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A CONTINUUM WHOSE HYPERSPACE OF SUBCONTINUA IS NOT g-CONTRACTIBLE

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Abstract. A topological space Y is said to be g-contractible provided that there exists a continuous onto function $f : Y \to Y$ such that f is homotopic to a constant function. Answering a question by Sam B. Nadler, Jr., in this paper we construct a metric continuum Z such that its hyperspace of subcontinua $C(Z)$ is not g-contractible.

1. Introduction

A *continuum* is a compact connected metric space. A *map* is a continuous function. A topological space Y is said to be *g-contractible* provided that there exists an onto map $f : Y \to Y$ such that f is homotopic to a constant map. For a continuum X, $C(X)$ (resp., 2^X) denotes the hyperspace of subcontinua (resp., nonempty closed subsets) of X, with the Hausdorff metric.

Clearly, every contractible space is g-contractible. A simple closed curve is an easy example of a g-contractible and non-contractible continuum. In fact, the Hahn-Mazurkiewicz Theorem (see [7, Theorem 8.14]) implies that any locally connected continuum is g-contractible. The notion of g-contractibility was introduced by D. P. Bellamy in [1]. In [5], S. B. Nadler, Jr., studied g-contractibility in hyperspaces. He proved that, for any continuum X, 2^X is g-contractible ([5, 3.9] or [6, Theorem 4.10]), and if X is a continuum such that X contains an open subset with uncountably many components, then $C(X)$ is g-contractible ([5, 3.12] or [6, Theorem 4.12]). Nadler also asked if $C(X)$ is g-contractible for any continuum X ([5, 3.10] or [6, Question 4.11]). In this paper we answer Nadler's question in the negative by constructing an example of a continuum Z such that $C(Z)$ is not g-contractible.

2. Auxiliary results

Lemma 1. *If $C(X)$ is g-contractible, then there exists an onto map $f : C(X) \to C(X)$ and there exists a map $\psi : C(X) \times [0,1] \to C(X)$ such that:*

- *$\psi(A,0) = f(A)$ and $\psi(A,1) = X$ for each $A \in C(X)$,*
- *if $A \in C(X)$ and $0 \le s \le t \le 1$, then $\psi(A,s) \subset \psi(A,t)$.*

Proof. Suppose that $C(X)$ is g-contractible. Then there exist:

- an onto map $f : C(X) \to C(X)$,

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- an element $C \in C(X)$ and
- a map $G : C(X) \times [0,1] \to C(X)$,

such that $G(A,0) = f(A)$ and $G(A,1) = C$ for each $A \in C(X)$.

Since $C(X)$ is arcwise connected ([6, Theorem 1.13]) we can take a map $\alpha : [0,1] \to C(X)$ such that $\alpha(0) = C$ and $\alpha(1) = X$. Let $F : C(X) \times [0,1] \to C(X)$ be given by

$$F(A,t) = \begin{cases} G(A,2t), & \text{if } t \in [0, \frac{1}{2}], \\ \alpha(2t-1), & \text{if } t \in [\frac{1}{2}, 1]. \end{cases}$$

Clearly, F is a map such that $F(A,0) = f(A)$ and $F(A,1) = X$ for each $A \in C(X)$.

Now, let $\psi : C(X) \times [0,1] \to C(X)$ be given by

$$\psi(A,t) = \bigcup \{F(A,s) : s \in [0,t]\}.$$

It is easy to show that ψ has the required properties. □

The proof of the following lemma is similar to the proof of Theorem (2) of [2].

Lemma 2. *Let $f : X \to Y$ be an onto map between continua. Let $q \in Y$. If X is connected im kleinen at each point of $f^{-1}(q)$, then Y is connected im kleinen at q.*

The following result is an easy consequence of Theorem 2 of [3].

Lemma 3. *Let X be a continuum and let $p \in X$ be a point such that X is connected im kleinen at p. Then $C(X)$ is connected im kleinen at each element A that satisfies $p \in A$.*

3. THE EXAMPLE

The example is constructed in the euclidean plane $\mathbb{R}^2$. For each subset A of $\mathbb{R}^2$, let $-A$ denote the set $-A = \{-p \in \mathbb{R}^2 : p \in A\}$. Let $\theta = (0,0) \in \mathbb{R}^2$. Let $X_0 = \{0\} \times [-1,1]$. For each $n \geq 1$, let $X_n = \{\frac{1}{n}\} \times [0, \frac{1}{n}]$. For each $n \geq 1$, let L_n be a homeomorphic copy of the real line such that: (a) $L_n \subset (\frac{1}{n+1}, \frac{1}{n}) \times [0,1]$, (b) $L_n \cap ([0,1] \times \{1\}) \neq \emptyset$, and (c) $\mathrm{cl}_{\mathbb{R}^2}(L_n) = L_n \cup X_n \cup X_{n+1}$. Let $Y_0 = X_0 \cup (\bigcup\{X_n : n \geq 1\}) \cup (\bigcup\{L_n : n \geq 1\})$.

Finally, put $X = (-Y_0) \cup Y_0$.

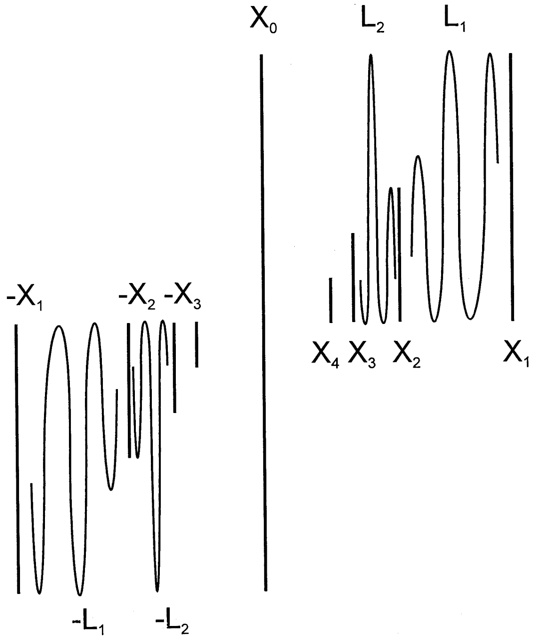
Clearly, X is a continuum (shown on the next page).

Proof. Now we prove that $C(X)$ is not g-contractible. Suppose, to the contrary, that $C(X)$ is g-contractible. Let $f : C(X) \to C(X)$ and $\psi : C(X) \times [0,1] \to C(X)$ be as in Lemma 1. From Lemma 3, it follows that $C(X)$ is connected im kleinen at any element $A \in C(X)$ such that $A \cap [(\bigcup\{L_n : n \geq 1\}) \cup (\bigcup\{-L_n : n \geq 1\})] \neq \emptyset$. Thus, if A is an element of $C(X)$ such that $C(X)$ is not connected im kleinen at A, then A is contained at some arc X_n or at some arc $-X_n$ $(n \geq 0)$.

Now we prove the following.

If $\mathcal{A}$ is a locally connected subcontinuum of $C(X)$, then $\mathcal{A}$ intersects finitely many sets of the form $C(X_n)$ (and finitely many sets of the form $C(-X_n)$).

Suppose, to the contrary, that there is a sequence of integers $n_1 < n_2 < \ldots$ such that $\mathcal{A} \cap C(X_{n_k}) \neq \emptyset$ for each $k \geq 1$. Fix an element $B_k \in \mathcal{A} \cap C(X_{n_k})$. Then $B_k \to \{\theta\}$. Thus $\{\theta\} \in \mathcal{A}$. Since $\mathcal{A}$ is locally connected, there exists a subcontinuum $\mathcal{B}$ of $\mathcal{A}$ such that $\{\theta\} \in \mathcal{B}$, $B_k \in \mathcal{B}$ for some k and $H(\{\theta\}, B) < \frac{1}{4}$ for each $B \in \mathcal{B}$, where H is the Hausdorff metric for $C(X)$. Let $B_0 = \bigcup \mathcal{B}$. From [6, Lemma 1.43],



B_0 is a subcontinuum of X such that $\theta \in B_0$, where the diameter of B_0 is less than $\frac{1}{2}$ and B_0 intersects X_{n_k}. From the construction of X one can see that the existence of such B_0 is impossible. This contradiction completes the proof of the claim.

Next we show that there exists a sequence of subcontinua $\{A_n\}_{n=1}^{\infty}$ of X such that $A_n \to \{\theta\}$, $f(A_n) \to \{\theta\}$ and $f(A_n) \subset \bigcup\{X_m : m \geq 1\}$ for each n.

Each continuum of one of the forms X_i or $-X_i$ is an arc, thus the hyperspaces $C(X_i)$ and $C(-X_i)$ are 2-cells ([6, Example 0.54]). Therefore, the sets $f[C(X_i)]$ and $f[C(-X_i)]$ are locally connected (see [7, Proposition 8.16]). Given $n \geq 1$, by the previous claim, there exists a positive integer k_n such that, $n \leq k_n$ and $f[C(X_0) \cup C(X_1) \cup C(-X_1) \cup ... \cup C(X_n) \cup C(-X_n)] \cap C(X_{k_n}) = \emptyset$. Fix a point $p \in X_{k_n}$; then it is easy to check that $C(X)$ is not connected im kleinen at $\{p\}$. According to Lemma 2, there exists an element $A_n \in C(X)$ such that $C(X)$ is not connected im kleinen at A_n and $f(A_n) = p$. Thus, A_n is contained in some set of the form X_r or in some set of the form $-X_r$ for some $r \geq 0$. By the choice of k_n, $r \geq n$. This implies that $H(\{\theta\}, A_n) < \frac{1}{n}$. Since $k_n \geq n$ and $p \in X_{k_n}$, $H(\{\theta\}, f(A_n)) = H(\{\theta\}, \{p\}) \leq \frac{1}{n}$. This completes the construction of A_n and proves the claim.

In a similar way, it is possible to construct a sequence of subcontinua $\{B_n\}_{n=1}^{\infty}$ of X such that $B_n \to \{\theta\}$, $f(B_n) \to \{\theta\}$ and $f(B_n) \subset \bigcup\{-X_m : m \geq 1\}$ for each n.

We are ready to obtain the final contradiction.

Since f is continuous, $f(A_n) \to f(\{\theta\})$. Thus, $f(\{\theta\}) = \{\theta\}$. We know that $\psi(\{\theta\}, 0) = f(\{\theta\}) = \{\theta\}$ and $\psi(\{\theta\}, 1) = X$; then the number $t_0 = \max\{t \in [0, 1] : \psi(\{\theta\}, t) = \{\theta\}\}$ is in the interval $[0, 1)$. Thus, $\psi(\{\theta\}, t_0) = \{\theta\}$. From the continuity of the map ψ, there exists a number $s \in (t_0, 1)$ such that $H(\{\theta\}, \psi(\{\theta\}, s)) < \frac{1}{8}$. Hence, $\{\theta\}$ is properly contained in $\psi(\{\theta\}, s)$.

On the other hand, $\lim \psi(B_n, s) = \psi(\{\theta\}, s) = \lim \psi(A_n, s)$, so there exists $R \geq 1$ such that, for each $r \geq R$, $\max\{H(\psi(\{\theta\}, s), \psi(A_r, s)), H(\psi(\{\theta\}, s), \psi(B_r, s))\} < \frac{1}{8}$. Given $r \geq R$, $\psi(A_r, s)$ is a subcontinuum of X such that it contains the set $\psi(A_r, 0) = f(A_r) \subset \bigcup\{X_m : m \geq 1\}$ and the diameter of $\psi(A_r, s)$ is less than $\frac{1}{2}$. From the construction of X, it follows that $\psi(A_r, s)$ is contained in $[0, 1] \times [0, 1]$. It follows that $\psi(\{\theta\}, s) \subset [0, 1] \times [0, 1]$. Using B_r instead of A_r, it follows that $\psi(\{\theta\}, s) \subset [-1, 0] \times [-1, 0]$. Hence, $\psi(\{\theta\}, s) = \{\theta\}$. This is a contradiction that completes the proof that $C(X)$ is not g-contractible.

References

[1] D. P. Bellamy, *The cone over the Cantor set - continuous maps from both directions*, Proc. Topology Conference (Emory University, Atlanta, Ga., 1970), J. W. Rogers, Jr., ed., 8-25. MR **49:**6190

[2] R. Engelking and A. Lelek, *Cartesian products and continuous images*, Colloq. Math. 8, (1961), 27-29. MR **24:**A1115

[3] J. T. Goodykoontz, Jr., *More on connectedness im kleinen and local connectedness in $C(X)$*, Proc. Amer. Math. Soc. 65, (1977), 357-364. MR **56:**9475

[4] I. Krzemińska and J. R. Prajs, *A non-g-contractible uniformly path connected continuum*, Topology Appl. 91, (1999), 151-158. MR **2000a:**54044

[5] S. B. Nadler, Jr., *Some problems concerning hyperspaces*, Topology Conference (V. P. I. and S. U.), Lecture Notes in Math., vol. 375, Springer-Verlag, New York, 1974, R. F. Dickman, Jr. and P. Fletcher, ed., 190-197. MR **51:**6692

[6] S. B. Nadler, Jr., *Hyperspaces of Sets, A text with research questions*, Pure and Applied Mathematics, A Series of Monographs and Textbooks, vol. 49, Marcel Dekker, Inc., New York, 1978. MR **58:**18330

[7] S. B. Nadler, Jr., *Continuum Theory, An introduction*, Pure and Applied Mathematics, A Series of Monographs and Textbooks, vol. 158, Marcel Dekker, Inc., New York, 1992. MR **93m:**54002

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