

## ON THE CELLULAR DECOMPOSITION OF THE EXCEPTIONAL LIE GROUP $G_2$

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ABSTRACT. The present note is to give a cellular decomposition of the compact connected exceptional Lie group  $G_2$ .

### 1. INTRODUCTION

Let us denote by  $G_2$  the compact connected exceptional Lie group of rank 2. By definition  $G_2$  is the automorphism group  $\text{Aut}(\mathfrak{C})$ , where  $\mathfrak{C}$  is the Cayley algebra. It has been long known that  $G_2$  has the homotopy type of

$$S^3 \cup e^5 \cup e^6 \cup e^8 \cup e^9 \cup e^{11} \cup e^{14}.$$

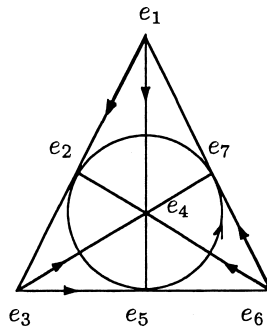
The purpose of the present note is to give a cellular decomposition of  $G_2$ :

$$G_2 = S^3 \cup e^5 \cup e^6 \cup e^8 \cup e^9 \cup e^{11} \cup e^{14}.$$

### 2. PRELIMINARY

Let us recall known results on  $G_2$  which will be needed later.

The Cayley algebra  $\mathfrak{C}$  is isomorphic to  $\mathbb{R}^8$  as an  $\mathbb{R}$ -module and we denote its basis by  $\{e_0, e_1, e_2, e_3, e_4, e_5, e_6, e_7\}$ . Notice that the multiplication is not associative. The element  $e_0$  is the unit of the algebra which we denote by 1. The multiplication of the remaining basis is given in the following diagram:




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Then the exceptional Lie group  $G_2$  is defined to be

$$G_2 = \{g \in SO(8) \mid g(x)g(y) = g(xy), x, y \in \mathfrak{C}\} = \text{Aut}(\mathfrak{C}).$$

Since  $g(1) = 1$  for any  $g \in G_2$ , we may regard  $G_2$  as a subgroup of  $SO(7)$ . So from now on, we express an element of  $G_2$  as that of  $SO(7)$ . The subgroup of elements in  $G_2$  fixing  $e_1$  is known to be isomorphic to  $SU(3)$ . The subgroup of elements in  $G_2$  fixing  $e_1$  and  $e_2$  is known to be isomorphic to  $SU(2)$ . Thus we regard  $SU(3)$  and  $SU(2)$  as subgroups of  $G_2$ . Let  $S^6$  be the unit sphere of  $\mathbb{R}^7$  whose basis is  $\{e_i \mid 1 \leq i \leq 7\}$  and  $S^5$  be the unit sphere of  $\mathbb{R}^6$  whose basis is  $\{e_i \mid 2 \leq i \leq 7\}$ . Then there are two principal fibre bundles over them:

$$\begin{aligned} SU(3) &\longrightarrow G_2 \xrightarrow{p_1} S^6, \\ SU(2) &\longrightarrow SU(3) \xrightarrow{p_2} S^5, \end{aligned}$$

where  $p_i(g) = g(e_i)$  for  $i = 1, 2$ . Let  $H$  be the subgroup of  $G_2$  defined by

$$H = G_2 \cap (SO(3) \oplus SO(4)).$$

**Lemma 2.1.**  $gh^{-1} \in SU(2)$  for any  $h \in H$  and  $g \in SU(2)$ .

*Proof.* It is obvious, since  $SU(2) = G_2 \cap (\{1\} \oplus SO(4))$ , where  $\{1\}$  denotes the subgroup of  $SO(3)$  consisting of the identity element.  $\square$

In the remainder of the section, we will construct cells of  $G_2$ . Let  $D^i$  ( $1 \leq i \leq 3$ ) be the  $i$ -dimensional discs defined respectively by

$$\begin{aligned} D^3 &= \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1^2 + x_2^2 + x_3^2 \leq 1\}, \\ D^2 &= \{(y_1, y_2) \in \mathbb{R}^2 \mid y_1^2 + y_2^2 \leq 1\}, \\ D^1 &= \{z_1 \in \mathbb{R} \mid z_1^2 \leq 1\}. \end{aligned}$$

We define  $V^3, V^5$  and  $V^6$  as follows:

$$V^3 = D^3, \quad V^5 = D^3 \times D^2, \quad V^6 = D^3 \times D^2 \times D^1,$$

and put  $X, Y$  and  $Z$  as

$$X = \sqrt{1 - x_1^2 - x_2^2 - x_3^2}, \quad Y = \sqrt{1 - y_1^2 - y_2^2}, \quad Z = \sqrt{1 - z_1^2}.$$



*Proof.* We express the map  $(p_2\varphi_5)|_{V^5 \setminus \partial V^5}$  as follows:

$$\begin{pmatrix} 0 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \end{pmatrix} = p_2\varphi_5(x_1, x_2, x_3, y_1, y_2) = \begin{pmatrix} 0 \\ 1 - 2X^2Y^2 \\ 2x_1XY^2 \\ 2Y(y_1X^2 - x_1y_2X) \\ -2Y(x_1y_1X + y_2X^2) \\ -2x_2XY \\ -2x_3XY \end{pmatrix},$$

and hence

$$\begin{pmatrix} 0 \\ 1 - a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \end{pmatrix} = 2XY \begin{pmatrix} 0 \\ XY \\ x_1Y \\ y_1X - x_1y_2 \\ -x_1y_1 - y_2X \\ -x_2 \\ -x_3 \end{pmatrix}.$$

Since  $X > 0$ ,  $Y > 0$  and  $1 - a_2 > 0$ , an easy calculation from the second component in the above equation gives the following equation:

$$(3.1) \quad XY = \frac{\sqrt{1 - a_2}}{\sqrt{2}},$$

from which we easily obtain

$$(3.2) \quad x_2 = \frac{-a_6}{\sqrt{2(1 - a_2)}},$$

$$(3.3) \quad x_3 = \frac{-a_7}{\sqrt{2(1 - a_2)}}.$$

Further we obtain two more equalities from the above equation:

$$\begin{aligned} (1 - a_2)^2 + a_3^2 &= 4X^2Y^4(x_1^2 + X^2), \\ a_4^2 + a_5^2 &= 4X^2Y^2(y_1^2 + y_2^2)(x_1^2 + X^2) = 4X^2Y^2(1 - Y^2)(x_1^2 + X^2). \end{aligned}$$

Using these two equalities, we obtain

$$(3.4) \quad Y^2 = \frac{(1 - a_2)^2 + a_3^2}{(1 - a_2)^2 + a_3^2 + a_4^2 + a_5^2}.$$

It follows from (3.1) and (3.4) that

$$(3.5) \quad X^2 = \frac{(1 - a_2)((1 - a_2)^2 + a_3^2 + a_4^2 + a_5^2)}{2((1 - a_2)^2 + a_3^2)}.$$

It follows from (3.2), (3.3) and (3.5) that

$$(3.6) \quad x_1^2 = \frac{a_3^2((1 - a_2)^2 + a_3^2 + a_4^2 + a_5^2)}{2(1 - a_2)((1 - a_2)^2 + a_3^2)}.$$

Since  $Y > 0$ , (3.4) implies that

$$(3.7) \quad Y = \frac{\sqrt{(1 - a_2)^2 + a_3^2}}{\sqrt{(1 - a_2)^2 + a_3^2 + a_4^2 + a_5^2}}.$$

Since  $X > 0$ , (3.5) implies that

$$(3.8) \quad X = \frac{\sqrt{(1-a_2)((1-a_2)^2 + a_3^2 + a_4^2 + a_5^2)}}{\sqrt{2((1-a_2)^2 + a_3^2)}}.$$

Since the signs of  $x_1$  and  $a_3$  are the same, (3.6) implies that

$$(3.9) \quad x_1 = \frac{a_3\sqrt{(1-a_2)^2 + a_3^2 + a_4^2 + a_5^2}}{\sqrt{2(1-a_2)((1-a_2)^2 + a_3^2)}}.$$

Now we determine  $y_2$ ; we have

$$a_4x_1 + a_5X = -2XY(x_1^2 + X^2)y_2.$$

Substituting the equations (3.1), (3.8) and (3.9) in the above equation, we obtain

$$(3.10) \quad y_2 = \frac{-a_3a_4 - (1-a_2)a_5}{\sqrt{((1-a_2)^2 + a_3^2)((1-a_2)^2 + a_3^2 + a_4^2 + a_5^2)}}.$$

Finally we determine  $y_1$ ; we have

$$a_4X - a_5x_1 = 2XY(x_1^2 + X^2)y_1.$$

Substituting the equations (3.1), (3.8) and (3.9) in the above equation, we obtain

$$(3.11) \quad y_1 = \frac{(1-a_2)a_4 - a_3a_5}{\sqrt{((1-a_2)^2 + a_3^2)((1-a_2)^2 + a_3^2 + a_4^2 + a_5^2)}}.$$

Thus we have expressed  $x_1, x_2, x_3, y_1, y_2$  in terms of  $a_2, \dots, a_7$ , that is, the inverse map has been constructed, which completes the proof.  $\square$

**Proposition 3.2.**  $e^0 \cup e^3 \cup e^5 \cup e^8$  is a cellular decomposition of  $SU(3)$ .

*Proof.* First we will show that  $\dot{e}^i \cap \dot{e}^j = \emptyset$  if  $i \neq j$ . We consider the following three cases:

- (1) For the case where  $i = 0$  and  $j = 3$ , it is obvious that  $\dot{e}^0 \cap \dot{e}^3 = \emptyset$  since  $e^0 \cup e^3$  is a cellular decomposition of  $SU(2)$ .
- (2) For the case where  $i \in \{0, 3\}$  and  $j \in \{5, 8\}$ , we have  $p_2(\dot{e}^i) = e_2$  and  $p_2(\dot{e}^j) = S^5 \setminus \{e_2\}$ . Then we have  $\dot{e}^i \cap \dot{e}^j = \emptyset$ .
- (3) For the case where  $i = 5$  and  $j = 8$ , suppose that  $A \in \dot{e}^5 \cap \dot{e}^8$ . Since  $\dot{e}^8 = \dot{e}^5 \dot{e}^3$ , we can put  $A = A_1A_2$  where  $A_1 \in \dot{e}^5$  and  $A_2 \in \dot{e}^3$ . We have  $A = A_1$  since  $p_2(A) = p_2(A_1A_2) = p_2(A_1)$  and  $p_2|_{\dot{e}^5}$  is monic. Then we have  $A_2 = 1 \in \dot{e}^3$ , which is a contradiction. Thus  $\dot{e}^5 \cap \dot{e}^8 = \emptyset$ .

Next, we will check that the boundaries of the cells are included in the lower dimensional cells. It is obvious that the boundary  $\dot{e}^3$  is included in  $e^0$ . Observe that the boundary  $\dot{e}^5$  is a union of the following two sets:

$$\begin{aligned} &\{BAB^{-1} \mid A \in A(\dot{D}^3), B \in B(D^2)\}, \\ &\{BAB^{-1} \mid A \in A(D^3), B \in B(\dot{D}^2)\}. \end{aligned}$$

The first set contains only the identity element since  $A$  is the identity element. Lemma 2.1 implies that the second set is contained in  $SU(2)$  since  $B$  is contained in  $H$ . Thus we have  $\dot{e}^5 \subset e^3$ . Further we have  $\dot{e}^8 = \dot{e}^5 \cup e^3 \cup \dot{e}^3 \subset e^3 \cup e^3 \cup e^0 = e^3 \cup e^5$ .

Finally, we will show that the inclusion map  $e^0 \cup e^3 \cup e^5 \cup e^8 \rightarrow SU(3)$  is epic. Let  $g \in SU(3)$ . If  $p_2(g) = e_2$ , then  $g$  is contained in  $SU(2) = e^0 \cup e^3$ . Suppose that  $p_2(g) \neq e_2$ . There is an element  $h \in e^5$  such that  $p_2(h) = p_2(g)$ . Then we have

$h^{-1}g \in SU(2) = e^0 \cup e^3$ , since  $p_2(h^{-1}g) = e_2$ . Therefore we have  $g \in h(e^0 \cup e^3) \subset e^0 \cup e^3 \cup e^5 \cup e^8$ . □

4. A CELLULAR DECOMPOSITION OF  $G_2$

First we need to show

**Lemma 4.1.** *The composite map  $p_1\varphi_6 : (V^6, \partial V^6) \rightarrow (S^6, \{e_1\})$  is a relative homeomorphism.*

*Proof.* We express the map  $(p_1\varphi_6)|_{V^6 \setminus \partial V^6}$  as follows:

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \end{pmatrix} = p_1\varphi_6(x_1, x_2, x_3, y_1, y_2, z_1) = \begin{pmatrix} 1 - 2X^2Y^2Z^2 \\ 2x_1XY^2Z \\ 2z_1X^2Y^2Z \\ -2XYZ(x_1y_1 + y_2X) \\ -2XYZ(y_1z_1X - x_1y_2z_1 + x_2Z) \\ -2XYZx_3 \\ -2XYZ(y_1XZ - x_1y_2Z - x_2z_1) \end{pmatrix},$$

and hence

$$\begin{pmatrix} 1 - a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \end{pmatrix} = 2XYZ \begin{pmatrix} XYZ \\ x_1Y \\ z_1XY \\ -x_1y_1 - y_2X \\ -y_1z_1X + x_1y_2z_1 - x_2Z \\ -x_3 \\ -y_1XZ + x_1y_2Z + x_2z_1 \end{pmatrix}.$$

We set for simplicity

$$\alpha_1 = (1 - a_1)^2 + a_3^2, \quad \alpha_2 = (1 - a_1)^2 + a_2^2 + a_3^2, \quad \alpha_3 = (1 - a_1)^2 + a_2^2 + a_3^2 + a_4^2, \\ \beta_1 = a_3a_5 + (1 - a_1)a_7, \quad \beta_2 = a_2a_4 + a_3a_5 + (1 - a_1)a_7.$$

Since  $1 - a_1 > 0$ , we have  $\alpha_i > 0$  for  $i = 1, 2, 3$ . By an easy calculation one can obtain the following three equations:

(4.1) 
$$Z^2 = \frac{(1 - a_1)^2}{\alpha_1},$$

(4.2) 
$$X^2Y^2 = \frac{1 - a_1}{2Z^2} = \frac{\alpha_1}{2(1 - a_1)},$$

(4.3) 
$$z_1^2 = 1 - Z^2 = \frac{a_3^2}{\alpha_1}.$$

Since  $Z \geq 0$  and  $1 - a_1 \geq 0$ , (4.1) implies that

(4.4) 
$$Z = \frac{1 - a_1}{\sqrt{\alpha_1}}.$$

Since the signs of  $z_1$  and  $a_3$  are the same, (4.3) implies that

(4.5) 
$$z_1 = \frac{a_3}{\sqrt{\alpha_1}}.$$

We easily have

$$(4.6) \quad x_3 = \frac{-a_6}{2XYZ} = \frac{-a_6}{\sqrt{2(1-a_1)}}.$$

Next we determine  $X$  and  $Y$ ; we have

$$\begin{aligned} (a_4^2 + (a_5z_1 + a_7Z)^2)Y^2 &= 4X^2Y^4Z^2(y_1^2 + y_2^2)(x_1^2 + X^2) \\ &= 4X^2Y^4Z^2(1 - Y^2)(x_1^2 + X^2), \\ a_2^2(1 - Y^2) &= 4X^2Y^4Z^2(1 - Y^2)x_1^2. \end{aligned}$$

It follows from these two equalities that

$$(a_4^2 + (a_5z_1 + a_7Z)^2)Y^2 - a_2^2(1 - Y^2) = 4X^4Y^4Z^2(1 - Y^2).$$

Substituting the equations (4.2), (4.4) and (4.5) in the above equation, we obtain

$$a_4^2Y^2 + \frac{(a_3a_5 + (1 - a_1)a_7)^2}{\alpha_1}Y^2 + a_2^2Y^2 - a_2^2 = \alpha_1(1 - Y^2),$$

whence we have

$$Y^2 = \frac{\alpha_1\alpha_2}{\alpha_1\alpha_3 + \beta_1^2}.$$

Since  $Y \geq 0$ , we have

$$(4.7) \quad Y = \frac{\sqrt{\alpha_1\alpha_2}}{\sqrt{\alpha_1\alpha_3 + \beta_1^2}}.$$

Since  $X \geq 0$ , (4.2) and (4.7) imply that

$$(4.8) \quad X = \frac{\sqrt{\alpha_1\alpha_3 + \beta_1^2}}{\sqrt{2(1-a_1)\alpha_2}}.$$

We easily have

$$(4.9) \quad x_1 = \frac{a_2}{2XY^2Z} = \frac{a_2\sqrt{\alpha_1\alpha_3 + \beta_1^2}}{\sqrt{2(1-a_1)\alpha_1\alpha_2}}.$$

Now we determine  $y_1$  and  $y_2$ ; we have

$$a_4x_1 + a_5z_1X + a_7XZ = -2XYZ(x_1^2 + X^2)y_1,$$

into which substituting (4.4), (4.5), (4.7), (4.8) and (4.9) we obtain

$$(4.10) \quad y_1 = \frac{-\beta_2\sqrt{\alpha_1}}{\sqrt{\alpha_2(\alpha_1\alpha_3 + \beta_1^2)}}.$$

Quite similarly the equation

$$a_4X - a_5x_1z_1 - a_7x_1Z = -2XYZ(x_1^2 + X^2)y_2$$

gives rise to

$$(4.11) \quad y_2 = \frac{a_2\beta_1 - a_4\alpha_1}{\sqrt{\alpha_2(\alpha_1\alpha_3 + \beta_1^2)}}.$$

Finally we determine  $x_2$ ; we have

$$a_5Z - a_7z_1 = -2XYZx_2,$$

which gives

$$(4.12) \quad x_2 = \frac{a_3 a_7 - (1 - a_1) a_5}{\sqrt{2(1 - a_1) \alpha_1}}.$$

Thus we have expressed  $x_1, x_2, x_3, y_1, y_2, z_1$  in terms of  $a_1, \dots, a_7$ , that is, the inverse map has been constructed and this completes the proof.  $\square$

The following is our main result.

**Theorem 4.2.** *The cell complex  $e^0 \cup e^3 \cup e^5 \cup e^6 \cup e^8 \cup e^9 \cup e^{11} \cup e^{14}$  thus constructed gives a cellular decomposition of  $G_2$ .*

*Proof.* First we show that  $\dot{e}^i \cap \dot{e}^j = \emptyset$  if  $i \neq j$ . We consider the following three cases:

- (1) For the case where  $i, j \in \{0, 3, 5, 8\}$ , both cells  $e^i$  and  $e^j$  are in  $SU(3)$  and  $e^0 \cup e^3 \cup e^5 \cup e^8$  is a cellular decomposition of  $SU(3)$ . Then we have  $\dot{e}^i \cap \dot{e}^j = \emptyset$  if  $i \neq j$ .
- (2) For the case where  $i \in \{0, 3, 5, 8\}$  and  $j \in \{6, 9, 11, 14\}$ , we have  $p_1(\dot{e}^i) = \{e_1\}$  and  $p_1(\dot{e}^j) = S^6 \setminus \{e_1\}$ . Then we have  $\dot{e}^i \cap \dot{e}^j = \emptyset$ .
- (3) For the case where  $i, j \in \{6, 9, 11, 14\}$ , suppose that  $A \in \dot{e}^i \cap \dot{e}^j$ . Since  $\dot{e}^i = \dot{e}^6 \dot{e}^{i-6}$  and  $\dot{e}^j = \dot{e}^6 \dot{e}^{j-6}$ , we can put  $A = A_1 A_2 = A'_1 A'_2$  where  $A_1, A'_1 \in \dot{e}^6$ ,  $A_2 \in \dot{e}^{i-6}$  and  $A'_2 \in \dot{e}^{j-6}$ . We have  $A_1 = A'_1$ , since  $p_1(A_1) = p_1(A_1 A_2) = p_1(A'_1 A'_2) = p_1(A'_1)$  and  $p_1|_{e^6}$  is monic. Then we have  $A_2 = A'_2$  and the first case shows that  $i - 6 = j - 6$ , that is,  $i = j$ . Thus  $\dot{e}^i \cap \dot{e}^j = \emptyset$  if  $i \neq j$ .

Next, we will check that the boundaries of the cells are included in the lower dimensional cells. In the proof of Proposition 3.2, it is proved that the boundaries  $\dot{e}^3, \dot{e}^5$  and  $\dot{e}^8$  are included in the lower dimensional cells. Observe that the boundary  $\dot{e}^6$  is a union of the following three sets:

$$\begin{aligned} & \{CBAB^{-1}C^{-1} \mid A \in A(\dot{D}^3), B \in B(D^2), C \in C(D^1)\}, \\ & \{CBAB^{-1}C^{-1} \mid A \in A(D^3), B \in B(\dot{D}^2), C \in C(D^1)\}, \\ & \{CBAB^{-1}C^{-1} \mid A \in A(D^3), B \in B(D^2), C \in C(\dot{D}^1)\}. \end{aligned}$$

The first set contains only the identity element, since  $A$  is the identity element. Lemma 2.1 implies that the second set is contained in  $SU(2)$ , since  $B$  and  $C$  are contained in the subgroup  $H$ . We consider the third set. If  $C = C(1) = 1$ , it is obvious that  $CBAB^{-1}C^{-1} = BAB^{-1} \in e^5$ . Suppose that  $C = C(-1)$ . It is easy to check that

$$\begin{aligned} CB(y_1, y_2)C^{-1} &= B(y_1, -y_2), \\ CA(x_1, x_2, x_3)C^{-1} &= A(-x_1, x_2, -x_3). \end{aligned}$$

Thus the third set is contained in  $e^5$ , since we have

$$\begin{aligned} & CB(y_1, y_2)A(x_1, x_2, x_3)B(y_1, y_2)^{-1}C^{-1} \\ &= (CB(y_1, y_2)C^{-1})(CA(x_1, x_2, x_3)C^{-1})(CB(y_1, y_2)^{-1}C^{-1}) \\ &= B(y_1, -y_2)A(-x_1, x_2, -x_3)B(y_1, -y_2)^{-1}. \end{aligned}$$

We have  $\dot{e}^9 = e^6 \dot{e}^3 \cup \dot{e}^6 e^3 \subset e^6 e^0 \cup e^5 e^3 = e^6 \cup e^8$ . We also have  $\dot{e}^{11} = \dot{e}^6 e^5 \cup e^6 \dot{e}^5 \subset e^5 e^5 \cup e^6 e^3 = e^5 \cup e^9$ , and  $\dot{e}^{14} = \dot{e}^6 e^5 e^3 \cup e^6 \dot{e}^5 e^3 \cup e^6 e^5 \dot{e}^3 \subset e^5 e^5 e^3 \cup e^6 e^3 e^3 \cup e^6 e^5 = e^8 \cup e^9 \cup e^{11}$ .

Finally, we will show that the inclusion map  $e^0 \cup e^3 \cup e^5 \cup e^6 \cup e^8 \cup e^9 \cup e^{11} \cup e^{14} \rightarrow G_2$  is epic. Let  $g \in G_2$ . If  $p_1(g) = e_1$ , then  $g$  is contained in  $SU(3) = e^0 \cup e^3 \cup e^5 \cup e^8$ . Suppose that  $p_1(g) \neq e_1$ . There is an element  $h \in e^6$  such that  $p_1(h) = p_1(g)$ . Thus we have  $h^{-1}g \in SU(3) = e^0 \cup e^3 \cup e^5 \cup e^8$  since  $p_1(h^{-1}g) = e_1$ . Therefore we have  $g \in h(e^0 \cup e^3 \cup e^5 \cup e^8) \subset e^0 \cup e^3 \cup e^5 \cup e^6 \cup e^8 \cup e^9 \cup e^{11} \cup e^{14}$ .  $\square$

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