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A DIMENSIONAL RESULT FOR RANDOM SELF-SIMILAR SETS

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ABSTRACT. A very important property of a deterministic self-similar set is that its Hausdorff dimension and upper box-counting dimension coincide. This paper considers the random case. We show that for a random self-similar set, its Hausdorff dimension and upper box-counting dimension are equal *a.s.*

1. INTRODUCTION

The notation of self-similarity is of increasing interest in many theoretical and applied settings, such as random processes, turbulence, and dynamical systems. Self-similar sets are usually fractal sets, i.e. sets of non-integer Hausdorff dimension. If a compact set K is the union of sets $K_1, K_2, \cdots, K_N$, which are similar to K in the ratio $r_i \leq 1, i = 1, 2, \cdots, N$, and $K_i \cap K_j$ is "almost empty"(precisely open set condition), then $\dim_H K = \overline{\dim}_B K = \alpha$, where α is determined by the following equation:

$$\sum_{i=1}^{N} r_i^{\alpha} = 1.$$

This was made precise by Hutchinson in [6]. It is much more difficult to determine the dimension for self-similar sets with overlap. Falconer [3] (see also [4], Chapter 3) proved that a self-similar set (overlap or not) has a very nice property, that is $\dim_H K = \overline{\dim}_B K$.

The above-mentioned notion of deterministic self-similarity is very restrictive and does not apply to many phenomena. Recently there have been some models of random sets possessing the self-similar property. In particular, Falconer [2], Graf [5] and Mauldin and Williams [8] independently investigated random fractal sets by randomizing each step in Hutchinson's construction. Much more recently, Hutchinson and Ruschendorff [7] extended the contraction mapping method to prove various existence and uniqueness properties of random self-similar sets. For other references on this topic, see [1, 9] and the references therein.

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In this paper, we prove the analogous result to one of [3], that is, there exists a constant a such that $a.s.$

$$\dim_H K(\omega) = \overline{\dim}_B K(\omega) = a,$$

where $K(\omega)$ is the random self-similar set. The proofs in [3] cannot extend to the random case directly. In the deterministic case, each ball centered in K with radius at most the diameter of K, contains a subset of K which is the image of K under a map with Lipschitz constant comparable to the radius of the ball. This is an important point that underlies the argument in [3]. In the random setting although this is not true for individual realization of K, it is true in a certain probabilistic sense.

2. Notations and definitions

In this section, we recall the definitions of various dimensions and random self-similar sets and give some properties. A more detailed introduction and the proofs of the properties may be found in [1], [4] and [7].

Let $K \subset \mathbf{R^d}$. For any $s \geq 0$, the s-dimensional Hausdorff measure of K is given in the usual way by

$$\mathbf{H}^s(K) = \lim_{\delta \to 0} \inf\{\sum_i |U_i|^s : K \subset \bigcup_i U_i,\ 0 < |U_i| < \delta\},$$

where $|\cdot|$ denotes the diameter of a set. This leads to the definition of the Hausdorff dimension of K:

$$\dim_H K = \inf\{s : \mathbf{H}^s(K) < \infty\} = \sup\{s :\ \mathbf{H}^s(K) > 0\}. \tag{1}$$

Many other definitions of dimensions are encountered in the literature. Let $M_\epsilon(K)$ be the smallest number of closed balls of radius ϵ that cover K, and let $N_\epsilon(K)$ be the maximum number of disjoint closed balls of radius ϵ with centers in K. We define the lower and upper box-counting dimensions by

$$\underline{\dim}_B K = \liminf_{\epsilon \to 0} \frac{\log M_\epsilon(K)}{-\log \epsilon}, \tag{2}$$

$$\overline{\dim}_B K = \limsup_{\epsilon \to 0} \frac{\log M_\epsilon(K)}{-\log \epsilon}. \tag{3}$$

In [4], it was shown that

$$\underline{\dim}_B K = \liminf_{\epsilon \to 0} \frac{\log N_\epsilon(K)}{-\log \epsilon} = \liminf_{n \to \infty} \frac{\log N_{2^{-n}}(K)}{n \log 2} = \liminf_{n \to \infty} \frac{\log M_{2^{-n}}(K)}{n \log 2}, \tag{4}$$

$$\overline{\dim}_B K = \limsup_{\epsilon \to 0} \frac{\log N_\epsilon(K)}{-\log \epsilon} = \limsup_{n \to \infty} \frac{\log N_{2^{-n}}(K)}{n \log 2} = \limsup_{n \to \infty} \frac{\log M_{2^{-n}}(K)}{n \log 2}. \tag{5}$$

Finally, we can define packing dimension of K by

$$\dim_P K = \inf\{\sup \overline{\dim}_B A_i, K \subset \bigcup_i A_i\}. \tag{6}$$

It is well known (see [4]) that

$$\dim_H K \leq \underline{\dim}_B K \leq \overline{\dim}_B K, \tag{7}$$

$$\dim_H K \leq \dim_P K \leq \overline{\dim}_B K. \tag{8}$$

Let $J \subset \mathbf{R}^d$ be a fixed compact set with $J = \overline{int(J)}$. We are given a positive integer $N \geq 2$ and a probability measure μ on Sim^N, where Sim is the space of all similarities of $\mathbf{R}^d$ equipped with the usual topology of uniform convergence on compact sets. In this paper, we assume the following:

Assumptions.

(I) $S_i(J) \subset J$ for all $i = 1, 2, \cdots, N$ and μ-$a.s.$ $(S_1, S_2, \cdots, S_N)$.

(II) There exist $0 < \hat{r}_1 \leq \hat{r}_2 < 1$ such that $\hat{r}_1 \leq Lip(S_i) \leq \hat{r}_2$ for all $i = 1, 2, \cdots, N$ and μ-$a.s.$ $(S_1, S_2, \cdots, S_N)$.

In the sequel we often make use of symbolic dynamics. Let $\Sigma = \{1, 2, \cdots, N\}^{\mathbf{N}}$ be the code space over the indices $1, 2, \cdots, N, \Sigma_n = \{1, 2, \cdots, N\}^n$ the space of all sequence of length n, $n \geq 0$, $\Sigma_0 = \emptyset$, and $\Sigma_* = \bigcup_n \Sigma_n$. For $\tau \in \Sigma_n$, denote by $|\tau| = n$ the length of τ, and by $\tau|k$ the truncation of τ to the first k entries, $k \leq n$. For any $\tau \in \Sigma_*, \sigma \in \Sigma_* \cup \Sigma$, define $\tau * \sigma = (\tau_1, \tau_2, \cdots, \tau_{|\tau|}, \sigma_1, \sigma_2, \cdots)$. We write $\tau \prec \sigma$ if there exists a $\eta \in \Sigma_* \cup \Sigma$ with $\sigma = \tau * \eta$.

Define the space $\Omega = (Sim^N)^{\Sigma_*}$. Let $\Im$ be the product σ-algebra on Ω. Taking $\mathbf{P}$ the product measure with μ on each component, we get our primary probability space $(\Omega, \Im, \mathbf{P})$.

For $\omega \in \Omega$ and $\tau \in \Sigma_*$ write

$$\omega(\tau) = (S_{\tau*1}(\omega), S_{\tau*2}(\omega), \cdots, S_{\tau*N}(\omega))$$

and $S_\emptyset(\omega) = id$. By $\Im_k$ we denote the σ-algebra generated by all S_τ with $|\tau| \leq k$. For brevity, for any $\tau \in \Sigma_*$, write

$$\begin{aligned}
&\overline{S_\tau} = S_{\tau|1} \circ S_{\tau|2} \circ \cdots \circ S_{\tau||\tau|}, \\
&r_\tau = Lip(S_\tau), \\
&\overline{r}_\tau = Lip\overline{S}_\tau = r_{\tau|1} r_{\tau|2} \cdots r_{\tau||\tau|}, \\
&J_\tau = \overline{S}_\tau(J).
\end{aligned}$$

Further, let us introduce shift operators $\Delta_\tau : \Sigma_* \cup \Sigma \to \Sigma_* \cup \Sigma$ by

$$\Delta_\tau(\sigma) = \tau * \sigma,$$

and write

$$T_\tau \omega = \omega(\Delta_\tau).$$

Then we have $S_\sigma(T_\tau \omega) = S_{\tau*\sigma}(\omega)$.

Let us define a random mapping $\pi_\omega : \Sigma \to J$ by

$$\pi_\omega(\sigma) = \lim_{n\to\infty} \overline{S}_{\sigma|n}(\omega)(x_0). \tag{9}$$

This limit exists for $\mathbf{P}$-$a.s.$ $\omega \in \Omega$ and does not depend on the choice of x_0. We call a random compact set $K(\omega) = \pi_\omega(\Sigma)$ the random self-similar set.

We call a random subset $\Gamma \subset \Sigma_*$ a Markov stopping, if:

(1) For each $\sigma \in \Sigma$ and $\omega \in \Omega$, there exists a unique $\tau \in \Gamma(\omega)$ with $\tau \prec \sigma$, and

(2) $\{\omega \in \Omega : \tau \in \Gamma(\omega)\} \in \Im_{|\tau|}$ for all $\tau \in \Sigma_*$.

If Γ is a Markov stopping, let $\Im_\Gamma$ be the sub-σ-algebra of $\Im$ generated by $\{S_\eta$: there is a $\tau \in \Gamma$ with $\eta \prec \tau$ $\}$.

The set $K(\omega)$ fulfills the following invariance (see [1, 7]).

Theorem 1. *For* $\mathbf{P}$*-a.s.* $\omega \in \Omega$*, we have:*

(1) $K(\omega) = \bigcap_{n=0}^{\infty} \bigcup_{\sigma \in \Sigma_n} \overline{S}_\sigma(J)$,

(2) $K(\omega) = \bigcup_{\tau \in \Gamma} S_\tau(\omega) K(T_\tau \omega)$*, where the* $K(T_\tau \omega)$ *are i.i.d. copies of* $K(\omega)$ *and independent of* $\Im_\Gamma$.

3. Main result and its proof

In this section, we state and prove the main theorem of this paper.

Lemma 1. *Suppose the Assumptions are satisfied and let* $K(\omega)$ *be the random self-similar set. Then there exists a constant* a *such that* $\mathbf{P}$*-a.s.*

$$\dim_H K(\omega) = a.$$

Proof. Suppose there exists b such that

$$0 < \mathbf{P}\{\omega : \dim_H K(\omega) < b\} < 1.$$

By (2) of Theorem 1, $K(\omega) = \bigcup_{i=1}^{N} S_i(\omega) K(T_i \omega)$, thus

$$\dim_H K(\omega) = \max_{1 \le i \le N} \dim_H S_i(\omega) K(T_i \omega) = \max_{1 \le i \le N} \dim_H K(T_i \omega).$$

We have

$$\begin{aligned} &\mathbf{P}\{\omega : \dim_H K(\omega) < b\} \\ =&\mathbf{P}\{\omega : \max_{1 \le i \le N} \dim K(T_i \omega) < b\} \\ =&\prod_{i=1}^{N} \mathbf{P}\{\omega : \dim_H K(T_i \omega) < b\} \\ =&\left(\mathbf{P}\{\omega : \dim_H K(\omega) < b\}\right)^N, \end{aligned}$$

and this leads to a contradiction.

For any $\sigma \in \Sigma_*$ and $\epsilon > 0$, let

$$I_\epsilon^\sigma(\omega) = \{\tau \in \Sigma_* : |\overline{S}_\tau(T_\sigma \omega)(J)| \le \epsilon,\ |\overline{S}_{(\tau||\tau|-1)}(T_\sigma \omega)(J)| > \epsilon\}.$$

Define

$$\begin{aligned} &D_\epsilon^\sigma(\omega) = \{A \subset I_\epsilon^\sigma(\omega) : \forall \zeta, \eta \in A,\ \overline{S}_\zeta(T_\sigma \omega)(J) \cap \overline{S}_\eta(T_\sigma \omega)(J) = \emptyset\}, \\ &\overline{N}_\epsilon^{(\sigma)}(\omega) = \max_{A \in D_\epsilon^\sigma(\omega)} \sharp A. \end{aligned}$$

Let $A_\epsilon^\sigma(\omega)$ be a member of $D_\epsilon^\sigma(\omega)$ which realizes $\overline{N}_\epsilon^{(\sigma)}(\omega)$ and write

$$A_\epsilon^\sigma(\omega) = \{\tau_{\sigma,1}(\omega), \tau_{\sigma,2}(\omega), \cdots, \tau_{\sigma, \overline{N}_\epsilon^{(\sigma)}(\omega)}(\omega)\}.$$

For simplicity, write $I_\epsilon(\omega) = I_\epsilon^\emptyset(\omega), D_\epsilon(\omega) = D_\epsilon^\emptyset(\omega), \overline{N}_\epsilon(\omega) = \overline{N}_\epsilon^{(\emptyset)}(\omega), A_\epsilon(\omega) = A_\epsilon^\emptyset(\omega)$.

Lemma 2. *Let the Assumptions be satisfied. There exists a constant* $c > 0$ *such that* $\mathbf{P}$*-a.s. for any* $\epsilon > 0$,

$$\overline{N}_\epsilon(\omega) \ge N_\epsilon(K(\omega)),\ \ \overline{N}_\epsilon(\omega) \le c M_\epsilon(K(\omega)).$$

Proof. Let $B(x_1,\epsilon), B(x_2,\epsilon), \cdots, B(x_{N_\epsilon(K(\omega))},\epsilon)$ be $N_\epsilon(K(\omega))$ disjoint closed balls of radius ϵ and center in $K(\omega)$. For any $1 \le i \le N_\epsilon(K(\omega))$, by Theorem 1 (1), there exists $\mathbf{i} \in \Sigma$ such that

$$\bigcap_{n=1}^{\infty} \overline{S}_{(\mathbf{i}|n)}(\omega)(J) = x_i.$$

Choose n_0 such that

$$\overline{S}_{(\mathbf{i}|n_0)}(\omega)(J) \subset B(x_i,\epsilon), \quad \overline{S}_{(\mathbf{i}|n_0-1)}(\omega)(J)\backslash B(x_i,\epsilon) \neq \emptyset.$$

Then we have $|\overline{S}_{(\mathbf{i}|n_0-1)}(\omega)(J)| > \epsilon$.

Now we consider two cases:

(i) If $|\overline{S}_{(\mathbf{i}|n_0)}(\omega)(J)| \le \epsilon$, choose $\tau^{(i)} = (\mathbf{i}|n_0)$.

(ii) If $|\overline{S}_{(\mathbf{i}|n_0)}(\omega)(J)| > \epsilon$, then there exists $l \ge 1$ such that

$$|\overline{S}_{(\mathbf{i}|n_0+l)}(\omega)(J)| \le \epsilon, \quad |\overline{S}_{(\mathbf{i}|n_0+l-1)}(\omega)(J)| > \epsilon,$$

choose $\tau^{(i)} = (\mathbf{i}|n_0+l)$. Therefore we have

$$\overline{N}_\epsilon(\omega) \ge N_\epsilon(K(\omega)). \tag{10}$$

Now we prove the second inequality.

Let $\overline{S}_{\tau^{(i)}}(\omega)(J), i = 1,2,\cdots,\overline{N}_\epsilon(\omega)$, be $\overline{N}_\epsilon(\omega)$ disjoint sets satisfying for any $1 \le i \le \overline{N}_\epsilon(\omega)$,

$$|\overline{S}_{\tau^{(i)}}(\omega)(J)| \le \epsilon, \quad |\overline{S}_{(\tau^{(i)}||\tau^{(i)}|-1)}(\omega)(J)| > \epsilon.$$

Since $int(J) \neq \emptyset$, there exist $x \in J$ and $c_1 > 0$ such that $B(x,c_1) \subset J$; then $\overline{S}_{\tau^{(i)}}(\omega)(J)$ contains a closed ball with radius $c_1 \cdot \overline{r}_{\tau^{(i)}}$.

By Assumption (II),

$$c_1 \cdot \overline{r}_{\tau^{(i)}} \ge \frac{\hat{r}_1}{|J|} \cdot c_1 \cdot \overline{r}_{\tau^{(i)}||\tau^{(i)}|-1}|J| \ge \frac{\hat{r}_1}{|J|} \cdot c_1 \cdot \epsilon.$$

Let $B(x_1,\epsilon), B(x_2,\epsilon), \cdots, B(x_{M_\epsilon(K(\omega))},\epsilon)$ be $M_\epsilon(K(\omega))$ closed balls of radius ϵ such that $\bigcup_{i=1}^{M_\epsilon(K(\omega))} B(x_i,\epsilon) \supset K(\omega)$. For any $1 \le i \le M_\epsilon(K(\omega))$, define

$$B_i(\omega) = \{j : 1 \le j \le \overline{N}_\epsilon(\omega), \overline{S}_{\tau^{(j)}}(\omega)(J) \cap B(x_i,\epsilon) \neq \emptyset\}.$$

Then for any $j \in B_i(\omega)$, $\overline{S}_{\tau^{(j)}}(\omega)(J) \subset B(x_i, 2\epsilon)$. By volume estimating, we have

$$\lambda_d(B(x_i,2\epsilon)) \ge \sharp B_i(\omega) \cdot \lambda_d(B(0, \frac{\hat{r}_1}{|J|} \cdot c_1 \cdot \epsilon)),$$

where λ_d is the d-dimensional Lebesgue measure. That is,

$$\sharp B_i(\omega) \le \frac{(2\epsilon)^d}{(\frac{\hat{r}_1}{|J|})^d \cdot c_1^d \cdot (\epsilon)^d} = \frac{2^d \cdot |J|^d}{\hat{r}_1^d \cdot c_1^d} := c.$$

Thus,

$$M_\epsilon(K(\omega)) \ge \frac{1}{c}\overline{N}_\epsilon(\omega). \tag{11}$$

From Lemma 2 and (5), we immediately get the following corollary.

Corollary 1. *Let the Assumptions be satisfied. Then* $\mathbf{P}$*-a.s.*

$$\overline{\dim}_B K(\omega) = \limsup_{n\to\infty} \frac{\log \overline{N}_{2^{-n}}(\omega)}{n \log 2}.$$

Now let us prove the main result of this paper.

Theorem 2. *Let the Assumptions be satisfied. Then there exists a constant* a *such that* $\mathbf{P}$*-a.s.,*

$$\dim_H K(\omega) = \overline{\dim}_B K(\omega) = a.$$

Proof. By Lemma 1, we have that there exists a constant a such that $\dim_H K(\omega) = a$, $\mathbf{P}$-$a.s.$

For any $n \geq 1$, choose $\epsilon = 2^{-n}$, and define a random set $K_n(\omega) \subset K(\omega)$ as follows. Let

$$\begin{aligned}
F_{n,0}(\omega) &= J,\\
F_{n,1}(\omega) &= \bigcup_{\sigma\in A_\epsilon(\omega)} \overline{S}_\sigma(\omega)(J),\\
F_{n,2}(\omega) &= \bigcup_{\sigma\in A_\epsilon(\omega)} \bigcup_{\tau\in A_\epsilon^\sigma(\omega)} \overline{S}_{\sigma*\tau}(\omega)(J),\\
&\cdots\\
F_{n,k}(\omega) &= \bigcup_{\sigma_1\in A_\epsilon(\omega)} \bigcup_{\sigma_2\in A_\epsilon^{\sigma_1}(\omega)} \cdots \bigcup_{\sigma_k\in A_\epsilon^{\sigma_1*\sigma_2*\cdots*\sigma_{k-1}}(\omega)} \overline{S}_{\sigma_1*\sigma_2*\cdots*\sigma_k}(\omega)(J).
\end{aligned}$$

Let

$$K_n(\omega) = \bigcap_{k=0}^{\infty} F_{n,k}(\omega).$$

By the construction of $K(\omega)$, we have that $\{F_{n,k}(\omega), k \geq 0\}$ is a random recursive construction. This was studied by Mauldin and Williams [8] (see also [2]). On the other hand, by (9) and Theorem 1, it is clear that $K_n(\omega) \subset K(\omega)$, $\mathbf{P}$-$a.s.$ By Theorem 1.3 and Theorem 3.6 of [8], we have for $\mathbf{P}$-$a.s.$ ω,

$$\dim_H K_n(\omega) = b,$$

where b satisfies the following equation:

$$\mathbf{E}[\sum_{i=1}^{\overline{N}_{2^{-n}}(\omega)} \big(Lip(\overline{S}_{\tau_{\emptyset,i}(\omega)}(\omega))\big)^b] = 1.$$

Therefore

$$\mathbf{E}[\sum_{i=1}^{\overline{N}_{2^{-n}}(\omega)} \big(Lip(\overline{S}_{\tau_{\emptyset,i}(\omega)}(\omega))\big)^a] \leq 1,$$

that is,

$$\mathbf{E}\overline{N}_{2^{-n}}(\omega) \leq \frac{|J|^a \cdot 2^{na}}{\hat{r_1}^a}.$$

For any $\delta > 0$,

$$\sum_{n=1}^{\infty} \mathbf{P}\{\omega : \overline{N}_{2^{-n}}(\omega) \geq |J|^a \cdot 2^{n(a+\delta)}\} \leq \sum_{n=1}^{\infty} \frac{\mathbf{E}\overline{N}_{2^{-n}}(\omega)}{|J|^a \cdot 2^{n(a+\delta)}} \leq \frac{1}{\hat{r_1}^a} \cdot \sum_{n=1}^{\infty} 2^{-n\delta} < \infty.$$

By the Borel-Cantelli Lemma, we have

$$\mathbf{P}\{\omega : \overline{N}_{2^{-n}}(\omega) \geq |J|^a \cdot 2^{n(a+\delta)}\ i.o.\} = 0. \tag{12}$$

By Corollary 1 and (12), we have for $\mathbf{P}$-$a.s.\,\omega$,

$$\overline{\dim}_B K(\omega) \leq \limsup_{n\to\infty} \frac{\log |J|^a \cdot 2^{n(a+\delta)}}{n \log 2} = a + \delta.$$

Since δ is arbitrary, we have for $\mathbf{P}$-$a.s.\ \omega$

$$\dim_H K(\omega) = \overline{\dim}_B K(\omega) = a,$$

and we finish the proof of Theorem 2.

By (7), (8) and Theorem 2, we immediately have the following corollary.

Corollary 2. *Let the Assumptions be satisfied. Then there exists a constant a, such that for $\mathbf{P}$-a.s. ω,*

$$\dim_H K(\omega) = \dim_P K(\omega) = \underline{\dim}_B K(\omega) = \overline{\dim}_B K(\omega) = a.$$

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