

ABELIAN POINT STABILIZERS IN TRANSITIVE PERMUTATION GROUPS

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ABSTRACT. In this note we prove that if the point stabilizer A in a transitive permutation group of degree m is abelian, then the exponent of A is less than m . This extends an earlier result of Andrea Lucchini, who proved this in the case where A is cyclic.

The following pretty result of Andrea Lucchini appears in [2].

Theorem (Lucchini). *Suppose that A is a point stabilizer in a transitive permutation group of degree $m > 1$. If A is cyclic, then $|A| < m$.*

In this brief note, we use some of Lucchini's methods to prove a somewhat stronger result.

Theorem A. *Suppose that A is a point stabilizer in a transitive permutation group of degree $m > 1$. If A is abelian, then the exponent $\exp(A) < m$.*

Lucchini's proof depends on a theorem of A. Delgado and A. Chermak [1], but we prefer to use the somewhat similar (but more powerful) result of V. I. Zenkov [3], which we state here for the convenience of the reader. (We mention that the proof of Zenkov's theorem is quite short, and it does not rely on the classification of simple groups.)

Theorem (Zenkov). *Let A and B be abelian subgroups of a finite group G . Then there exists an element $g \in G$ such that $A \cap B^g \subseteq \mathbf{F}(G)$, the Fitting subgroup.*

We shall also need the following well known fact, which can be viewed as a corollary of Zenkov's theorem. (More is true, however, and even more elementary proofs are available.)

Corollary B. *Let A be an abelian group that acts faithfully and coprimely on a nontrivial group N . Then $|A| < |N|$.*

Proof. Let $G = NA$ be the semidirect product and let π be the set of prime divisors of $|A|$. Since N is a π' -group, it follows that $\mathbf{O}_\pi(G)$ is contained in the (trivial) kernel of the action of A on N . Thus $\mathbf{O}_\pi(G) = 1$, and we conclude that $A \cap \mathbf{F}(G) = 1$. By Zenkov's theorem, therefore, there exists $g \in G$ such that $A \cap A^g = 1$. Since $A < G$, the subset AA^g is not the whole group G , and it follows that $|G| > |AA^g| = |A|^2$. Thus $|N| = |G : A| > |A|$, as required. \square

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Proof of Theorem A. Let G be a transitive permutation group of degree m and suppose that a point stabilizer A is abelian. We show by induction on m that $\exp(A) \leq m$ and that this inequality is strict if $m > 1$. This assertion is clear when $m = 1$, and so we can assume $m > 1$. We can also assume, of course, that $A > 1$.

If $\mathbf{F}(G) = 1$, then by Zenkov's theorem $A \cap A^g = 1$ for some element $g \in G$. As in the proof Corollary B, it follows that $|G| > |AA^g| = |A|^2$, and therefore $\exp(A) \leq |A| < |G : A| = m$, and we are done in this case. We can thus assume that $\mathbf{F}(G)$ is nontrivial, and we choose a minimal normal subgroup N of G such that N is an elementary abelian p -group for some prime p .

Now $N \not\subseteq A$ since $\text{core}_G(A) = 1$, and thus $NA > A$. Writing $u = |G : NA|$, we see that $u < m$. Let $K = \text{core}_G(NA)$, so that G/K is a transitive permutation group of degree u with point stabilizer NA/K . Since $N \subseteq K$, we see that $NA = KA$ and $NA/K = KA/K \cong A/(K \cap A) = A/J$, where we have written $J = K \cap A$. Since A/J is abelian, it follows by the inductive hypothesis that $\exp(A/J) \leq u$, with equality only if $u = 1$. Also, we note that $K = N(K \cap A) = NJ$ by Dedekind's lemma, and $m/u = |NA : A| = |N : N \cap A|$ is a power of p .

First, suppose that K is nonabelian. Write $Z = \mathbf{Z}(K) \triangleleft G$, and note that ZJ is abelian since J is abelian and $J \subseteq K$. It follows that $N \not\subseteq Z$, since $NJ = K$ is nonabelian. Since N is minimal normal in G , we conclude that $N \cap Z = 1$.

Now write $V = NZ \cap J$. We have $N \subseteq NZ \subseteq K = NJ$, and thus by Dedekind's lemma, $NZ = N(NZ \cap J) = NV$. Also, NZ is abelian, and thus V centralizes both N and J . Therefore, $V \subseteq \mathbf{Z}(NJ) = \mathbf{Z}(K) = Z \subseteq NZ = NV$. By Dedekind's lemma again, we have $Z = (N \cap Z)V = V \subseteq A$. But $Z \triangleleft G$ and A has trivial core, and it follows that $1 = Z = \mathbf{Z}(K) = \mathbf{Z}(NJ)$. Since J is abelian, we have $\mathbf{C}_J(N) = 1$, and J acts faithfully on N . In particular, we see that $N \cap A \subseteq \mathbf{C}_J(N) = 1$, and thus $m/u = |N|$.

Now recall that N is a p -group and let $P \in \text{Syl}_p(J)$. Then NP is a Sylow p -subgroup of $NJ = K$, and we note that K/N is abelian since NA/N is abelian. Thus $NP/N \triangleleft G/N$, and so $NP \triangleleft G$, and hence $\mathbf{Z}(NP) \triangleleft G$. But $N \cap \mathbf{Z}(NP) > 1$ since NP is a p -group, and it follows by the minimality of N that $N \subseteq \mathbf{Z}(NP)$. Thus $P \subseteq \mathbf{C}_J(N) = 1$, and therefore J is a p' -group. But J is abelian and acts faithfully on the p -group N , and so by Corollary B, we have $|J| < |N| = m/u$. Since $\exp(A/J) \leq u$, we have $\exp(A) \leq \exp(A/J)|J| < u(m/u) = m$, and we are done in this case.

We can now assume that K is abelian. We observe that $u > 1$, since otherwise $NA = G$ and thus $G = K$ is abelian, which implies that $A = 1$, contrary to assumption. We will show that $\exp(J) \leq p$, and thus (since $u > 1$) we have $\exp(A) \leq \exp(A/J)\exp(J) < up$. Also, since $1 < m/u$ is a power of p , we have $p \leq m/u$, and the result will follow.

Let $x \in K = NJ$ and write $x = ab$, where $a \in N$ and $b \in J$. Since K is abelian and N is an elementary abelian p -group, we have $x^p = b^p \in J$. Thus J contains the subgroup $\{x^p \mid x \in K\}$, which is characteristic in K . Since $\text{core}_G(A) = 1$, this subgroup is trivial, and thus K is an elementary abelian p -group. In particular, since $J \subseteq K$, we see that $\exp(J) \leq p$, as claimed, and this completes the proof. \square

REFERENCES

1. A. Chermak and A. Delgado, A measuring argument for finite groups, Proc. Amer. Math. Soc. **107** (1989) 907–914. MR **90c**:20001

2. A. Lucchini, On the order of transitive permutation groups with cyclic point-stabilizer, *Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei (9) Mat. Appl.* **9** (1998) 241–243. MR **2000k**:20004
3. V. I. Zenkov, Intersections of abelian subgroups in finite groups, *Math. Notes* **56** (1994) 869–871. MR **95m**:20021

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