PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 130, Number 8, Pages 2319–2324
S 0002-9939(02)06297-4
Article electronically published on January 17, 2002

L^1 CONVERGENCE OF THE RECONSTRUCTION FORMULA FOR THE POTENTIAL FUNCTION

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(Communicated by Carmen C. Chicone)

ABSTRACT. It is known that the potential function of the Sturm-Liouville problem can be reconstructed from the nodal data by a pointwise limit. We show that this convergence is in fact L^1.

1. INTRODUCTION

Consider the Sturm-Liouville problem

$$-y'' + q(x)y = \lambda y$$

such that

$$\begin{cases} y(0)\cos\alpha + y'(0)\sin\alpha = 0, \\ y(1)\cos\beta + y'(1)\sin\beta = 0, \end{cases}$$

where $q \in L^1(0,1)$, and $\alpha, \beta \in [0,\pi)$.

In 1988, J.R. McLaughlin [8] showed that knowledge of the nodal set of the eigenfunctions alone can determine the potential function of the Sturm-Liouville problem up to a constant. This is the so-called inverse nodal problem [1]. C.F. Yang [13] showed that this uniqueness result is valid for any $q \in L^1(0,1)$. Independently C.L. Shen [10] had similar ideas for the density function ρ of the string equation $y'' + \lambda\rho y = 0$. See also [1, 11, 12]. The uniqueness result holds for the situation that ρ is only assumed to be of bounded variation [3].

The inverse nodal problem in two dimension is much more difficult. Hald and McLaughlin [2] proved the uniqueness result for the potential function of the Schrödinger operator defined on a rectangular domain in $\mathbf{R}^2$. Some results for density function of string equation on a membrane can be found in [9, 7].

In this note we concern ourselves with the reconstruction of the potential function of the one-dimensional Sturm-Liouville problem. Law-Shen-Yang [4], improving a result of X.F. Yang [13], gave a reconstruction formula for the potential function q.

Let λ_n be the n-th eigenvalue, $s_n = \sqrt{\lambda_n}$ and $x_i^{(n)}$ be the i-th nodal point of the n-th eigenfunction y_n. In other words, $y_n(x_i^{(n)}) = 0$, $i = 1, 2, ..., n-1$. Let $I_i^{(n)} = (x_i^{(n)}, x_{i+1}^{(n)})$, and $l_i^{(n)}$ be the nodal length where $l_i^{(n)} = |I_i^{(n)}| = x_{i+1}^{(n)} - x_i^{(n)}$.

Received by the editors November 21, 2000 and, in revised form, February 20, 2001.
2000 *Mathematics Subject Classification.* Primary 34A55; Secondary 34B24.
Key words and phrases. Inverse nodal problem, potential function, reconstruction formula.

Define $x_0^{(n)} = 0$, $x_n^{(n)} = 1$. We also define the function $j_n(x)$ to be the largest index j such that $0 \le x_j^{(n)} \le x$. Thus, $j = j_n(x)$ if and only if $x \in [x_j^{(n)}, x_{j+1}^{(n)})$.

Theorem 1.1 ([4]). *The potential function $q \in L^1(0,1)$ satisfies*

$$q(x) = \lim_{n\to\infty} 2s_n^2 \left(\frac{s_n l_j^{(n)}}{\pi} - 1 \right)$$

for a.e. $x \in (0,1)$, with $j = j_n(x)$.

Note that Theorem 1.1, with the asymptotic expression for s_n (see [5, Lemma 2] and [6, Lemma 2.2]) implies that $q(x) = \lim_{n\to\infty} F_n(x)$, where F_n is determined only by the nodal data and the constant $\int_0^1 q$:

(a) If $\alpha = \beta = 0$ or $\alpha,\ \beta > 0$, then

$$F_n(x) = 2n^2\pi^2 \left\{ n l_j^{(n)} - 1 + \frac{l_j^{(n)}}{n\pi^2} \left(\frac{1}{2} \int_0^1 q - \text{scot}\alpha + \text{scot}\beta \right) \right\} .$$

(b) If $\alpha = 0 < \beta$ or $\beta = 0 < \alpha$, then

$$F_n(x) = 2(n - \frac{1}{2})^2\pi^2 \left\{ (n - \frac{1}{2}) l_j^{(n)} - 1 + \frac{l_j^{(n)}}{n\pi^2} (\frac{1}{2} \int_0^1 q - \text{scot}\alpha + \text{scot}\beta) \right\} .$$

Here, $\text{scot}\gamma = 0$ if $\gamma = 0$; $\text{scot}\gamma = \cot\gamma$ otherwise.

Since q is in L^1, one may ask whether the convergence is in fact L^1. The answer is affirmative.

Theorem 1.2. *F_n converges to q in L^1.*

2. Proof

Lemma 2.1 ([5]). *Let $q \in L^1$. Then*

$$l_i^{(n)} = \frac{\pi}{s_n} + \frac{1}{2s_n^2} \int_{x_i^{(n)}}^{x_{i+1}^{(n)}} (1 + \alpha_0 \cos(2s_n y))\, q(y) dy + o(\frac{1}{s_n^3}) \ , \tag{2.1}$$

where $\alpha_0 = 1$ if $\alpha > 0$, $\alpha_0 = -1$ if $\alpha = 0$.

In the above lemma, the order estimate is independent of i. As a result,

$$\frac{s_n l_i^{(n)}}{\pi} = 1 + O(\frac{1}{s_n}) = 1 + O(\frac{1}{n}) \ .$$

Lemma 2.2. *Suppose the sequence $f_k \in C[0,1]$ converges to f in L^1. For any $\epsilon > 0$, then with $j = j_n(x)$,*

$$\| \frac{s_n}{\pi} \int_{x_j^{(n)}}^{x_{j+1}^{(n)}} (f_k(y) - f(y)) dy \|_1 < \epsilon$$

for all sufficiently large n and k.

Proof. By Lemma 2.1 and the observation that the integral $\int_{x_j^{(n)}}^{x_{j+1}^{(n)}} (f_k(y) - f(y))\, dy$ is constant on any nodal interval $I_j^{(n)}$,

$$\begin{aligned}
&\int_0^1 \left| \frac{s_n}{\pi} \int_{x_j^{(n)}}^{x_{j+1}^{(n)}} (f_k(y) - f(y))\, dy \right| \, dx \\
&= \sum_{i=0}^{n-1} \frac{s_n l_i^{(n)}}{\pi} \left| \int_{x_i^{(n)}}^{x_{i+1}^{(n)}} (f_k(y) - f(y))\, dy \right| \\
&\le \sum_{i=0}^{n-1} (1 + O(\frac{1}{n})) \int_{x_i^{(n)}}^{x_{i+1}^{(n)}} |f_k(y) - f(y)| \, dy \\
&= (1 + O(\frac{1}{n})) \int_0^1 |f_k(y) - f(y)| \, dy \ ,
\end{aligned}$$

and hence converges to 0 as $k \longrightarrow \infty$. □

Lemma 2.3. *Suppose $q \in L^1(0,1)$. Then as $n \longrightarrow \infty$, with $j = j_n(x)$,*

$$\left\| \frac{s_n}{\pi} \int_{x_j^{(n)}}^{x_{j+1}^{(n)}} q(y) dy - q(x) \right\|_1 \longrightarrow 0 \ .$$

Proof. We first show that the result holds if q is continuous on [0,1]. Let $M = \max_{x \in (0,1)} |q(x)|$ and $x_0 \in [x, y]$. By intermediate value theorem, there exists $\xi \in (x, y)$ such that

$$\left| \frac{1}{y - x} \int_x^y q - q(x_0) \right| = |q(\xi) - q(x_0)| \ .$$

Due to the uniform continuity of q there is a $\delta > 0$ such that the difference is small whenever $|y - x| < \delta$. Hence, given $\epsilon > 0$, when n is large enough such that $l_j^{(n)} < \delta$ and $|\frac{s_n l_j^{(n)}}{\pi} - 1| < \min\{\frac{\epsilon}{M}, 1\}$, with $j = j_n(x)$, we have

$$\begin{aligned}
&\left| \frac{s_n}{\pi} \int_{x_j^{(n)}}^{x_{j+1}^{(n)}} q(y)\, dy - q(x) \right| \\
&\quad \le \quad \left| \frac{s_n(x_{j+1}^{(n)} - x_j^{(n)})}{\pi} \left[\frac{1}{x_{j+1}^{(n)} - x_j^{(n)}} \int_{x_j^{(n)}}^{x_{j+1}^{(n)}} q - q(x) \right] \right| + |q(x)| \left| \frac{s_n l_j^{(n)}}{\pi} - 1 \right| \\
&\quad \le \quad 2\epsilon \ .
\end{aligned}$$

Therefore, if $q \in C[0,1]$, then $\| \frac{s_n}{\pi} \int_{x_j^{(n)}}^{x_{j+1}^{(n)}} q(y)\, dy - q(x) \|_1$ can be arbitrarily small for all large n.

Since $C[0,1]$ is dense in $L^1(0,1)$, for any $q \in L^1(0,1)$ there exists a sequence $q_k \in C[0,1]$ that converges to q in $L^1(0,1)$. Now

$$\begin{aligned}
&\int_0^1 \left| \frac{s_n}{\pi} \int_{x_j^{(n)}}^{x_{j+1}^{(n)}} q(y)dy - q(x) \right| \, dx \\
&\le \int_0^1 \left| \frac{s_n}{\pi} \int_{x_j^{(n)}}^{x_{j+1}^{(n)}} (q(y) - q_k(y))dy \right| \, dx \\
&\quad + \int_0^1 \left| \frac{s_n}{\pi} \int_{x_j^{(n)}}^{x_{j+1}^{(n)}} q_k(y) \, dy - q_k(x) \right| \, dx \\
&\quad + \int_0^1 |q_k(x) - q(x)| \, dx \, .
\end{aligned}$$

For any $\epsilon > 0$, fix k large enough such that the first and last terms are smaller than ϵ. Then for all n large enough, the second term is smaller than 2ϵ by above. Hence, as $n \longrightarrow \infty$,

$$\left\| \frac{s_n}{\pi} \int_{x_j^{(n)}}^{x_{j+1}^{(n)}} q(y)dy - q(x) \right\|_1 \longrightarrow 0 \, .$$

□

Proof of Theorem 1.2. Since

$$\left| F_n(x) - 2s_n^2 (\frac{s_n l_j^{(n)}}{\pi} - 1) \right| = O(\frac{1}{n}),$$

by an asymptotic estimate of s_n, it suffices to show that as $n \longrightarrow \infty$,

$$\left\| 2s_n^2 (\frac{s_n l_j^{(n)}}{\pi} - 1) - q \right\|_1 \longrightarrow 0.$$

By (2.1), we have

$$2s_n^2 \left(\frac{s_n l_j^{(n)}}{\pi} - 1 \right) = \frac{s_n}{\pi} \int_{x_j^{(n)}}^{x_{j+1}^{(n)}} q + \frac{\alpha_0 s_n}{\pi} \int_{x_j^{(n)}}^{x_{j+1}^{(n)}} \cos(2s_n y) q(y) \, dy + o(1) \, .$$

Hence we only need to prove that as $n \longrightarrow \infty$,

$$\left\| \frac{s_n}{\pi} \int_{x_j^{(n)}}^{x_{j+1}^{(n)}} q - q \right\|_1 \longrightarrow 0,$$

and

$$\left\| \frac{s_n}{\pi} \int_{x_j^{(n)}}^{x_{j+1}^{(n)}} \cos(2s_n y) q(y) \, dy \right\|_1 \longrightarrow 0 \, . \tag{2.2}$$

The first limit holds because of Lemma 2.3. By the proof of [4, Theorem 3.2], the sequence of functions

$$h_n(x) = \frac{s_n}{\pi} \int_{x_j^{(n)}}^{x_{j+1}^{(n)}} \cos(2s_n y) q(y) \, dy$$

converges to 0 for a.e. $x \in (0,1)$. Furthermore,

$$|h_n(x)| \leq \frac{s_n}{\pi} \int_{x_j^{(n)}}^{x_{j+1}^{(n)}} |q(y)| dy = g_n(x)$$

and

$$\begin{aligned} \int_0^1 g_n(x) dx &= \sum_{i=0}^{n-1} \frac{s_n l_i^{(n)}}{\pi} \int_{x_i^{(n)}}^{x_{i+1}^{(n)}} |q(y)| dy \\ &= \left(1 + O(\frac{1}{n})\right) \|q\|_1 \ . \end{aligned}$$

Thus we may apply the Lebesgue dominated convergence theorem to show that (2.2) is valid. The proof is complete. □

Acknowledgement

We thank Ngai-Ming Mok of the University of Hong Kong for suggesting the problem. The authors are partially supported by National Science Council, Taiwan, Republic of China.

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