_0)}{s - t_0} = \widehat{f}(t_0).$$

*Then either there exists a unique function  $N \in \bigcup_{\kappa'=0}^{\kappa} \mathbf{N}_{\kappa'}$ , or there exist infinitely many functions  $N \in \bigcup_{\kappa'=0}^{\kappa} \mathbf{N}_{\kappa'}$ , such that for  $s \in \mathcal{A} \setminus \mathcal{E}_N$  the relations*

$$(4) \quad \lim_{z \widehat{\rightarrow} s} N(z) = f(s), \quad \lim_{z \widehat{\rightarrow} s} \frac{\text{Im } N(z)}{\text{Im } z} \leq \widehat{f}(s)$$

*hold. If  $\mathcal{A}^i \neq \emptyset$  the first case prevails, that is, the function  $N$  is uniquely determined. If  $f$  is bounded on a compact nonempty subinterval  $\Delta$  of  $\mathcal{A}^i$ , then  $\Delta \cap \mathcal{E}_N = \emptyset$  and  $f$  is holomorphic on  $\Delta$ .*

If  $\mathcal{A}$  is a nonempty open interval,  $f$  is continuously differentiable on  $\mathcal{A}$  and we choose  $\widehat{f}(s) = f'(s)$ ,  $s \in \mathcal{A}$ , then there exists a unique generalized Nevanlinna function  $N$  such that (4) holds;  $N$  belongs to the class  $\mathbf{N}_\kappa$  and is holomorphic on  $\mathcal{A}$ . This statement, which is the main result of [AR], follows from the last claim of the theorem.

2. PROOF OF THE THEOREM

For the convenience of the reader we start with the following simple lemma (compare [LS, Lemma 1.1]).

**Lemma.** *If  $N \in \mathbf{N}_\kappa$  and for all  $s$  in some open interval  $\Delta \subset \mathbb{R} \setminus \mathcal{E}_N$  the limit  $\lim_{z \xrightarrow{s} z} \frac{\operatorname{Im} N(z)}{\operatorname{Im} z}$  is finite, then  $N$  is holomorphic on  $\Delta$ .*

*Proof.* The function  $N$  can be decomposed as

$$N(z) = N_0(z) + \int_{\Delta'} \frac{d\tau(\lambda)}{\lambda - z},$$

where  $\Delta'$  is an open interval whose closure is contained in  $\Delta$ ,  $\tau$  is a bounded nondecreasing function on  $\Delta'$  and  $N_0$  is a function of the class  $\mathbf{N}_\kappa$  which is holomorphic on  $\Delta'$  (see [KL2, Satz 3.1]). According to the assumption of the Lemma we have for all  $t \in \Delta'$

$$\int_{\Delta'} \frac{d\tau(\lambda)}{|\lambda - t|^2} < \infty.$$

For an arbitrary point  $\mu \in \Delta'$ ,  $\mu \neq t$ , we have

$$\left| \int_t^\mu \frac{d\tau(\lambda)}{|\lambda - t|^2} \right| \leq \int_{\Delta'} \frac{d\tau(\lambda)}{|\lambda - t|^2}$$

and, consequently,

$$\frac{|\tau(\mu) - \tau(t)|}{|\mu - t|^2} \leq \left| \int_t^\mu \frac{d\tau(\lambda)}{|\lambda - t|^2} \right| \leq \int_{\Delta'} \frac{d\tau(\lambda)}{|\lambda - t|^2}.$$

It follows that  $\tau(\mu) - \tau(t) = O((\mu - t)^2)$  if  $\mu \rightarrow t$ , and hence  $\tau$  is constant on  $\Delta'$ . □

In order to prove the Theorem, we denote by  $\Pi_\kappa$  the reproducing kernel Pontryagin space generated by the kernel  $K$ . This space can be realized as follows. Consider the linear space of all functions on  $\mathcal{A}$  of the form  $\sum_{s \in \mathcal{A}} K(s, \cdot)c_s$ ,  $c_s \in \mathbb{C}$ , where always in this proof in the sums which run over  $\mathcal{A}$  only for finitely many points  $s$  of  $\mathcal{A}$  the complex numbers  $c_s$  are different from zero. This space is equipped with the inner product

$$\left\langle \sum_{s \in \mathcal{A}} K(s, \cdot)c_s, \sum_{t \in \mathcal{A}} K(t, \cdot)d_t \right\rangle := \sum_{s, t \in \mathcal{A}} d_t^* K(s, t)c_s,$$

which, according to the assumption, has  $\kappa$  negative squares, and  $\Pi_\kappa$  is its Pontryagin space completion (see [ADRS]).

We define the linear relation  $S$  in  $\Pi_\kappa$  by

$$S := \left\{ \left\{ \sum_{s \in \mathcal{A}} K(s, \cdot)c_s, \sum_{s \in \mathcal{A}} sK(s, \cdot)c_s \right\} \mid \sum_{s \in \mathcal{A}} c_s = 0 \right\}.$$

Then  $S \subset S^*$ , hence  $S$  and its closure  $\overline{S}$  are symmetric. We show that for  $S$  the following statements are valid:

- (a) There exists a conjugation operator  $C$  on  $\Pi_\kappa$  which commutes with  $S$ .
- (b) There exists a fundamental symmetry on  $\Pi_\kappa$  which commutes with  $C$ .
- (c) The range of  $S - t_0$  is dense in  $\Pi_\kappa$ .
- (d)  $S$  has defect index  $(0, 0)$  or  $(1, 1)$ .
- (e) The selfadjoint extensions of  $S$  in  $\Pi_\kappa$  have a nonempty resolvent set.

Assumption 2 is used in the proof of (c), and this in turn is needed for the important property (e).

*Proof of (a):* Consider the following relation  $C_0$  in  $\Pi$ :

$$C_0 := \left\{ \left\{ \sum_{s \in \mathcal{A}} K(s, \cdot) c_s, \sum_{s \in \mathcal{A}} K(s, \cdot) c_s^* \right\} \right\}.$$

Since  $K(s, t)$  is real,  $C_0$  is anti-linear and has the conjugation property: for  $\{f, g\}, \{h, k\} \in C_0$  and  $\alpha, \beta \in \mathbb{C}$  we have

$$\{\alpha f + \beta h, \alpha^* g + \beta^* k\} \in C_0, \quad \langle f, h \rangle = \langle k, g \rangle.$$

Moreover  $\text{dom } C_0$  and  $\text{ran } C_0$  are dense in  $\Pi_\kappa$ . Similar arguments as in the proof of Shmuljan’s theorem (see, for example, [ADRS, Theorems 1.4.1 and 1.4.2]) show that the closure of  $C_0$  in  $\Pi_\kappa$  is the graph of a continuous conjugation operator on  $\Pi_\kappa$ , which we shall denote by  $C$ . It is easy to see that  $\{Cf, Cg\} \in S$  if and only if  $\{f, g\} \in S$ , which is the same as saying that  $S$  commutes with  $C$ .

*Proof of (b):* Since the values of the kernel  $K(\cdot, \cdot)$  are real, for any  $m \in \mathbb{N}$  the  $m \times m$ -matrices  $(K(s_j, s_k))_{i,j=1}^m$  are hermitian and have real entries. Thus the eigenvectors of these matrices can be chosen so that their entries are also real. It follows that there exist linear combinations  $\sum_{s \in \mathcal{A}} K(s, \cdot) c_s^{(j)}$ ,  $j = 1, \dots, \kappa$ , with real coefficients  $c_s^{(j)}$ , which span a  $\kappa$ -dimensional negative subspace of  $\Pi_\kappa$ . The space is invariant under the conjugation  $C$  and hence  $C$  commutes with the projection  $P_-$  in  $\Pi_\kappa$  onto this negative subspace. Consequently,  $C$  also commutes with  $J := (I - P_-) - P_-$  which is a fundamental symmetry on  $\Pi_\kappa$ .

*Proof of (c):* We first show

$$(5) \quad \text{ran}(S - w) = \begin{cases} \left\{ \sum_{s \in \mathcal{A}} K(s, \cdot) d_s \mid \sum_{s \in \mathcal{A}} \frac{d_s}{s - w} = 0 \right\}, & w \notin \mathcal{A}, \\ \left\{ \sum_{s \in \mathcal{A}, s \neq w} K(s, \cdot) d_s \right\}, & w \in \mathcal{A}. \end{cases}$$

To see this, we note that

$$\text{ran}(S - w) = \left\{ \sum_{s \in \mathcal{A}} (s - w) K(s, \cdot) c_s \mid \sum_{s \in \mathcal{A}} c_s = 0 \right\}.$$

Evidently, this implies equality (5) in the case  $w \notin \mathcal{A}$ , and in the case  $w \in \mathcal{A}$  it implies the inclusion  $\text{ran}(S - w) \subset \{ \sum K(s, \cdot) d_s \mid s \neq w \}$ . As to the converse inclusion, for given numbers  $d_s, s \neq w$ , where again only finitely many  $d_s$  are nonzero, we set  $c_s := d_s / (s - w)$  if  $s \neq w$ , and  $c_w := - \sum_{s \neq w} c_s$ . Then

$$K(w, \cdot) c_w + \sum_{s \in \mathcal{A}} K(s, \cdot) c_s \in \text{dom } S,$$

and if we apply  $S - w$  to this sum we get  $\sum_{s \in \mathcal{A}} K(s, \cdot) d_s$ .

In particular, (5) implies

$$\text{ran}(S - t_0) = \left\{ \sum_{s \in \mathcal{A}, s \neq t_0} K(s, \cdot) d_s \right\}.$$

We show that this set is dense in  $\Pi_\kappa$ . Let  $t_n$  be a sequence in  $\mathcal{A}$  with  $t_n \neq t_0$  which converges to  $t_0$ . The set is dense when we have shown that the sequence  $(K(t_n, \cdot))$  in  $\text{ran}(S - t_0)$  converges to  $K(t_0, \cdot)$ . According to [IKL, Theorem 2.4 (i)] this is the case if (and only if) for all  $u \in \mathcal{A}$  and  $n \rightarrow \infty$ ,

$$\langle K(t_n, \cdot), K(u, \cdot) \rangle \rightarrow \langle K(t_0, \cdot), K(u, \cdot) \rangle$$

and

$$\langle K(t_n, \cdot), K(t_n, \cdot) \rangle \rightarrow \langle K(t_0, \cdot), K(t_0, \cdot) \rangle.$$

By writing out the inner products, these relations follow from assumption 2 in the Theorem.

*Proof of (d):* The operator  $JS$  is symmetric in the Hilbert space  $(\Pi_\kappa, \langle J \cdot, \cdot \rangle)$  and commutes with the operator  $C$ , and  $C$  is also a conjugation operator with respect to this Hilbert space inner product. Hence  $JS$  has equal defect numbers, and then  $S$  also has equal defect numbers in the Pontryagin space  $\Pi_\kappa$ . For  $w \in \mathbb{C} \setminus \mathbb{R}$ , the linear functional

$$\sum_{s \in \mathcal{A}} K(s, \cdot) d_s \mapsto \sum_{s \in \mathcal{A}} \frac{d_s}{s - w}$$

is either continuous or not continuous in the space  $\Pi_\kappa$ . By (5), in the first case  $\text{ran}(S - w)$  is dense in  $\Pi_\kappa$  and the defect index of  $S$  is  $(0, 0)$ , and in the second case the defect index of  $S$  is  $(1, 1)$ . Hence, accordingly,  $S$  is selfadjoint or has canonical selfadjoint extensions in  $\Pi_\kappa$ . Recall that for an operator  $S$  in  $\Pi_\kappa$  an extension is termed canonical if it also acts in  $\Pi_\kappa$ , and it is called noncanonical if it acts in a space which contains  $\Pi_\kappa$  as a proper subspace.

*Proof of (e):* Let  $A$  be a selfadjoint extension of  $S$  in  $\Pi_\kappa$ , the case  $A = \overline{S}$  included. Then  $A - t_0$  and its inverse  $(A - t_0)^{-1}$  are selfadjoint relations. By (c) the latter is densely defined, hence it is a selfadjoint operator and therefore has a nonempty resolvent set (see for example [DS, p. 162]). It follows that  $A$  itself has a nonempty resolvent set, in fact, for all  $z \in \mathbb{C}$ ,

$$I + (z - t_0)(A - z)^{-1} = (I - (z - t_0)(A - t_0)^{-1})^{-1},$$

which implies  $z \in \rho(A)$  if and only if  $\frac{1}{z - t_0} \in \rho((A - t_0)^{-1})$ .

Now let  $A$  be any selfadjoint extension of  $S$  in a Pontryagin space  $\tilde{\Pi}_\kappa$  of negative index  $\kappa$  which contains  $\Pi_\kappa$  if the defect index of the closure  $\overline{S}$  is  $(1, 1)$ , or  $A = \overline{S}$  if  $S$  has defect index  $(0, 0)$ . We choose any point  $s_0 \in \mathcal{A}$ , and define for  $z \in \rho(A)$  the function

$$(6) \quad N(z) := f(s_0) + (z - s_0) \langle (I + (z - s_0)(A - z)^{-1})K(s_0, \cdot), K(s_0, \cdot) \rangle;$$

here the inner product in  $\tilde{\Pi}_\kappa$  is also denoted by  $\langle \cdot, \cdot \rangle$ . This function  $N$  is independent of the choice of the point  $s_0 \in \mathcal{A}$ . In order to see this we choose an arbitrary point  $s \in \mathcal{A}$ ,  $s \neq s_0$ . Then

$$K(s_0, \cdot) - K(s, \cdot) \in \text{dom } S$$

and

$$(S - z)(K(s_0, \cdot) - K(s, \cdot)) = (s_0 - z)K(s_0, \cdot) - (s - z)K(s, \cdot).$$

Applying  $(A - z)^{-1}$  to both sides and rearranging terms we find that

$$(I + (z - s_0)(A - z)^{-1})K(s_0, \cdot) = (I + (z - s)(A - z)^{-1})K(s, \cdot),$$

hence, if  $z \in \rho(A)$ ,

$$\begin{aligned} N(z) &= f(s_0) + (z - s_0) \langle (I + (z - s)(A - z)^{-1})K(s, \cdot), K(s_0, \cdot) \rangle \\ &= f(s_0) + (z - s_0) \langle K(s, \cdot), K(s_0, \cdot) \rangle \\ &\quad + (z - s) \langle K(s, \cdot), (z^* - s_0)(A - z^*)^{-1}K(s_0, \cdot) \rangle \\ &= f(s_0) + (z - s_0) \langle K(s, \cdot), K(s_0, \cdot) \rangle \\ &\quad + (z - s) \langle K(s, \cdot), (I + (z^* - s)(A - z^*)^{-1})K(s, \cdot) - K(s_0, \cdot) \rangle \\ &= f(s_0) + (z - s_0) \langle K(s, \cdot), K(s_0, \cdot) \rangle - (z - s) \langle K(s, \cdot), K(s_0, \cdot) \rangle \\ &\quad + (z - s) \langle (I + (z - s)(A - z)^{-1})K(s, \cdot), K(s, \cdot) \rangle \\ &= f(s) + (z - s) \langle (I + (z - s)(A - z)^{-1})K(s, \cdot), K(s, \cdot) \rangle. \end{aligned}$$

In general, this representation of the function  $N$  is not  $K(s, \cdot)$ -minimal, which is, for example, the case if  $S$  has an eigenvalue; see the first remark in Section 3. However, the inclusion  $\mathcal{E}_N \subset \sigma_0(A) \cap (\mathbb{R} \cup \mathbb{C}^+)$  does always hold. If  $s \in \mathcal{A} \setminus \sigma_0(A)$ , we choose an interval  $\Delta$  around  $s$  the closure of which does not contain any point of  $\sigma_0(A)$ . Then, with the spectral function  $E_A$  of  $A$  and  $u := K(s, \cdot)$  we can write

$$\begin{aligned} N(z) &= f(s) + (z - s) \langle u, u \rangle + (z - s)^2 \langle (A - z)^{-1}(I - E_A(\Delta))u, u \rangle \\ &\quad + (z - s)^2 \int_{\Delta} \frac{\langle E_A(d\lambda)u, u \rangle}{\lambda - z}. \end{aligned}$$

Now the first relation (4) follows easily, the second relation in (4) follows from  $\langle u, u \rangle = \widehat{f}(s)$  and

$$\lim_{z \xrightarrow{s} s} \frac{\operatorname{Im} N(z)}{\operatorname{Im} z} = \widehat{f}(s) - \langle E_A(\{s\})u, u \rangle \leq \widehat{f}(s).$$

The fact that the function  $N$  belongs to a generalized Nevanlinna class  $\mathbf{N}_{\kappa'}$  with  $\kappa' \leq \kappa$  follows from the relation

$$\begin{aligned} \frac{N(z) - N(w)^*}{z - w^*} &= \langle (I + (z - s_0)(A - z)^{-1})K(s_0, \cdot), (I + (w - s_0)(A - w)^{-1})K(s_0, \cdot) \rangle. \end{aligned}$$

In fact,  $\kappa'$  is the negative index of the subspace of  $\widetilde{\Pi}_{\kappa}$  which is spanned by the elements

$$(I + (z - s_0)(A - z)^{-1})K(s_0, \cdot), \quad z \in \rho(A).$$

If  $\mathcal{A}^i \neq \emptyset$ , since  $\mathcal{E}_N$  contains only finitely many points there exists a nonempty open interval  $\Gamma$  in  $\mathcal{A}^i$  which does not contain any point of  $\mathcal{E}_N$ . Then the second relation in (4) and the Lemma imply that all functions  $N$  are holomorphic in  $\Gamma$ . Since on  $\Gamma$  the functions  $N$  coincide with  $f$  they coincide everywhere. Finally, assume  $\Delta$  is as in the last statement of the Theorem. If  $\Delta$  would contain a point of  $\mathcal{E}_N$ , then  $N$  and hence also  $f$  would be unbounded in a real neighbourhood of this point, which is impossible since  $f$  is bounded there. The Theorem is proved.

## 3. FINAL REMARKS

1. Assume that  $t_1 \in \mathcal{A}$  is an accumulation point of  $\mathcal{A}$ . Then the following statements hold. Here  $s$  and  $t$  belong to  $\mathcal{A}$ .

(i)  $K(s, \cdot) \rightarrow K(t_1, \cdot)$  in  $\Pi_\kappa$  as  $s \rightarrow t_1$  if and only if

$$\lim_{s \rightarrow t_1} \frac{f(s) - f(t_1)}{s - t_1} = \widehat{f}(t_1) = \lim_{s \rightarrow t_1} \widehat{f}(s).$$

(ii)  $K(s, \cdot)$  converges to some element in  $\Pi_\kappa$  as  $s \rightarrow t_1$  if and only if

$$\lim_{s, t \rightarrow t_1} \left( \widehat{f}(s) + \widehat{f}(t) - 2 \frac{f(s) - f(t)}{s - t} \right) = 0$$

and  $\lim_{s \rightarrow t_1} \frac{f(s) - f(t_1)}{s - t_1}$  exists.

(iii) If the conditions in (ii) hold, then  $K(s, \cdot) \rightarrow K(t_1, \cdot)$  in  $\Pi_\kappa$  if and only if

$$\lim_{s \rightarrow t_1} \frac{f(s) - f(t_1)}{s - t_1} = \widehat{f}(t_1).$$

(iv) If the conditions in (ii) hold, then  $t_1$  is an eigenvalue of  $\overline{S}$  with a positive (negative, respectively) eigenvector in  $\Pi_\kappa$  if and only if

$$\widehat{f}(t_1) - \lim_{s \rightarrow t_1} \frac{f(s) - f(t_1)}{s - t_1} > 0 \quad (< 0, \text{ respectively}).$$

The first two statements follow directly from [IKL, Theorem 2.4]; in fact (i) was shown and used in the proof of the Theorem. Note that the existence of the limit

$$\lim_{s \rightarrow t_1} \frac{f(s) - f(t_1)}{s - t_1}$$

does not imply the existence of the limit

$$\lim_{s, t \rightarrow t_1} \frac{f(s) - f(t)}{s - t}.$$

As to the proof of (iii), the “only if” part follows from (i). For the “if” part set  $\varepsilon_{t_1} = \lim_{s \rightarrow t_1} K(s, \cdot)$ . Then for all  $t$ ,

$$\langle K(t_1, \cdot) - \varepsilon_{t_1}, K(t, \cdot) \rangle = 0,$$

which implies that the difference on the left in the inner product is zero. To see the latter equality, for  $t = t_1$  the left side of the equality equals

$$\lim_{s \rightarrow t_1} \left( \widehat{f}(t_1) - \frac{f(s) - f(t_1)}{s - t_1} \right)$$

and this is zero by assumption. For  $t \neq t_1$  the left side of the equality is equal to

$$\lim_{s \rightarrow t_1} \left( \frac{f(t_1) - f(t)}{t_1 - t} - \frac{f(s) - f(t)}{s - t} \right),$$

which is zero because the assumption implies that  $f$  is continuous at  $t_1$ . Concerning (iv), we have

$$\langle K(t_1, \cdot) - \varepsilon_{t_1}, K(t_1, \cdot) - \varepsilon_{t_1} \rangle = \widehat{f}(t_1) - \lim_{s \rightarrow t_1} \frac{f(s) - f(t_0)}{s - t_0}.$$

Moreover,  $K(s, \cdot) - K(t_1, \cdot) \in \mathcal{D}(S)$  and

$$\begin{aligned} \lim_{s \rightarrow t_1} (K(t_1, \cdot) - K(s, \cdot)) &= K(t_1, \cdot) - \varepsilon_{t_1}, \\ \lim_{s \rightarrow t_1} S(K(s, \cdot) - K(t_1, \cdot)) &= t_1(K(t_1, \cdot) - \varepsilon_{t_1}). \end{aligned}$$

These formulas readily imply (iv) with eigenvector  $K(t_1, \cdot) - \varepsilon_{t_1}$ . From

$$S^* = \{ \{f, g\} \in \Pi^2 \mid g(t) - tf(t) \equiv \text{constant on } \mathcal{A} \},$$

it follows that the eigenspaces  $\ker(S^* - \lambda)$  of  $S^*$  at  $\lambda \in \mathbb{C}$  are at most 1-dimensional; if  $\ker(S^* - \lambda)$  is 1-dimensional, then it is spanned by the function

$$f(t) = \begin{cases} \frac{1}{t - \lambda} & \lambda \notin \mathcal{A}, \\ \delta_{t\lambda} & \lambda \in \mathcal{A}, \end{cases}$$

where  $\delta$  is the Kronecker delta. Thus for the eigenvalues  $t_1$  satisfying (iv),

$$\dim \ker(S - t_1) = 1.$$

Denote by  $\mathcal{L}_0$  the span of all such eigenspaces. Since these eigenspaces are mutually orthogonal  $\mathcal{L}_0$  is a Pontryagin space with say  $\kappa_0$  negative squares,  $0 \leq \kappa_0 \leq \kappa$ . Moreover  $\mathcal{L}_0$  is invariant under the canonical selfadjoint extensions  $A$  of  $S$  and  $K(s_0, \cdot) \in \Pi_\kappa \ominus \mathcal{L}_0$  for any  $s_0 \in \mathcal{A}$  which is not one of the eigenvalues  $t_1$ . From the representation (6) of the function  $N$  we now see that

$$N(z) = f(s_0) + (z - s_0) \langle (I + (z - s_0)(A_0 - z)^{-1})K(s_0, \cdot), K(s_0, \cdot) \rangle_{\Pi_\kappa \ominus \mathcal{L}_0},$$

where  $A_0 = A|_{\Pi_\kappa \ominus \mathcal{L}_0}$ . Thus the functions  $N$  in the Theorem belong to  $\bigcup_{\kappa'=0}^{\kappa-\kappa_0} \mathbf{N}_{\kappa'}$ . The situation of (iv) prevails for the example  $\mathcal{A} := (-1, +1)$ ,

$$f(t) = 1, \quad \widehat{f}(t) := \begin{cases} 0 & \text{if } t \in (-1, +1) \setminus \{0\}, \\ -1 & \text{if } t = 0. \end{cases}$$

The corresponding kernel  $K$  has one negative square, the space  $\Pi_1$  is 1-dimensional and  $S$  is the zero operator and hence is selfadjoint, and the function  $N$  of the theorem is  $N(z) \equiv 1$  and belongs to the class  $\mathbf{N}_0$ .

2. The factorization theorem of [DLLS] for scalar generalized Nevanlinna functions readily implies that under the conditions of the Theorem there exist  $\kappa_1 (\leq \kappa)$  points  $\alpha_1, \dots, \alpha_{\kappa_1}$  and  $\kappa_2 (\leq \kappa)$  points  $\beta_1, \dots, \beta_{\kappa_2}$  such that

$$f_0(z) = \frac{\prod_{j=1}^{\kappa_1} (z - \alpha_j)(z - \alpha_j^*)}{\prod_{j=1}^{\kappa_2} (z - \beta_j)(z - \beta_j^*)} f(z)$$

where  $f_0 \in \mathbf{N}_0$ .

3. The Theorem also holds in the matrix case with almost the same proof. Recall that a function  $N$  belongs to the class  $\mathbf{N}_\kappa^{m \times m}$  if it is  $m \times m$ -matrix valued with meromorphic entries on  $\mathbb{C}^+$  and such that the Nevanlinna kernel (1) has  $\kappa$  negative squares, where  $N(w)^*$  is now the adjoint matrix of  $N(w)$ . The function  $N$  is assumed to be continued to  $\mathbb{C}^-$  and points in  $\mathbb{R}$  as in the scalar case (see (2)). The representation (3) for  $N \in \mathbf{N}_\kappa^{m \times m}$  takes the form

$$N(z) = N(\mu)^* + (z - \mu)\Gamma^*(I + (z - \mu)(A - z)^{-1})\Gamma,$$

where  $A$  and  $\mu$  are as for (3) and  $\Gamma : \mathbb{C}^m \rightarrow \Pi$  is a linear map. The minimality property that guarantees uniqueness of the representation is that

$$\overline{\text{span}} \{ \text{ran}(I + (z - \mu)(A - z)^{-1})\Gamma \mid z \in \rho(A) \} = \Pi.$$

The formulation of the matrix version of the Theorem only requires some adaptations: All functions are  $m \times m$ -matrix valued and the values of the functions  $f$  and  $\hat{f}$  on  $\mathcal{A}$  are assumed to be Hermitian matrices. We could not prove the equality of the defect numbers of the corresponding symmetric linear relation  $S$ . However, each maximal symmetric extension  $A$  of  $S$  leads as in the scalar case to a solution  $N$  of the matrix Loewner problem. The function  $N$  is first defined in the half plane where the resolvent of  $A$  exists and then extended to the other half plane by the relation (2). Another way to prove the matrix version of the Theorem is to reduce it to the scalar case as in the proof of [ABDR, Theorem 2.1].

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