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ISOMORPHISM OF COMMUTATIVE GROUP ALGEBRAS OF CLOSED p-GROUPS AND p-LOCAL ALGEBRAICALLY COMPACT GROUPS

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Abstract. Let G be an abelian group and let K be a field of char$K = p > 0$. It is shown via a universal algorithm that if the modified *Direct-Factor Problem* holds, then the K-isomorphism $KH \cong KG$ for some group H yields $H \cong G$ provided G is a closed p-group or a p-local algebraically compact group. In particular, this is the case when G is closed p-primary of arbitrary power, or G is p-local algebraically compact with cardinality at most $\aleph_1$ and K is in cardinality not exceeding $\aleph_1$. The last claim completely settles a question raised by W. May in Proc. Amer. Math. Soc. (1979) and partially extends our results published in Rend. Sem. Mat. Univ. Padova (1999) and Southeast Asian Bull. Math. (2001).

Introduction

In 1979, Warren May in [11] asked if a p-torsion abelian group being closed (in other words, torsion-complete) is an invariant property for the commutative group algebra of this group over an arbitrary field with characteristic p. We gave a partial answer in [2, 4] to this problem by restricting the field to have cardinality p and the group to have cardinality less than or equal to the first uncountable cardinal $\aleph_1$ (actually, we have established stronger affirmations). Here we shall develop our mixed algebraic-topological technique which generalizes the above-cited fact for a field with characteristic p of arbitrary power and which gives new perspectives for an investigation of a dual class of abelian groups called algebraically compact groups. For this purpose we first formulate the following well-known and long-standing question, namely

Direct-Factor Problem. *Does it follow that the p-component G_p of the abelian group G is a direct factor of the normed Sylow p-subgroup $S(KG)$ in the group algebra KG over a field K of characteristic p? As a consequence, does it follow that the p-mixed abelian group G is a direct factor of the group $V(KG)$ of all normalized units?*

Additional information, however, ensures the next crucial modification obtained in a general form from May (cf. [12]).

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Modified Direct-Factor Problem. *For G an abelian group and K a perfect field with* $\mathrm{char}K = p > 0$ *is it true that $S(KG)/G_p$ is totally projective? As a consequence, is $V(KG)/G$ then totally projective whenever G is p-mixed?*

In [5] we have proved (about this see also [1]) that the Direct-Factor Problem is valid in the case when G is a direct sum of closed p-groups or is a direct sum of p-local algebraically compact groups; the complementary factors are not described.

Main result

We come now to the main goal that motivates this article, namely

Central Theorem. *Suppose G is a closed abelian p-group or a p-local algebraically compact abelian group and K is a field of characteristic $p > 0$. If the Modified Direct-Factor Problem is fulfilled, then $KH \cong KG$ as K-algebras for any group H if and only if $H \cong G$.*

Proof. Without loss of generality we presume that $KG = KH$ and that K is perfect. Further we consider two cases, namely

Case 1. *G is p-torsion closed*. So, owing to the Modified Direct-Factor Hypothesis we may write $V(KG) \cong G \times V(KG)/G \cong H \times V(KH)/H \cong V(KH)$, where $V(KG)/G$ must be totally projective. On the other hand the separability of G trivially yields that $V(KG)$ is so, hence the same is valid for $V(KG)/G$ as a "subgroup". Therefore from [9], $V(KG)/G$ is a direct sum of cyclics and hence according to [10], H is itself semi-closed as a direct factor of the semi-closed $V(KG)$. Thus, we can write $H = T \times C$, where T is closed and C is a direct sum of cyclics. But C may be represented as a countable ascending union of pure and bounded subgroups, i.e. in other words we write $C = \bigcup_n C_n$, $C_n \subseteq C_{n+1}$ are pure in C and $C_n^{p^n} = 1$. Consequently, we derive $H = \bigcup_n H_n$, where $H_n = T \times C_n \subseteq H_{n+1}$ are pure closed subgroups in H. Furthermore $V(KG) = V(KH) = \bigcup_n V(KH_n)$ hence $G = \bigcup_n [V(KH_n) \cap G]$.

Now, we select a special sequence $(g_n) \in G$ with the properties: $g_n = \alpha_1^{(n)} h_1^{(n)} + \cdots + \alpha_{s_n}^{(n)} h_{s_n}^{(n)}$, where $\alpha_1^{(n)}, \ldots, \alpha_{s_n}^{(n)} \in K$ and $\alpha_1^{(n)} + \cdots + \alpha_{s_n}^{(n)} = 1$; $h_1^{(n)} = a_1^{(n)} c_1^{(n)} \in T \times C_n, \ldots, h_{s_n}^{(n)} = a_{s_n}^{(n)} c_{s_n}^{(n)} \in T \times C_n$, $c_1^{(n)} \neq \cdots \neq c_{s_n}^{(n)} \neq c_1^{(n)}$ and $\mathrm{order}(c_r^{(n)}) = \cdots = \mathrm{order}(c_{s_n}^{(n)}) = p^n$ for some natural number r with $r \leq s_n$, $s_n \in N$, such that (g_n) is a bounded, say $g_n^p = 1$, Cauchy sequence but $(h_r^{(n)}), \ldots, (h_{s_n}^{(n)})$ are not "*a priori*" bounded however. Now, we make the following choice: $h_1^{(n)^p} = \cdots = h_{r-1}^{(n)^p} = 1$, $\alpha_1^{(n)} + \cdots + \alpha_{r-1}^{(n)} = 1$; $h_r^{(n)^p} = \cdots = h_{s_n}^{(n)^p} \neq 1$, $\alpha_r^{(n)} + \cdots + \alpha_{s_n}^{(n)} = 0$ and

$$\begin{aligned}&\alpha_1^{(n+L)} h_1^{(n+L)} + \cdots + \alpha_{s_{n+L}}^{(n+L)} h_{s_{n+L}}^{(n+L)} \\ &\quad = [\alpha_1^{(n)} h_1^{(n)} + \cdots + \alpha_{s_n}^{(n)} h_{s_n}^{(n)}](\beta_1^{(n)} a_1^{(n)} + \cdots + \beta_{m_n}^{(n)} a_{m_n}^{(n)})\end{aligned}$$

for some natural number m_n and each integer $L \geq 0$ where $a_1^{(n)} \in H^{p^n}, \ldots, a_{m_n}^{(n)} \in H^{p^n}$, hence $\beta_1^{(n)} a_1^{(n)} + \cdots + \beta_{m_n}^{(n)} a_{m_n}^{(n)} \in V^{p^n}(KH) = V^{p^n}(KG)$ and so $g_{n+L} g_n^{-1} \in G \cap V^{p^n}(KG) = G^{p^n}$. Evidently, the choice of such a sequence is possible only when such $s_n \geq 3$ exists. If so, we observe further that Kulikov's criterion ([9], p. 38, Theorem 70.7) implies (g_n) converges to an element of G. Certainly the boundary is of the form $f_1 b_1^{(j)} + f_2 b_2^{(j)} + \cdots + f_t b_t^{(j)} \neq 1$ for a fixed $t \in N$ and a fixed $j \in N$

so that $b_1^{(j)}, \dots, b_t^{(j)} \in H_j$ and $f_1, \dots, f_t \in K$ with $f_1 + \cdots + f_t = 1$. Since

$$f_1 b_1^{(j)} + \cdots + f_t b_t^{(j)} \in [\alpha_1^{(n)} h_1^{(n)} + \cdots + \alpha_{s_n}^{(n)} h_{s_n}^{(n)}] G^{p^k}$$

for every $k \geq 1$ and every $n \geq k$, the above representation of G as a special countable union of groups yields a fixed element $1 \neq c^{(j)} \in C_j$ and some $c_m^{(n)} \in C_n$ for $r \leq m \leq s_n$ with the property $c^{(j)} \in c_m^{(n)} C^{p^n}$. But when $n > j$ we get $c^{(j)} c_m^{(n)^{-1}} \in C^{p^n} \cap C_n = 1$, hence $c^{(j)} = c_m^{(n)}$, a contradiction. Moreover, eventually $h_i^{(n)} \in h_m^{(n)} G^{p^k}$ for each $k \geq 1$, $n \geq k$ and some $1 \leq i \neq m \leq s_n$, whence $c_i^{(n)} \in c_m^{(n)} C^{p^k}$ and thus $c^{(j)} = c_i^{(n)}$ for every positive integer $n > j$. The same procedure works and for the other nonidentity elements of C_j from the boundary. Consequently C must be bounded. That is, by [9], H must be closed, as claimed.

Now if each bounded Cauchy sequence (g_n) in G is represented as $g_n = \alpha_n h_n = h_n$ where $1 = \alpha_n \in K$ and $h_n \in H$ or $g_n = \alpha_1^{(n)} h_1^{(n)} + \alpha_2^{(n)} h_2^{(n)}$ where $\alpha_1^{(n)}, \alpha_2^{(n)} \in K$ with $\alpha_1^{(n)} + \alpha_2^{(n)} = 1$ and $h_1^{(n)}, h_2^{(n)} \in H$, then it follows that every bounded Cauchy sequence of H is also of the two types presented and thus it holds obviously that an arbitrary restricted Cauchy sequence in H possesses a stabilization in H for almost all natural numbers, i.e. it is convergent in H. Furthermore, the above-mentioned Kulikov's theorem implies that H is itself closed, as stated.

Finally, in all cases, $G \cong H$ because the group Ulm-Kaplansky functions are invariants of the commutative modular group algebra (cf. [11]) and determine up to an isomorphism the closed p-groups (see for example [9] or [11]).

Case 2. *G is p-local algebraically compact.* By [3], we may assume that G and H are reduced, and the p-locality of H follows trivially. Moreover applying the Modified Direct-Factor Conjecture we deduce as above that $G \times V(FG)/G \cong H \times V(FH)/H$, where $V(FG)/G$ is totally projective p-torsion. But G algebraically compact does imply that G_p is separable (see [9]) whence the same is true for $S(FG)/G_p \cong V(FG)/G$ owing to the fact that G_p is nice in $S(FG)$ (for instance see [2, 12]). Consequently $V(FG)/G$ is a direct sum of p-cyclics. Thus by similar arguments to [10], H can be written as $H = A \times C$, where A is p-local algebraically compact and C is a direct sum of p-cyclics. That is why analogous to the above $C = \bigcup_n C_n$, $C_n \subseteq C_{n+1}$ are pure in C and $C_n^{p^n} = 1$. Furthermore $H = \bigcup_n H_n$, where $H_n = A \times C_n \subseteq H_{n+1}$ are pure algebraically compact subgroups in H. On the other hand $V(KG) = V(KH) = \bigcup_n V(KH_n)$ hence $G = \bigcup_n [V(KH_n) \cap G]$. After this, choose the same special sequence (g_n) defined as in Case 1 with the exception that g_n need not be bounded. Complying with the Kaplansky criterion ([9], p. 191, Theorem 39.1) we may derive that (g_n) is convergent to an element of G. In the spirit of the situation of similar reasons as above, C must be bounded. Finally we can conclude that H is algebraically compact p-local too (see [9]), and thus from an application of [3] it follows that G and H are isomorphic, as well. The proof is finished after all. □

Applications

The first actual consequence of the major theorem is the following assertion announced in [7, 8].

Corollary. *Let G be a closed abelian p-group and let K be a field with* $\mathrm{char} K = p \neq 0$. *Then for some group H it is true that $KH \cong KG$ as K-algebras if and only if $H \cong G$.*

Proof. Since length(G) = length$(H) \leq \omega$, the proof follows automatically by combining the main result in [6] concerning the Direct-Factor Conjecture along with our Central Theorem. □

Remark. The attainment above completely solves a problem of May [11] posed in these Proceedings in 1979.

In this direction, we can formulate and prove

Corollary. *Let G be an algebraically compact p-local abelian group and let K be a field of* char$K = p \neq 0$ *both with powers at most $\aleph_1$. Then for some group H it is valid that $KH \cong KG$ as K-algebras if and only if $H \cong G$.*

Proof. Since $|G| = |H| \leq \aleph_1$ and length(G_p) = length$(H_p) \leq \omega$, by employing our Direct-Factor Theorem in [4] plus the Central Theorem argued above we are done. □

The following commentary is valuable.

Remark. An example of a closed p-group with power $\aleph_1$ is given by us in [2] provided that the *Continuum Hypothesis* holds. By the same token we may construct an abelian closed p-primary group of arbitrary cardinality $\aleph_\alpha$ where α is any ordinal. Besides, the above corollaries were proved in [2, 4] where K is only the field of p-elements (i.e. the simple hence finite field in characteristic p) and G is of cardinality $\aleph_1$, but using an algebraic technique and more especially the so-called back-and-forth method together with the Direct-Factor Conjecture.

A problem of some interest and importance in abelian group theory is to determine the criterion illustrated when one countable ascending union of abelian groups can be a closed p-group or a p-mixed algebraically compact group, respectively. Such a necessary and sufficient condition will be useful for another new and elegant proof of the Central Theorem.

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