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INTEGRAL REPRESENTATION FOR A CLASS OF VECTOR VALUED OPERATORS

LAKHDAR MEZIANI

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ABSTRACT. Let S be a compact space and let X, $\|\cdot\|_X$ be a (real, for simplicity) Banach space. We consider the space $C_X = C(S, X)$ of all continuous X-valued functions on S, with the supremum norm $\|\cdot\|_\infty$.

We prove in this paper a Bochner integral representation theorem for bounded linear operators

$$T : C_X \longrightarrow X$$

which satisfy the following condition:

$$x^*, y^* \in X^*, f, g \in C_X : x^* \circ f = y^* \circ g \Longrightarrow x^* \circ Tf = y^* \circ Tg$$

where X^* is the conjugate space of X. In the particular case where $X = \mathbb{R}$, this condition is obviously satisfied by every bounded linear operator

$$T : C_{\mathbb{R}} \longrightarrow \mathbb{R}$$

and the result reduces to the classical Riesz representation theorem.

If the dimension of X is greater than 2, we show by a simple example that not every bounded linear $T : C_X \longrightarrow X$ admits an integral representation of the type above, proving that the situation is different from the one dimensional case.

Finally we compare our result to another representation theorem where the integration process is performed with respect to an operator valued measure.

1. INTRODUCTION

The problem of the integral representation for certain classes of linear operators has been studied for a long time by several authors.

Among the most celebrated theorems which have been proved in this domain, one can cite the Riesz representation theorem ([3], p. 265, and the references therein).

We can briefly describe the problem we will be concerned with in this paper as follows:

Let us consider a compact Hausdorff space S and a Banach space X, $\|\cdot\|_X$ (to simplify matters, take X real).

Now, we consider a real finite signed measure μ on the σ-field $\mathfrak{B}_S$ of the Borel sets of S, and form the Bochner integral

$$Tf = \int_S f(s)\, \mu(ds) \tag{1}$$

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for continuous X-valued functions

$$f : S \longrightarrow X.$$

Then it is a simple consequence of Bochner integral properties [4] that relation (1) defines a bounded X-valued operator on the space $C_X = C(S, X)$ of all continuous X-valued functions on S equipped with the supremum norm $\|\cdot\|_\infty$.

By analogy with the Riesz representation theorem, cited above, it is natural to ask the following:

For which bounded linear operators

$$T : C_X \longrightarrow X$$

do we have the integral form (1) with respect to a signed measure μ on $\mathfrak{B}_S$?

In the case $X = \mathbb{R}$ we have the integral form for every linear bounded operator $T : C_{\mathbb{R}} \longrightarrow \mathbb{R}$. This is indeed the classical theorem of F. Riesz.

In this paper we consider arbitrary Banach space X, and will give a condition under which bounded linear X-valued operators on C_X do have the integral form (1) with respect to a suitable signed measure.

2. The main result

First we make precise some notations we will use in the sequel:

1. For each $x^* \in X^*$, let us introduce the linear operator $U_{x^*} : C_X \longrightarrow C_{\mathbb{R}}$ defined by

$$f \in C_X, \qquad U_{x^*} f = x^* \circ f. \tag{2}$$

   It may be seen that U_{x^*} is bounded and we have

$$\|U_{x^*}\| = \|x^*\|. \tag{3}$$

2. We denote by C_{XX} the Banach space of all linear bounded operators $T : C_X \longrightarrow X$ normed by

$$T \in C_{XX}, \|T\| = \sup\{\|Tf\|_X : f \in C_X, \|f\|_\infty \leq 1\}. \tag{4}$$

3. Finally we will be interested in the set H_{XX} of all T in C_{XX} satisfying condition (C) below:

$$x^*, y^* \in X^*,\ f, g \in C_X :\ x^* \circ f = y^* \circ g \Rightarrow x^* \circ Tf = y^* \circ Tg.$$

   It is easy to see that H_{XX} is a closed subspace of C_{XX}. Note also that in the case $X = \mathbb{R}$ we have

$$H_{XX} = C_{XX} = C^*_{\mathbb{R}}. \tag{5}$$

We are now in a position to give the main theorem of this paper:

Theorem 2.1. *There is an isometric isomorphism between the Banach space H_{XX} and the conjugate space $C^*_{\mathbb{R}}$ for each nontrivial Banach space X.*

In other words, there exists a linear bijective mapping

$$\varphi : H_{XX} \longrightarrow C^*_{\mathbb{R}}$$

such that

$$\|\varphi(T)\| = \|T\| \tag{6}$$

for every $T \in H_{XX}$.

For the proof we prepare two lemmas.

Lemma 2.2. *There exists a functional $z^* \in X^*$ with $\|z^*\| = 1$ such that for every h in $C_{\mathbb{R}}$ we can find a function f in C_X with*

$$h = z^* \circ f \quad \text{and} \quad \|f\|_\infty = \|h\|_\infty \, ; \tag{7}$$

in other words U_{z^*} is onto.

Proof. Choose $\alpha \in X$ with $\|\alpha\|_X = 1$. Then by the Hahn-Banach theorem there exists a functional $z^* \in X^*$ such that

$$z^*(\alpha) = \|\alpha\|_X = 1 \quad \text{and} \quad \|z^*\| = 1.$$

We show that U_{z^*} is onto. Let $h \in C_{\mathbb{R}}$ and define f in C_X by

$$s \in S, \qquad f(s) = h(s) \cdot \alpha. \tag{8}$$

Then we have

$$\begin{aligned} U_{z^*} f(s) &= (z^* \circ f)(s) \\ &= z^*(h(s) \cdot \alpha) \\ &= h(s)\, z^*(\alpha) \\ &= h(s) \quad (\text{since } z^*(\alpha) = 1)\, ; \end{aligned}$$

therefore

$$U_{z^*} f = h.$$

From (8) and the choice of α, it is clear that $\|f\|_\infty = \|h\|_\infty$. ☐

Lemma 2.3. *Let V be a bounded linear functional on $C_{\mathbb{R}}$ (i.e. $V \in C_{\mathbb{R}}^*$). Then*

$$\|V\| = \sup\{\|V \circ U_{x^*}\| : \|x^*\| \le 1\}. \tag{9}$$

Proof. For each x^* in X^*, we have

$$\begin{aligned} \|V \circ U_{x^*}\| &\le \|V\| \cdot \|U_{x^*}\| \\ &= \|V\| \cdot \|x^*\| \quad \text{in view of (3)} \\ &\le \|V\| \quad \text{if} \quad \|x^*\| \le 1. \end{aligned}$$

Therefore

$$\sup\{\|V \circ U_{x^*}\| : \|x^*\| \le 1\} \le \|V\|. \tag{10}$$

To prove the reverse inequality, put λ for the right-hand side of (9) and let $\varepsilon > 0$ be given; then citing (4) with $X = \mathbb{R}$, there exists $h \in C_{\mathbb{R}}$, $\|h\|_\infty \le 1$, such that

$$\|V\| - \varepsilon < |Vh|. \tag{11}$$

By Lemma 2.2 there exist $z^* \in X^*$ and $f \in C_X$ such that

$$\|z^*\| = 1, \quad h = U_{z^*} f, \quad \text{and} \quad \|f\|_\infty = \|h\|_\infty.$$

It is noteworthy that z^* does not depend on ε. But the function f, like h, may depend on it. Now with these ingredients we have

$$|Vh| = |V \circ U_{z^*}(f)| \le \|V \circ U_{z^*}\|$$

since $\|f\|_\infty \le 1$.

Hence

$$|Vh| \le \lambda$$

which gives, in view of (10) and (11),

$$\|V\| - \varepsilon < \lambda \leq \|V\| .$$

Since $\varepsilon > 0$ is arbitrary we conclude that $\lambda = \|V\|$ and this proves (9). $\square$

Proof of Theorem 2.1. This will be presented in several steps. We will define linear mappings

$$\varphi : H_{XX} \longrightarrow C_{\mathbb{R}}^{*} \quad \text{and} \quad \theta : C_{\mathbb{R}}^{*} \longrightarrow H_{XX},$$

and will show that φ and θ are isometries with $\varphi^{-1} = \theta$.

Step 1. We begin with φ which is a priori more subtle.

Let T be in H_{XX} and let h be in $C_{\mathbb{R}}$. By Lemma 2.2 we have $h = z^* \circ f$ for some z^* in X^* with $\|z^*\| = 1$, and some f in C_X with $\|f\|_\infty = \|h\|_\infty$. Let us define $V : C_{\mathbb{R}} \longrightarrow \mathbb{R}$ by

$$h \in C_{\mathbb{R}}, \quad Vh = z^* \circ T(f) . \tag{12}$$

We check that V is a well defined linear bounded functional on $C_{\mathbb{R}}$ (i.e. $V \in C_{\mathbb{R}}^*$).

Take f, g in C_X such that $h = z^* \circ f = z^* \circ g$ as in Lemma 2.2. Then since T is in H_{XX}, condition (C) is valid for T, and we have $z^*(Tf) = z^*(Tg)$; therefore V is well defined by (12). On the other hand V is linear since T is as well.

Finally from (12) we can write

$$\begin{aligned} |Vh| &= |z^* \circ T(f)| \\ &\leq \|z^* \circ T\| \cdot \|f\|_\infty \\ &= \|z^* \circ T\| \cdot \|h\|_\infty , \quad \text{since } \|f\|_\infty = \|h\|_\infty ; \end{aligned}$$

this shows that V is bounded.

Now define $\varphi : H_{XX} \longrightarrow C_{\mathbb{R}}^*$ by

$$T \in H_{XX}, \quad \varphi(T) = V \tag{13}$$

where V is defined from T by (12).

φ is linear. Indeed if T_1, $T_2 \in H_{XX}$, and if $V_1 = \varphi(T_1)$, $V_2 = \varphi(T_2)$, $V = \varphi(T_1 + T_2)$, then, with $h = z^* \circ f \in C_{\mathbb{R}}$ as in (12), we have

$$\begin{aligned} Vh &= z^* \circ (T_1 + T_2)(f) \\ &= z^* \circ T_1(f) + z^* \circ T_2(f) \\ &= V_1(h) + V_2(h) . \end{aligned}$$

Thus

$$\varphi(T_1 + T_2) = \varphi(T_1) + \varphi(T_2) ,$$

likewise

$$\varphi(aT) = a\varphi(T) , \quad \forall a \in \mathbb{R} , \forall T \in H_{XX}.$$

Step 2. For every T in H_{XX} and for every x^* in X^*, we have

$$V \circ U_{x^*} = x^* \circ T \tag{14}$$

where

$$V = \varphi(T) .$$

For $f \in C_X$ and $x^* \in X$, write $x^* \circ f = z^* \circ g$, using Lemma 2.2. Then

$$\begin{aligned} V \circ U_{x^*}(f) &= V(x^* \circ f) \\ &= z^* \circ T(g) \quad \text{(by (12))} \\ &= x^* \circ T(f) \quad \text{(by condition (C))}. \end{aligned}$$

Since f is arbitrary, we deduce (14).

Now it is clear from (14) that

$$\sup\{\|V \circ U_{x^*}\| : \|x^*\| \leq 1\} = \sup\{\|x^* \circ T\| : \|x^*\| \leq 1\}.$$

Citing Lemma 2.3 for the left-hand side we find

$$\|V\| = \sup\{\|x^* \circ T\| : \ \|x^*\| \leq 1\},$$

and a simple estimation of the right-hand side leads to

$$\|V\| = \|T\|. \tag{15}$$

This proves that φ is an isometry.

Step 3. We turn to the definition of the mapping

$$\theta : C_{\mathbb{R}}^* \longrightarrow H_{XX}.$$

If $V \in C_{\mathbb{R}}^*$, there exists by Riesz representation theorem a unique signed real measure on the Borel $\sigma-$field $\mathfrak{B}_S$, such that

$$h \in C_{\mathbb{R}}, \quad Vh = \int_S h(s)\,\mu(ds). \tag{16}$$

Furthermore, $\|V\| = \|\mu\|$ (=total variation of μ) (see [3] for more details).

With this measure μ we define the Bochner integral $\int_S f(s)\,\mu(ds)$, for $f \in C_X$, which turns out to be the bounded operator $T : C_X \longrightarrow X$ given by

$$f \in C_X, \quad Tf = \int_S f(s)\,\mu(ds). \tag{17}$$

Now put

$$\theta(V) = T. \tag{18}$$

Note that θ is well defined and linear from $C_{\mathbb{R}}^*$ into H_{XX}. (The fact that $\theta(V) \in H_{XX}$ is a consequence of a well known fact according to which $x^*\left(\int_S f(s)\,\mu(ds)\right) = \int_S x^* \circ f(s)\,\mu(ds)$ for each $x^* \in X^*$.)

Now we deduce from (16) and (17) that $x^* \circ T = V \circ U_{x^*}$ for every $x^* \in X^*$.

Arguing as in *Step* 2, we get

$$\|T\| = \|\theta(V)\| = \|V\|,$$

that is, θ is an isometry.

Step 4. The mapping $\theta \circ \varphi$ is the identity operator of H_{XX}.

Indeed let $T \in H_{XX}$, so $V = \varphi(T) \in C_{\mathbb{R}}^*$. We must show that $\theta(V) = T$.

Let μ be the real measure on $\mathfrak{B}_S$ corresponding to V. Then by (18) we have, for $f \in C_X$, $\theta(V)f = \int_S f(s)\,\mu(ds)$. On the other hand, if $x^* \in X^*$ and $f \in C_X$, then

$$\begin{aligned} x^* \circ \theta(V)(f) &= x^* \left(\int_S f(s)\, \mu(ds) \right) \\ &= \int_S x^* \circ f(s)\, \mu(ds) \\ &= V(x^* \circ f) \\ &= x^* \circ T(f) \end{aligned}$$

where the last equality results from (14).

Since x^* is arbitrary we deduce from the Hahn-Banach theorem that

$$\theta(V) f = Tf \quad \forall f \in C_X,$$

that is, $\theta(V) = T$, as was to be shown.

In the same way one can see that $\varphi \circ \theta$ is the identity operator of $C_{\mathbb{R}}^*$.

This completes the proof of Theorem 2.1. □

3. INTEGRAL REPRESENTATION OF OPERATORS IN THE CLASS H_{XX}

As an immediate consequence of Theorem 2.1, we have the following result which gives an integral representation of operators in H_{XX}. We use the notations of section 1.

Recall that S is a compact space and $\mathfrak{B}_S$ is the $\sigma-$field of the Borel sets of S.

Theorem 3.1. *Let T be an operator in H_{XX}. Then there exists a unique signed real measure μ on the $\sigma-$field $\mathfrak{B}_S$, such that:*

(i) $\forall f \in C_X \quad Tf = \int_S f(s)\, \mu(ds)$,
(ii) $\|T\| = \|\mu\|$ *(total variation of μ),*

the integral (i) *being in the sense of Bochner.*

Proof. Let $V = \varphi(T)$ as in the proof of Theorem 2.1 (*Step* 1 (13)). Then, write V as

$$h \in C_{\mathbb{R}}, \quad Vh = \int_S h(s)\, \mu(ds),$$

where μ is the (unique) real measure on $\mathfrak{B}_S$ given by Riesz theorem.

From *Step* 2 of the proof of Theorem 2.1 we know that

$$V(x^* \circ f) = x^* \circ T(f)$$

for every $x^* \in X^*$ and $f \in C_X$ (see (14)).

But

$$\begin{aligned} V(x^* \circ f) &= \int_S x^* \circ f(s)\, \mu(ds) \\ &= x^* \left(\int_S f(s)\, \mu(ds) \right). \end{aligned}$$

Therefore

$$x^* \circ T(f) = x^* \left(\int_S f(s)\, \mu(ds) \right)$$

for every $x^* \in X^*$ and $f \in C_X$ this implies (i).

To prove (ii), note that for every $x^* \in X^*$, $V \circ U_{x^*} = x^* \circ T$ and use Lemma 2.3 once more. □

4. Some remarks

The following lemma gives a consolidated form of condition (C).

Lemma 4.1. *Let* T *be an operator in* C_{XX}. *Then* T *satisfies condition* (C) *if and only if the following holds for each* $x^* \in X^*$*:*

$$T(\ker U_{x^*}) \subset \ker x^*. \tag{19}$$

Proof. Condition (C) implies (19):

Let $x^* \in X^*$ and let $f \in \ker U_{x^*}$; then $U_{x^*}(f) = x^* \circ f = 0$. By condition (C) we may write

$$x^* \circ Tf = x^* \circ T(0),$$

thus $Tf \in \ker x^*$.

Condition (19) implies condition (C):

Assume condition (19), i.e.,

$$T(\ker U_{x^*}) \subset \ker x^*, \text{ for all } x^* \in X^*.$$

Let $x^*, y^* \in X^*$, $f, g \in C_X$ and $x^* \circ f = y^* \circ g$. We need to show that

$$x^*Tf = y^*Tg.$$

Step 1: If $f = g$, then $(x^* - y^*)(f) = 0$ and thus $(x^* - y^*)(Tf) = 0$ by (19). It follows that $x^*(Tf) = y^*(Tg)$ in this case.

Step 2: If $x^* = y^*$, then $x^*(f - g) = 0$ and thus $x^*T(f - g) = 0$ by (19). Again $x^*(Tf) = y^*(Tg)$. If x^* and y^* are linearly dependent, then Step 2 can be applied.

Suppose next that x^* and y^* are linearly independent. Then there exists $x, y \in X$ so that

$$\begin{aligned} x^*(x) &= 1 = y^*(y), \\ x^*(y) &= 0 = y^*(x), \\ x^*(x+y) &= 1 = y^*(x+y). \end{aligned}$$

Set $h(s) = (x^* \circ f(s)) \cdot (x + y) = (y^* \circ g(s)) \cdot (x + y)$. Then $h \in C_X$ and $x^* \circ h(s) = x^* \circ f(s) = y^* \circ g(s) = y^* \circ h(s)$. By Step 1 $x^*Th = y^*Th$ and by Step 2, applied to the preceding equations,

$$x^*(Th) = x^*(Tf) \text{ and } y^*(Th) = y^*(Tg).$$

Combining the last three equalities, we get

$$x^* \circ Tf = y^* \circ Tg,$$

and condition (C) follows as wanted. ☐

Corollary 4.2. *An operator* $T : C_X \to X$ *has a Bochner integral representation if and only if* T *satisfies condition* (19).

Proof. If $Tf = \int_S f d\mu$ and $x^* \circ f = 0$, then $x^*(Tf) = x^*\left(\int_S f d\mu\right) = \int_S x^* f d\mu = 0$ which gives (19).

Conversely assume the validity of (19) for T; then condition (C) is satisfied by Lemma 4.1. Therefore, by appealing to Theorem 3.1, we get a Bochner integral representation for T. ☐

Remark 4.3. First of all let us recall that every X-valued bounded operator on C_X which has the integral form (1) satisfies condition (C). Note also that in the case $X = \mathbb{R}$ every bounded operator T on $C_{\mathbb{R}}$ satisfies condition (C), in other words $H_{\mathbb{R}\mathbb{R}} = C_{\mathbb{R}}^*$.

Now we show that if the dimension of X is greater than 2 the inclusion $H_{XX} \subset C_{XX}$ is strict. In view of Theorem 3.1 this means that the Riesz theorem fails to be valid on all of C_{XX} as in the one dimensional case, but is verified only for operators in the class H_{XX}. We prove the validity of this remark by:

Example 4.4. Let $X = \mathbb{R}^n$ be the Euclidean space of dimension $n \geq 2$. Let T be the bounded operator on C_X defined by the following recipe.

Fix $0 < \alpha < 1$ and also fix s_0 in the compact space S.

If $g = (g_1, g_2, \ldots, g_n)$ is a continuous $\mathbb{R}^n$-valued function on S, let us put

$$Tg = (\alpha g_1(s_0), g_2(s_0), \ldots, g_n(s_0)). \tag{20}$$

It is easy to check that T is a bounded $\mathbb{R}^n$-valued operator on $C_{\mathbb{R}^n}$.

We show that T does not satisfy condition (C).

Let z^* be the continuous functional on $\mathbb{R}^n$ given by

$$(y_1, y_2, \ldots, y_n) \in \mathbb{R}^n, \quad z^*(y_1, y_2, \ldots, y_n) = y_1 + y_2 + \ldots + y_n.$$

By a simple argument, construct a continuous function

$$h = (h_1, h_2, \ldots, h_n) : S \longrightarrow \mathbb{R}^n$$

such that

$$h_1(s) + h_2(s) + \ldots + h_n(s) = 0 \quad \forall s \in S, \tag{21}$$

$$h_1(s_0) \neq 0. \tag{22}$$

Then relation (21) says that $z^* \circ h \equiv 0$, i.e., $h \in KerU_{z^*}$.

By (20) we have

$$z^* \circ T(h) = \alpha h_1(s_0) + h_2(s_0) + \ldots + h_n(s_0),$$

and in view of (21) and (22) we conclude that $z^* \circ T(h) \neq 0$, that is $Th \notin Kerz^*$.

It follows that (19) of Lemma 4.1 is not satisfied, which at once implies that condition (C) fails for T.

This shows that T is not in the class H_{XX} for $X = \mathbb{R}^n, n \geq 2$.

It may be noticed more directly that there is no Borel measure μ on $\mathfrak{B}_S$ for which we have

$$Tg = \int_S g(s)\,\mu(ds) = \left(\int_S g_1(s)\,\mu(ds), \ldots, \int_S g_n(s)\,\mu(ds)\right). \tag{23}$$

Indeed if there were such measure, we would have $\mu = \delta_{s_0}$ (=Dirac mass on s_0); but it is obvious that (23) is impossible in this case.

5. A COMPARISON RESULT

For the sake of completeness, it may be of interest to compare Theorem 3.1 to other integral representation theorems, where the integration process is performed not with respect to scalar measure but with respect to vector measure (see [2] or [5] for the framework of this integration process).

In what follows we propose a comparison to Dinculeanu-Singer theorem where the integral representation is defined with respect to an operator valued measure.

Before citing the theorem, let us make precise some notations and facts which will be used in the sequel.

The symbols S, $\mathfrak{B}_S$, $C(S,X) = C_X$ have the meaning of the preceding sections. On the other hand let X, Y be Banach spaces and let

$$G : \mathfrak{B}_S \to \mathfrak{L}(X; Y^{**})$$

be a finitely additive vector measure on $\mathfrak{B}_S$ with values in the Banach space $\mathfrak{L}(X; Y^{**})$ of all bounded linear operators from X into the second conjugate space Y^{**} of Y. For each $y^* \in Y^*$ let us define the set function $G_{y^*} : \mathfrak{B}_S \to X^*$ by

$$G_{y^*}(E)(x) = y^* G(E)(x), \tag{24}$$

where $E \in \mathfrak{B}_S$ and $x \in X$.

Then it is known that G_{y^*} is a finitely additive X^* -valued measure and moreover if it is regular, then it is countably additive (see [1], chap. VI).

Let us observe that formula (24) allows us to define a family of scalar measures $\Lambda^x_{y^*}$ on $\mathfrak{B}_S$ via

$$E \in \mathfrak{B}_S,\ x \in X,\ y^* \in Y^* : \Lambda^x_{y^*}(E) = G_{y^*}(E)(x). \tag{25}$$

We are now in a position to give the Dinculeanu-Singer theorem. The version given below will suffice for our purpose.

Theorem 5.1 (Dinculeanu-Singer)**.** *Every bounded linear operator $T : C_X \to Y$ determines a unique finitely additive measure $G : \mathfrak{B}_S \to \mathfrak{L}(X; Y^{**})$ such that*

$$Tf = \int_S f dG \text{ for all } f \in C_X. \tag{26}$$

Moreover, for each $y^* \in Y^*$, G_{y^*} is a regular countably additive bounded measure on $\mathfrak{B}_S$ with values in X^* and we have

$$T^* y^* = G_{y^*}. \tag{27}$$

In (27) *we use the identification between the dual space C^*_X and the Banach space* $\operatorname{rcab}(\mathfrak{B}_S, X^*)$ *of all regular countably additive bounded measures on $\mathfrak{B}_S$ with values in X^*.*

For the proof see [2] or [5] in the general setting of topological vector spaces.

As an immediate consequence we have:

Proposition 5.2. *The scalar measures $\Lambda^x_{y^*}$, defined by* (25) *with the G given in Theorem* 5.1*, are regular.*

Proof. The measures G_{y^*} are regular by Theorem 5.1. ☐

Now we take $Y = X$ and we consider a bounded operator $T : C_X \to X$. Then we have the following comparison result.

Theorem 5.3. *Assume that T satifies condition* (C) (*or equivalently condition* (19))*. Let μ be the scalar measure giving the Bochner integral form* (1) *for T according to Theorem* 3.1 *and let G be the operator valued measure giving the integral form* (26)*, by virtue of Dinculeanu-Singer theorem. Then we have*

$$G(.) = \mu(.) \cdot \gamma \tag{28}$$

where γ is the canonical isomorphism of X into X^{**}.

Before giving the proof let us say that equation (28) has the following meaning:

$$G(A)(x) = \mu(A) \cdot \gamma(x) \quad \text{for all } A \in \mathfrak{B}_S \text{ and } x \in X. \tag{29}$$

Since for every y^* in X^*, $\gamma(x)(y^*) = y^*(x)$, this in turn is equivalent to

$$G_{y^*}(A)(x) = \mu(A) \cdot y^*(x) \text{ for all } A \text{ in } \mathfrak{B}_S,\ x \in X \text{ and } y^* \text{ in } X^*. \tag{30}$$

Proof of Theorem 5.3. Let f in C_X be of the form $f(s) = g(s) \cdot x$, where $g : S \to \mathbb{R}$ is a continuous scalar function on S and x a fixed vector in X. Then by Theorems 3.1 and 5.1

$$Tf = \int_S g(s) \cdot x dG = \int_S g(s) \cdot x d\mu.$$

Applying $y^* \in X^*$ to Tf gives

$$y^*(Tf) = T^* y^*(f) = \int_S f(s)\, dG_{y^*} = \int_S g(s) \cdot x dG_{y^*},$$

where the second equality follows from (27). But $\int_S g(s) \cdot x dG_{y^*} = \int_S g(s)\, d\Lambda^x_{y^*}$ as may be seen by first taking g as a simple function (using (25)) and then extending it with standard integration tools (see [5], equation 9, p. 381).

Now, going back to the Bochner form of Tf and applying again an $y^* \in X^*$, we obtain

$$y^* Tf = \int_S g(s)\, y^*(x)\, d\mu.$$

Finally we deduce from the two preceding forms of y^*Tf that

$$\int_S g(s)\, y^*(x)\, d\mu = \int_S g(s)\, d\Lambda^x_{y^*}$$

for every continuous $g : S \to \mathbb{R}$.

But, in view of Theorem 3.1 and Proposition 5.2 respectively, the measures $y^*(x)\,\mu$ and $\Lambda^x_{y^*}$ are scalar regular. Thus by appealing to the classical Riesz theorem, we conclude that

$$y^*(x)\,\mu(A) = \Lambda^x_{y^*}(A) = G_{y^*}(A)(x),$$

which is (30), and consequently (28) is proved. □

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Department of Mathematics, Faculty of Science, University of Batna, Algeria
E-mail address: `mezianilakhdar@hotmail.com`