

BOUNDEDNESS OF THE BERGMAN TYPE OPERATORS ON MIXED NORM SPACES

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ABSTRACT. Conditions sufficient for boundedness of the Bergman type operators on certain mixed norm spaces $L_{p,q}(\varphi)$ ($0 < p < 1, 1 < q < \infty$) of functions on the unit ball of C^n are given, and this is used to solve Gleason's problem for the mixed norm spaces $H_{p,q}(\varphi)$ ($0 < p < 1, 1 < q < \infty$).

1. INTRODUCTION

For $z = (z_1, z_2, \dots, z_n)$, $w = (w_1, w_2, \dots, w_n) \in C^n$, we shall denote the inner product of z, w by $\langle z, w \rangle = \sum_{j=1}^n z_j \overline{w_j}$ and the norm of z by $\|z\| = \sqrt{\langle z, z \rangle}$. Let $B = B_n = \{z \in C^n : \|z\| < 1\}$ be the open unit ball in C^n . The space of holomorphic functions on B will be denoted by $H(B)$. A positive continuous function φ on $[0,1)$ is normal (see [1]) if there exist $0 < a < b, 0 \leq r_0 < 1$ such that:

- (1) $\frac{\varphi(r)}{(1-r)^a}$ is nonincreasing for $r_0 \leq r < 1$ and $\lim_{r \rightarrow 1} \frac{\varphi(r)}{(1-r)^a} = 0$;
- (2) $\frac{\varphi(r)}{(1-r)^b}$ is nondecreasing for $r_0 \leq r < 1$ and $\lim_{r \rightarrow 1} \frac{\varphi(r)}{(1-r)^b} = \infty$.

For $0 < p < \infty$, $0 < q < \infty$ and a normal function φ , let $L_{p,q}(\varphi)$ denote the spaces of measurable functions on B with

$$\|f\|_{p,q,\varphi} = \left\{ \int_0^1 r^{2n-1} (1-r)^{-1} \varphi^p(r) M_q^p(r, f) dr \right\}^{1/p} < \infty,$$

where

$$M_q(r, f) = \left\{ \int_{\partial B} |f(r\zeta)|^q d\sigma(\zeta) \right\}^{1/q}.$$

Let $1 < q < \infty$. Equipped with the above norm, $L_{p,q}(\varphi)$ is a Banach space for $p \geq 1$. When $0 < p < 1$, $\|\cdot\|_{p,q,\varphi}^p$ is a quasinorm on $L_{p,q}(\varphi)$. Thus, $L_{p,q}(\varphi)$ is a metric space if supplied with the distance $d(f, g) = \|f - g\|_{p,q,\varphi}^p$ and the vector space operations are continuous in this metric. That this d is complete is proved in the same way as in the familiar case $p \geq 1$. There is nothing to guarantee that $\|\alpha f\|_{p,q,\varphi}^p = |\alpha| \|f\|_{p,q,\varphi}^p$, however, and so $\|\cdot\|_{p,q,\varphi}^p$ may not be a norm. So $L_{p,q}(\varphi)$ is a Frechet space but not a Banach space. The reader is referred to [2] for the

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basic theory of Frechet and Banach spaces. For $s \in R, t > 0$, the operator $P_{s,t}$ on $L_{p,q}(\varphi)$ is given by

$$P_{s,t}f(z) = c_{n,t}(1 - \|z\|^2)^s \int_B \frac{(1 - \|w\|^2)^{t-1} f(w)}{(1 - \langle z, w \rangle)^{n+t+s}} dV(w), \quad \forall f \in L_{p,q}(\varphi),$$

where the complex power is understood to be principal branches,

$$c_{n,t} = C_{n+t-1}^n = \frac{\Gamma(n+t)}{\Gamma(t)\Gamma(n+1)}.$$

G.B. Ren and J.H. Shi in [1] show that: If $t > b > a > -s$, then $P_{s,t}$ is a bounded operator of $L_{p,q}(\varphi)$ into $L_{p,q}(\varphi)$ ($1 \leq p < \infty, 1 \leq q < \infty$), but the problem which is still unsolved is the case $0 < p < 1$. We know that the main tools for proving the above results are Hölder’s inequality and the results due to Forelli and Rudin (e.g. see [3], Proposition 2.7 or Lemma 3.1, etc.). Because Hölder’s inequality can be used only in the case $1 < p < \infty$, the method in [5], [6], [1],[7] and [8] cannot deal with the case $0 < p < 1$. In order to overcome this difficulty, we will exploit the Hardy-Littlewood analytical technique in [9]. Here we discuss the boundedness of the operator $P_{s,t}$ on $L_{p,q}(\varphi)$ ($0 < p < 1, 1 < q < \infty$) and obtain a sufficient condition. One of the main points of the paper is to extend some results for a Banach space setting to a Frechet space setting. Our main result is:

Theorem A. *Let $0 < p < 1, 1 < q < \infty$. If $t > b > a > -s$ and φ is a normal function, then $P_{s,t} : L_{p,q}(\varphi) \rightarrow L_{p,q}(\varphi)$ is a bounded operator.*

Throughout this paper, the letter C stands for a positive different constant.

2. BACKGROUND AND PRELIMINARIES

The following lemmas will be needed in the proof of Theorem A.

Lemma 1 ([4], Proposition 1.4.10). *For $z \in B, c$ real, $t > -1$, define*

$$J_{c,t}(z) = \int_B \frac{(1 - \|w\|^2)^t}{|1 - \langle z, w \rangle|^{n+1+t+c}} dV(w).$$

Then:

- (1) *when $c < 0$, $J_{c,t}$ is bounded in z ;*
- (2) *when $c > 0$, $J_{c,t}(z) \sim (1 - \|z\|^2)^{-c}$, where \sim denotes $(1 - \|z\|^2)^c J_{c,t}(z)$ has a positive limit ($\|z\| \rightarrow 1$).*

Lemma 2 ([1], Lemma 2.3). *Let φ be a normal function. If $s + t > b > a > s$, then*

$$\int_0^1 \frac{\varphi^p(\rho) d\rho}{(1 - \rho)^{ps+1}(1 - r\rho)^{pt}} \leq C \frac{\varphi^p(r)}{(1 - r)^{p(s+t)}} \quad (0 \leq r < 1, p > 0).$$

Lemma 3 ([1], Lemma 2.1). *If $s + t > 0, 1 \leq q < \infty$, then*

$$M_q(\rho, P_{s,t}f) \leq C(1 - \rho^2)^s \int_0^1 \frac{r^{2n-1}(1 - r^2)^{t-1}}{(1 - r\rho)^{t+s}} M_q(r, f) dr.$$

This follows from Hölder’s inequality and Lemma 1 (see [1]).

In order to prove our main result, we first prove the following lemma.

Lemma 4. *If $t > b > a > -s, 0 < p < 1$ and $1 < q < \infty$, then*

$$M_q^p(\rho, P_{s,t}f) \leq C(1 - \rho^2)^{ps} \int_0^1 \frac{r^{p(2n-1)}(1 - r)^{pt-1}}{(1 - r\rho)^{p(t+s)}} M_q^p(r, f) dr.$$

Proof. Denote $s_k = 1 - 2^{-k}$. By the monotonicity of the integral means $M_q^p(r, f)$ with respect to r , Lemma 3 and the elementary inequality $(a + b)^p \leq a^p + b^p$ ($a, b \geq 0, 0 < p < 1$), we obtain

$$\begin{aligned} M_q^p(\rho, P_{s,t}f) &\leq C(1 - \rho^2)^{ps} \left(\int_0^1 \frac{r^{2n-1}(1-r)^{t-1}}{(1-r\rho)^{t+s}} M_q(r, f) dr \right)^p \\ &= C(1 - \rho^2)^{ps} \left(\sum_{k=1}^{\infty} \int_{s_{k-1}}^{s_k} \frac{r^{2n-1}(1-r)^{t-1}}{(1-r\rho)^{t+s}} M_q(r, f) dr \right)^p \\ &\leq C(1 - \rho^2)^{ps} \left(\sum_{k=1}^{\infty} \frac{s_k^{2n-1}(1-s_{k-1})^t}{(1-s_k\rho)^{t+s}} M_q(s_k, f) \right)^p \\ &\leq C(1 - \rho^2)^{ps} \sum_{k=1}^{\infty} \frac{s_k^{p(2n-1)}(1-s_{k-1})^{pt}}{(1-s_k\rho)^{p(t+s)}} M_q^p(s_k, f) \\ &\leq C(1 - \rho^2)^{ps} \int_0^1 \frac{r^{p(2n-1)}(1-r)^{pt-1}}{(1-r\rho)^{p(t+s)}} M_q^p(r, f) dr. \end{aligned}$$

This completes the proof of the lemma.

3. PROOF OF THE MAIN THEOREM

We can now prove the main result of this paper.

Proof of Theorem A. For any $f \in L_{p,q}(\varphi)$, by Lemma 3, Lemma 2, Lemma 4 and Fubini's Theorem, we have

$$\begin{aligned} \|P_{s,t}f\|_{p,q,\varphi}^p &\leq \int_0^1 (1-\rho)^{-1} \varphi^p(\rho) M_q^p(\rho, P_{s,t}f) d\rho \\ &\leq C \int_0^1 \left(\int_0^1 \frac{r^{p(2n-1)}(1-r)^{pt-1}}{(1-r\rho)^{p(t+s)}} M_q^p(r, f) dr \right) (1-\rho)^{ps-1} \varphi^p(\rho) d\rho \\ &= C \int_0^1 \left(\int_0^1 \frac{\varphi^p(\rho)}{(1-\rho)^{1-ps}(1-r\rho)^{p(t+s)}} d\rho \right) r^{p(2n-1)}(1-r)^{pt-1} M_q^p(r, f) dr \\ &\leq C \int_0^1 r^{p(2n-1)}(1-r)^{-1} \varphi^p(r) M_q^p(r, f) dr. \end{aligned}$$

Use the change of variables $r = \rho^{\frac{1}{p}}$ and note that $p < 1$, $\rho^{\frac{1}{p}} < \rho$. Thus

$$(1 - \rho^{\frac{1}{p}})^{-1} < (1 - \rho)^{-1} \quad \text{and} \quad M_q^p(\rho^{\frac{1}{p}}, f) < M_q^p(\rho, f).$$

Finally, since φ is normal,

$$\varphi^p(\rho^{\frac{1}{p}}) \leq \frac{(1 - \rho^{\frac{1}{p}})^{bp}}{(1 - \rho)^{bp}} \varphi^p(\rho).$$

That is, $\varphi^p(\rho^{\frac{1}{p}}) \leq C\varphi^p(\rho)$. So

$$\begin{aligned} \|P_{s,t}f\|_{p,q,\varphi}^p &\leq C \int_0^1 \rho^{2n-1} (1 - \rho^{\frac{1}{p}})^{-1} \varphi^p(\rho^{\frac{1}{p}}) M_q^p(\rho^{\frac{1}{p}}, f) d\rho \\ &\leq C \int_0^1 \rho^{2n-1} (1 - \rho)^{-1} \varphi^p(\rho) M_q^p(\rho, f) d\rho = C \|f\|_{p,q,\varphi}^p. \end{aligned}$$

Hence the proof of Theorem A follows.

Remark. Let

$$P_{s,t}^{\sim} f(z) = c_{n,t}(1 - \|z\|^2)^s \int_B \frac{(1 - \|w\|^2)^{t-1} |f(w)|}{|1 - \langle z, w \rangle|^{n+t+s}} dV(w), \quad \forall f \in L_{p,q}(\varphi).$$

The proof of Theorem A actually gives us a little more, that is, there exists a constant C such that $\|P_{s,t}^{\sim} f\|_{p,q,\varphi} \leq C\|f\|_{p,q,\varphi}, \forall f \in L_{p,q}(\varphi)$.

4. AN APPLICATION

We denote the holomorphic mixed norm space $L_{p,q}(\varphi) \cap H(B)$ by $H_{p,q}(\varphi)$, and the norm in $H_{p,q}(\varphi)$ is equivalent to

$$\|f\|_{p,q,\varphi} = \left\{ \int_0^1 (1-r)^{-1} \varphi^p(r) M_q^p(r, f) dr \right\}^{1/p}.$$

In this section, we investigate Gleason’s problem on $H_{p,q}(\varphi)$. To the best of our knowledge, this problem has been answered respectively by Zhu in [10], Ortega in [11], Choe in [5], and Ren and Shi in [1], [12], [13] in the case $1 \leq p < \infty$. So a natural question comes up: How is the case $0 < p < 1$? As an application of Theorem A, we obtain the following:

Theorem B. *Gleason’s problem can be solved on $H_{p,q}(\varphi)$ ($0 < p < 1, 1 < q < \infty$). More precisely, for any integer $m > 1$, there exist bounded linear operators A_α on $H_{p,q}(\varphi)$ such that if $f \in H_{p,q}(\varphi), D^\alpha f(0) = 0$ ($|\alpha| \leq m - 1$), then $f(z) = \sum_{|\alpha|=m} z^\alpha A_\alpha f(z)$ on B , with $\alpha = (\alpha_1, \dots, \alpha_n)$ a multi-index, $|\alpha| = \alpha_1 + \dots + \alpha_n$.*

The fundamental ideas used in arguing this theorem come from the references [10] and [13].

Proof. Assume $m = 1$. By Leibenson’s technique, we get

$$f(z) = \sum_{k=1}^n z_k \int_0^1 \frac{\partial f}{\partial z_k}(rz) dr,$$

where $f \in H_{p,q}(\varphi)$ and $f(0) = 0$. Set $A_k f(z) = \int_0^1 \frac{\partial f}{\partial z_k}(rz) dr, z \in B$. A_k is obviously linear, so it remains to show that A_k is bounded on $H_{p,q}(\varphi)$ ($0 < p < 1, 1 < q < \infty$). Given $f \in H_{p,q}(\varphi)$, let $f_r(z) = f(rz), r \in (0, 1)$. We have $P_{0,t} f_r = f_r$; see [1]. Letting $r \rightarrow 1^-$, the boundedness of $P_{0,t}$ implies that $P_{0,t} f = f$. Thus

$$f(z) = c_{n,t} \int_B \frac{(1 - \|w\|^2)^{t-1} f(w)}{(1 - \langle z, w \rangle)^{n+t}} dV(w),$$

where $t > b, t \in \mathcal{N}$.

Differentiating under the integral gives

$$\frac{\partial f}{\partial z_k}(z) = c_{n,t} \int_B \frac{\overline{w_k}(1 - \|w\|^2)^{t-1} f(w)}{(1 - \langle z, w \rangle)^{n+t+1}} dV(w).$$

This implies that

$$\begin{aligned} A_k f(z) &= C \int_0^1 dr \int_B \frac{\overline{w_k}(1 - \|w\|^2)^{t-1} f(w)}{(1 - r\langle z, w \rangle)^{n+t+1}} dV(w) \\ &= C \int_B \overline{w_k}(1 - \|w\|^2)^{t-1} f(w) dV(w) \int_0^1 \frac{1}{(1 - r\langle z, w \rangle)^{n+t+1}} dr \\ &= C \int_B \frac{\overline{w_k}(1 - \|w\|^2)^{t-1} f(w)}{(1 - \langle z, w \rangle)^{n+t}} Q(z, w) dV(w), \end{aligned}$$

where

$$Q(z, w) = \frac{1 - (1 - \langle z, w \rangle)^{n+t}}{\langle z, w \rangle} = \sum_{k=1}^{n+t-1} (1 - \langle z, w \rangle)^k.$$

Note that $Q(z, w)$ is a polynomial in z and \overline{w} . Thus we can find a constant $C > 0$ such that $|A_k f(z)| \leq C |P_{0,t}^\sim f(z)|$. By the Remark after Theorem A, we see that A_k is bounded on $H_{p,q}(\varphi)$ ($0 < p < 1, 1 < q < \infty$). For m in the general case, it can be proved by induction. Therefore, the proof is complete.

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