

## GRAPHICAL CONVERGENCE OF SUMS OF MONOTONE MAPPINGS

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(Communicated by Jonathan M. Borwein)

ABSTRACT. This paper gives sufficient conditions for graphical convergence of sums of maximal monotone mappings. The main result concerns finite-dimensional spaces and it generalizes known convergence results for sums. The proof is based on a duality argument and a new boundedness result for sequences of monotone mappings which is of interest on its own. An application to the epi-convergence theory of convex functions is given. Counterexamples are used to show that the results cannot be directly extended to infinite dimensions.

### 1. INTRODUCTION

Throughout this paper,  $X$  will stand for a reflexive Banach space and  $X^*$  for its dual, unless otherwise specified. Recall that a set-valued mapping  $T : X \rightrightarrows X^*$  is called *monotone* if

$$x_1^* \in T(x_1), x_2^* \in T(x_2) \implies \langle x_1 - x_2, x_1^* - x_2^* \rangle \geq 0,$$

where  $\langle \cdot, \cdot \rangle$  denotes the pairing between  $X$  and  $X^*$ . If a monotone mapping cannot be properly extended to another monotone mapping it is called *maximal monotone*. Such mappings are very important in variational analysis and optimization. For example, the subdifferential of a closed convex function is maximal monotone; see Rockafellar [21]. The *graph* of a set-valued mapping  $T : X \rightrightarrows X^*$  is the set  $\text{gph } T = \{(x, x^*) \in X \times X^* \mid x^* \in T(x)\}$  and the *domain*  $\text{dom } T$  and the *range*  $\text{rge } T$  of  $T$  are defined as the projections of  $\text{gph } T$  to  $X$  and  $X^*$ , respectively.

A sequence  $\{C_n\}_{n \in \mathbb{N}}$  of sets is said to *converge* to a set  $C$ , denoted by  $C_n \rightarrow C$ , if:

- (i) for every  $x \in C$  there is a sequence  $\{x_n\}_{n \in \mathbb{N}}$  with  $\lim x_n = x$  and  $x_n \in C_n$  for  $n$  sufficiently large;
- (ii) the cluster points of every sequence  $\{x_n\}_{n \in \mathbb{N}}$  with  $x_n \in C_n$  for  $n$  sufficiently large belong to  $C$ .

Above and in what follows, the convergence of points is taken in the strong topology, although other choices can also be important. For instance, if in (ii) the convergence is taken with respect to the weak topology on  $X$ , we say that the sequence

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Received by the editors June 17, 2000.

2000 *Mathematics Subject Classification*. Primary 47H05, 78M99.

*Key words and phrases*. Maximal monotone operators, set-valued mappings, graphical convergence, epiconvergence, subdifferential.

The first author was supported by the Academy of Finland under grant No. 70468.

converges in the *Mosco sense* [16]. *Graphical convergence* of a sequence  $\{S_n\}_{n \in \mathbb{N}}$  of set-valued mappings is defined as the convergence of their graphs and it is denoted by  $S_n \xrightarrow{g} S$ . In approximating variational problems, graphical convergence is recognized as the natural kind of convergence for set-valued mappings. It is the key in the study of consistency properties of various approximation schemes as judged by the convergence of solutions. For references on the convergence theory of sets and set-valued mappings see Attouch [1], Attouch and Théra [5], Aubin and Frankowska [9], Beer [12] or Rockafellar and Wets [23].

In this paper we will focus on the following question: given two sequences of maximal monotone mappings  $S_n \xrightarrow{g} S$  and  $T_n \xrightarrow{g} T$ , when can we conclude that  $S_n + T_n \xrightarrow{g} S + T$ ? Such questions are important in practice where one is often faced with sums of mappings. This is the case, for example, when one mapping in the sum represents some kind of penalty or barrier approximation to a constraint (see Attouch [2]), or when an infinite-dimensional problem of the form  $T(x) \ni 0$  is approximated by problems  $T(x) + N_{X_n}(x) \ni 0$  where  $N_{X_n}$  is the *normal cone mapping* of a finite-dimensional subspace  $X_n$  of  $X$ .

A set-valued mapping  $T : X \rightrightarrows X^*$  is said to be *locally bounded* at a point  $x \in \text{dom} T$  if there is a neighborhood  $V \ni x$  and a bounded set  $B$  such that  $T(V) \subset B$ . The following was obtained in Rockafellar [19].

**Theorem 1.1** (Rockafellar). *A maximal monotone mapping  $T$  is locally bounded at a point  $x \in \text{dom} T$  if and only if  $x \in \text{int} \text{dom} T$ .*

In studying graphical convergence of monotone mappings, the following concept turns out to be crucial.

**Definition 1.2.** A sequence  $\{T_n\}$  of set-valued mappings is *uniformly locally bounded* at a point  $x \in \bigcap_n \text{dom} T_n$  if there is a neighborhood  $V \ni x$  and a bounded set  $B$  such that

$$T_n(V) \subset B \quad \text{for all } n \in \mathbb{N}.$$

It is not hard to see that  $S_n + T_n \xrightarrow{g} S + T$  whenever  $S_n \xrightarrow{g} S$ ,  $T_n \xrightarrow{g} T$  and  $\{S_n\}_{n \in \mathbb{N}}$  (or  $\{T_n\}_{n \in \mathbb{N}}$ ) is equicontinuous. In [4], Attouch, Riahi and Théra obtained a convergence result for general maximal monotone mappings.

**Theorem 1.3** (Attouch, Riahi and Théra). *Let  $X$  and  $X^*$  be locally uniformly convex and let  $S_n, S, T_n, T : X \rightrightarrows X^*$  be maximal monotone such that  $S_n \xrightarrow{g} S$  and  $T_n \xrightarrow{g} T$ . If there is a point  $x \in \text{dom} S \cap \text{dom} T$  at which  $\{S_n\}_{n \in \mathbb{N}}$  is uniformly locally bounded, then*

$$S_n + T_n \xrightarrow{g} S + T.$$

Our main result is the following.

**Theorem 1.4.** *Assume that  $\dim X < \infty$  and let  $S_n, S, T_n, T : X \rightrightarrows X^*$  be maximal monotone such that  $S_n \xrightarrow{g} S$ ,  $T_n \xrightarrow{g} T$ , and  $0 \in \text{int}(\text{dom} S - \text{dom} T)$ . Then for  $n$  large enough  $0 \in \text{int}(\text{dom} S_n - \text{dom} T_n)$  and*

$$S_n + T_n \xrightarrow{g} S + T.$$

*Moreover, for any  $(x, x^*) \in \text{gph}(S + T)$  there is a sequence  $x_n \rightarrow x$  along with  $u_n^* \in S_n(x_n)$  and  $v_n^* \in T_n(x_n)$  such that  $\{u_n^*\}_{n \in \mathbb{N}}$  and  $\{v_n^*\}_{n \in \mathbb{N}}$  are bounded and  $u_n^* + v_n^* \rightarrow x^*$ .*

To show that Theorem 1.4 can fail in infinite dimensions, consider the following situation which comes up in Galerkin approximation schemes of functional problems; see for example Mosco [16], Glowinski, Lions and Trémolières [14] or Zeidler [25]. Let  $X$  be a separable reflexive Banach space and let  $X_n \subset X$  be the subspace spanned by the first  $n$  basis vectors in some enumerated basis. The sets  $X_n$  converge to  $X$  in the sense of Mosco (see [16]), which implies that  $N_{X_n} \xrightarrow{g} N_X$ , the zero mapping (see [1]). If we define  $S_n = N_{X_n}$  and  $T_n = N_{\{\bar{x}\}}$ , where  $\bar{x} \in X$  is such that  $\bar{x} \notin X_n$  for all  $n$ , we have  $\text{gph}(S_n + T_n) = \emptyset$  for all  $n$ , so that  $S_n + T_n$  cannot converge to  $N_X + N_{\{\bar{x}\}} = N_{\{\bar{x}\}}$  even though  $S_n \xrightarrow{g} N_X$ ,  $T_n \xrightarrow{g} N_{\{\bar{x}\}}$  and  $\text{dom } N_X - \text{dom } N_{\{\bar{x}\}} = X$ .

The main advantage of Theorem 1.4 over Theorem 1.3 is that the condition for convergence is given in terms of the limit mappings  $S$  and  $T$  only. The condition in Theorem 1.3 concerns the sequence  $\{S_n\}_{n \in \mathbb{N}}$  in addition to the limit mappings. Also, the condition  $0 \in \text{int}(\text{dom } S - \text{dom } T)$  is symmetric in  $S$  and  $T$  and it can be satisfied even if both  $\text{int}(\text{dom } S)$  and  $\text{int}(\text{dom } T)$  are empty.

Since  $\text{dom } S \cap \text{int } \text{dom } T \neq \emptyset$  implies  $0 \in \text{int}(\text{dom } S - \text{dom } T)$ , Theorem 1.4 yields the following.

**Corollary 1.5.** *Assume that  $\dim X < \infty$  and let  $S_n, S, T_n, T : X \rightrightarrows X^*$  be maximal monotone operators such that  $S_n \xrightarrow{g} S$  and  $T_n \xrightarrow{g} T$ . If  $\text{dom } S \cap \text{int } \text{dom } T \neq \emptyset$ , then*

$$S_n + T_n \xrightarrow{g} S + T.$$

In addition to being simpler, the condition  $\text{dom } S \cap \text{int } \text{dom } T \neq \emptyset$  in Corollary 1.5 is weaker than the uniform boundedness condition in Theorem 1.3 (see Theorem 1.1). However, the results of Section 2 will show that in finite dimensions the two conditions are in fact equivalent.

Theorem 1.4 generalizes [15, Theorem 9] of McLinden and Bergstrom which deals with the case where  $S_n$  and  $T_n$  are subdifferentials of convex functions; see Section 4. The main topic in [15] was epi-convergence of convex functions which is closely related to graphical convergence of subdifferentials. In Section 4, we will show how Theorem 1.4 gives a simple proof of one of the main results in [15].

The proof of Theorem 1.4 itself will be given in Section 3. It is based on a dualization trick and on the analysis of uniform local boundedness in Section 2. Connections with epi-convergence of closed convex functions will be studied in Section 4.

## 2. UNIFORM LOCAL BOUNDEDNESS

The purpose of this section is to derive a condition for the uniform local boundedness of a sequence of monotone mappings. The key to this is the estimate in Lemma 2.1 below. The unit ball will be denoted by  $\mathbb{B}$  and the ball with center  $x$  and radius  $r$  will be denoted by  $\mathbb{B}(x, r)$ . The *convex hull* of any set  $C \subset X$  will be denoted by  $\text{co } C$ .

**Lemma 2.1.** *Let  $T : X \rightrightarrows X^*$  be monotone and let  $C \subset X$ ,  $\rho_0, \rho_1 > 0$  be such that  $C \subset \rho_0 \mathbb{B}$  and  $T(x) \cap \rho_1 \mathbb{B} \neq \emptyset$  for every  $x \in C$ . Then for any  $V \subset X$  and  $\epsilon > 0$  such that  $V + \epsilon \mathbb{B} \subset \text{co } C$ , one has  $T(V) \subset \rho \mathbb{B}$  for  $\rho = \rho_1(2\rho_0 - \epsilon)/\epsilon$ .*

*Proof.* Here  $V + \epsilon \mathbb{B} \subset \rho_0 \mathbb{B}$ , so that  $V \subset (\rho_0 - \epsilon) \mathbb{B}$ . Consider any  $x \in V$  and  $v \in T(x)$ . For each  $x' \in C$  there is a  $v' \in T(x')$  with  $|v'| \leq \rho_1$ . By the monotonicity

of  $T$  we have

$$\langle v, x' - x \rangle \leq \langle v', x' - x \rangle \leq |v'|(|x'| + |x|) \leq \rho_1(\rho_0 + (\rho_0 - \epsilon)).$$

Since  $v$  was an arbitrary element of  $T(x)$ , the support function  $\sigma_{T(x)}$  of the set  $T(x)$  has

$$\sigma_{T(x)}(w) \leq \rho_1(2\rho_0 - \epsilon) \quad \forall w \in C - x.$$

The same inequality must then hold for all  $w \in \text{co}(C - x) = \text{co}C - x$  by the convexity of the support function. Because  $V + \epsilon\mathbb{B} \subset \text{co}C$  we have  $\epsilon\mathbb{B} \subset \text{co}C - x$ . Therefore

$$\sigma_{T(x)}(w) \leq \rho_1(2\rho_0 - \epsilon) \quad \forall w \in \epsilon\mathbb{B},$$

from which it follows by the positive homogeneity of the support function that

$$\sigma_{T(x)}(w) \leq (\rho_1(2\rho_0 - \epsilon)/\epsilon)|w| = \rho|w| \quad \forall w,$$

or in other words  $\sigma_{T(x)} \leq \sigma_{\rho\mathbb{B}}$ . This means by duality that  $T(x) \subset \rho\mathbb{B}$ .  $\square$

Lemma 2.1 can be used to derive the bound  $B$  in Theorem 1.1; see [19]. (The maximality of  $T$  is only needed to show that  $T$  cannot be bounded at the boundary of its domain.) In finite dimensions, Lemma 2.1 yields the following result on uniform local boundedness.

**Theorem 2.2.** *Assume that  $\dim X < \infty$ , let  $T_n, T : X \rightrightarrows X^*$  be monotone mappings such that  $T_n \xrightarrow{g} T$ , and let  $\bar{x} \in \text{int dom } T$ . Then there is an  $\bar{n}$  such that  $\{T_n\}_{n \geq \bar{n}}$  is uniformly locally bounded at  $\bar{x}$ .*

*Proof.* Take  $\epsilon > 0$  with  $\mathbb{B}(\bar{x}, 4\epsilon) \subset \text{dom } T$  and let  $V = \mathbb{B}(\bar{x}, \epsilon)$ , so that  $(V + 2\epsilon\mathbb{B}) + \epsilon\mathbb{B} \subset \text{dom } T$ . Since  $\dim X < \infty$  it is possible then to find  $\delta > 0$  and  $a_1, \dots, a_m \in X$  such that  $V + 2\epsilon\mathbb{B} \subset \text{co}\{a_1, \dots, a_m\} \subset \text{dom } T$  and

$$(2.1) \quad |x_k - a_k| \leq \delta \quad \forall k \quad \implies \quad V + \epsilon\mathbb{B} \subset \text{co } C \quad \text{for } C = \{x_1, \dots, x_m\}.$$

Any set  $C$  such as in (2.1) has  $C \subset \rho_0\mathbb{B}$ , where  $\rho_0 = \delta + \max\{|a_1|, \dots, |a_m|\}$ . Choose  $\rho_1$  large enough that  $T(a_k) \cap \text{int } \rho_1\mathbb{B} \neq \emptyset$  for all  $k$ . Since  $T_n \xrightarrow{g} T$  we have

$$T_n(\mathbb{B}(a_k, \delta)) \cap \rho_1\mathbb{B} \neq \emptyset$$

for all  $n$  large enough (see for example [23, Section 5E]). This means that for each large enough  $n$  there are  $x_k \in \mathbb{B}(a_k, \delta)$  with  $T_n(x_k) \cap \rho_1\mathbb{B} \neq \emptyset$ . Then by letting  $C = \{x_1, \dots, x_m\}$  we get from (2.1) and Lemma 2.1 that  $T_n(V) \subset \rho\mathbb{B}$  for  $\rho = \rho_1(2\rho_0 - \epsilon)/\epsilon$ .  $\square$

*Remark 2.3.* Theorem 2.2 shows that, in the finite-dimensional case, the conditions in Theorem 1.3 and Corollary 1.5 are equivalent. In particular, Theorems 1.3 and 2.2 combine to give an alternative proof of Corollary 1.5.

To show that the above result can fail in infinite dimensions, consider again the situation described after Theorem 1.4. The mappings  $N_{X_n}$  converge graphically to  $N_X$  whose domain is the whole space  $X$ . Thus, the condition  $\bar{x} \in \text{int dom } N_X$  is satisfied for any  $\bar{x} \in X$ , but since  $\text{int dom } N_{X_n} = \emptyset$  for all  $n$ , Theorem 1.1 says that the mappings  $N_{X_n}$  cannot be bounded anywhere.

It would be interesting to see whether Theorem 2.2 would hold in infinite-dimensional spaces if graphical convergence was replaced by the “graph-distance convergence” introduced in Attouch, Moudafi and Riahi [3]; see also Attouch and Wets [8] and Tossings [24].

3. GRAPHICAL CONVERGENCE OF SUMS

The idea of our proof of Theorem 1.4 comes from the recently developed duality theory for generalized equations; see [17] and the references therein. The following framework was introduced by Attouch and Théra [6].

**Theorem 3.1** (Attouch and Théra). *Let  $X$  and  $X^*$  be any linear spaces and let  $S, T : X \rightrightarrows X^*$  be set-valued mappings. Then the inclusion*

$$(\mathcal{P}) \quad S(x) + T(x) \ni 0$$

has a solution if and only if

$$(\mathcal{D}) \quad T^{-1}(x^*) - S^{-1}(-x^*) \ni 0$$

has one. Furthermore, the solutions  $\bar{x}$  of  $(\mathcal{P})$  satisfy  $\bar{x} \in T^{-1}(\bar{x}^*)$ , where  $\bar{x}^*$  is a solution of  $(\mathcal{D})$ .

Recall that the *resolvent*  $(J + \lambda T)^{-1}$  of a maximal monotone mapping  $T$  is everywhere defined and single-valued for every  $\lambda > 0$ ; see [22]. Here  $J$  stands for the *duality mapping*; see for example [25]. In Hilbert spaces  $J = I$  and the resolvents are Lipschitz continuous with Lipschitz constant 1. We will use the following characterization due to Attouch [1] of graphical convergence.

**Theorem 3.2** (Attouch). *Let  $T_n, T : X \rightrightarrows X^*$  be maximal monotone and let  $\lambda > 0$  be arbitrary. Then  $T_n \xrightarrow{g} T$  if and only if  $(T_n + \lambda J)^{-1} \xrightarrow{p} (T + \lambda J)^{-1}$  (pointwise convergence).*

We will also need a result concerning ranges of sums of monotone mappings. We will say that a monotone mapping  $T$  is *star-monotone* if for any  $(x, x^*) \in \text{dom } T \times \text{rge } T$

$$\inf_{(y, y^*) \in \text{gph } T} \langle y - x, y^* - x^* \rangle > -\infty.$$

The following result was obtained in the Hilbert space setting by Brézis and Haraux [13] and extended to reflexive Banach spaces by Reich [18].

**Theorem 3.3.** *Let  $S, T : X \rightrightarrows X^*$  be star-monotone mappings such that  $S + T$  is maximal monotone. Then*

$$\text{int rge}(S + T) = \text{int}(\text{rge } S + \text{rge } T).$$

Several examples of star-monotone mappings were given in [13] but for our purposes it suffices to know that this class contains the resolvents of monotone mappings.

We are now ready to prove Theorem 1.4.

*Proof of Theorem 1.4.* Assume first that  $S_n + T_n$  are maximal monotone for  $n$  large enough. Let  $y \in X$  be arbitrary. To prove the graphical convergence it suffices by Theorem 3.2 to show that the sequence  $\{x_n\}_{n \in \mathbb{N}}$  of solutions to

$$2x + S_n(x) + T_n(x) \ni y$$

converges to the solution  $x$  of

$$2x + S(x) + T(x) \ni y.$$

Defining  $S_n^y(x) = S_n(x) - y$  and  $S^y(x) = S(x) - y$ , we can write these inclusions as

$$(\mathcal{P}_n) \quad (I + S_n^y)(x) + (I + T_n)(x) \ni 0$$

and

$$(P) \quad (I + S^y)(x) + (I + T)(x) \ni 0.$$

By Theorem 3.1 the unique solutions to  $(P_n)$  and  $(P)$  can be expressed as  $x_n = (I + T_n)^{-1}(x_n^*)$  and  $x = (I + T)^{-1}(x^*)$ , respectively, where  $x_n^*$  and  $x^*$  are (not necessarily unique) solutions to the dual problems

$$(D_n) \quad C_n(x^*) \ni 0$$

and

$$(D) \quad C(x^*) \ni 0,$$

respectively, where

$$C_n(x^*) = (I + T_n)^{-1}(x^*) - (I + S_n^y)^{-1}(-x^*)$$

and

$$C(x^*) = (I + T)^{-1}(x^*) - (I + S^y)^{-1}(-x^*).$$

By Theorem 3.2  $C_n \xrightarrow{p} C$ , which implies  $C_n \xrightarrow{g} C$ , since for a sequence of equi-continuous mappings, pointwise convergence is equivalent to graphical convergence; see for example [23, Section 5F].

To prove that  $x_n \rightarrow x$ , it is enough to show that every subsequence of  $\{x_n\}_{n \in \mathbb{N}}$  has a subsequence converging to  $x$ . For notational simplicity, we denote by  $\{x_n\}_{n \in \mathbb{N}}$  such an arbitrary subsequence. By Theorem 3.3 we have

$$\text{int rge } C = \text{int}(\text{rge}(T + I)^{-1} - \text{rge}(S^y + I)^{-1}) = \text{int}(\text{dom } T - \text{dom } S).$$

Being sums of continuous maximal monotone mappings,  $C_n$  and  $C$  are maximal monotone. Thus, if  $0 \in \text{int}(\text{dom } S - \text{dom } T)$ , then by Theorem 2.2 there is an  $\bar{n}$  such that  $\{(C_n)^{-1}\}_{n \geq \bar{n}}$  is uniformly locally bounded at 0. Since  $x_n^* \in (C_n)^{-1}(0)$ , the sequence  $\{x_n^*\}$  is bounded, so we can find a subsequence  $\{n_k\}_{k \in \mathbb{N}}$  such that  $\lim_k x_{n_k}^*$  exists, and then  $C_n \xrightarrow{g} C$  implies  $C(\lim_k x_{n_k}^*) \ni 0$ . Since the sequence of mappings  $(I + T_{n_k})^{-1}$  is equi-continuous and pointwise convergent to  $(I + T)^{-1}$ , we have that

$$\lim_k x_{n_k} = \lim_k (I + T_{n_k})^{-1}(x_{n_k}^*) = (I + T)^{-1}(\lim_k x_{n_k}^*) = x.$$

To finish the proof of the first statement it suffices to note that since  $C_n$  are maximal monotone, the uniform local boundedness of  $\{(C_n)^{-1}\}_{n \geq \bar{n}}$  at 0 implies by Theorem 1.1 that

$$0 \in \text{int rge } C_n = \text{int}(\text{dom } T_n - \text{dom } S_n) \quad \forall n \geq \bar{n},$$

which in turn guarantees the maximal monotonicity of  $S_n + T_n$  for  $n \geq \bar{n}$ ; see [4].

To prove the last statement we let  $(x, x^*) \in \text{gph}(S + T)$  be arbitrary and choose  $y = 2x + x^*$ . Then  $2x + S(x) + T(x) \ni y$  and the sequence  $\{x_n\}_{n \in \mathbb{N}}$  constructed above converges to  $x$ . From  $(D_n)$  we get

$$y - x_n^* - x_n \in S_n(x_n), \quad x_n^* - x_n \in T_n(x_n),$$

where  $\{x_n^*\}_{n \in \mathbb{N}}$  is bounded. Thus,  $u_n^* = y - x_n^* - x_n$  and  $v_n^* = x_n^* - x_n$  will have the desired properties.  $\square$

4. CONNECTIONS WITH EPI-CONVERGENCE

Recall that the *epigraph* of a function  $f : X \rightarrow \mathbb{R} \cup \{+\infty\}$  on  $X$  is the set

$$\text{epi } f = \{(x, \alpha) \in X \times \mathbb{R} \mid f(x) \leq \alpha\}.$$

A function  $f$  is said to be *closed* if  $\text{epi } f$  is a closed set, and *proper* if  $\text{epi } f \neq \emptyset$ . A sequence  $\{f_n\}_{n \in \mathbb{N}}$  of functions is said to *epi-converge* to  $f$ , denoted by  $f_n \xrightarrow{e} f$ , if  $\text{epi } f_n \rightarrow \text{epi } f$ , or equivalently (see for example [23, 7.2]), if for every  $x \in X$

- (i)  $\liminf_n f_n(x_n) \geq f(x)$  for every sequence  $x_n \rightarrow x$ ;
- (ii)  $\limsup_n f_n(z_n) \leq f(x)$  for some sequence  $z_n \rightarrow x$ .

If  $f_n$  are convex and  $f_n \xrightarrow{e} f$ , then  $f$  is closed and convex [23, 7.4, 4.15]. The *subdifferential*  $\partial f : X \rightrightarrows X^*$  of a convex function  $f$  is defined by

$$\partial f(x) = \{x^* \in X^* \mid f(y) \geq f(x) + \langle x^*, y - x \rangle \quad \forall y \in X\}.$$

It is well known that the subdifferential of a closed proper convex function is maximal monotone [21].

The following result of Attouch (see for example [1]) provides a close connection between epi-convergence of convex functions and graphical convergence of their subdifferentials.

**Theorem 4.1** (Attouch). *Let  $f_n, f$  be closed proper convex functions. Then  $f_n \xrightarrow{e} f$  if and only if  $\partial f_n \xrightarrow{g} \partial f$  and the normalization condition holds: there are  $(x_n, x_n^*) \in \partial f_n$  and  $(x, x^*) \in \partial f$  such that  $x_n \rightarrow x, x_n^* \rightarrow x^*$  and  $f_n(x_n) \rightarrow f(x)$ .*

Combining Theorems 1.4 and 4.1 we get a simple proof of the following epi-convergence result of McLinden and Bergstrom [15, Theorem 5]. For related results see also Aze and Penot [11].

**Theorem 4.2** (McLinden and Bergstrom). *Assume that  $\dim X < \infty$  and let  $f_n$  and  $g_n$  be closed convex functions on  $X$  such that  $f_n \xrightarrow{e} f$  and  $g_n \xrightarrow{e} g$ . If  $0 \in \text{int}(\text{dom } f - \text{dom } g)$ , then  $f_n + g_n \xrightarrow{e} f + g$ .*

*New argument.* By [6, Theorem 4.9], the condition  $0 \in \text{int}(\text{dom } f - \text{dom } g)$  implies

$$0 \in \text{int}(\text{dom } \partial f - \text{dom } \partial g).$$

Therefore, by virtue of Theorem 1.4 we have for  $n$  large enough

$$\partial(f_n + g_n) = \partial f_n + \partial g_n \xrightarrow{g} \partial f + \partial g = \partial(f + g),$$

where the equalities follow from the classical sum rule of subdifferentiation [20, Theorem 23.8]. So by Theorem 4.1, it suffices to find  $(x_n, x_n^*) \in \text{gph } \partial(f_n + g_n)$  and  $(x, x^*) \in \text{gph } \partial(f + g)$  such that  $(x_n, x_n^*) \rightarrow (x, x^*)$  and  $(f_n + g_n)(x_n) \rightarrow (f + g)(x)$ . The sequence  $\{(x_n, u_n^* + v_n^*)\}_{n \in \mathbb{N}}$  given at the end of Theorem 1.4 will do the job. Indeed, the epi-convergence of  $f_n$  to  $f$  implies

$$(4.1) \quad \liminf_n f_n(x_n) \geq f(x)$$

as well as the existence of a sequence  $z_n \rightarrow x$  such that  $\limsup_n f_n(z_n) \leq f(x)$ . From  $u_n^* \in \partial f_n(x_n)$  we get

$$f_n(z_n) \geq f_n(x_n) + \langle u_n^*, z_n - x_n \rangle.$$

Since the sequence  $\{u_n^*\}_{n \in \mathbb{N}}$  is bounded and  $z_n - x_n \rightarrow 0$ , the last term tends to zero, and thus

$$f(x) \geq \limsup_n f_n(z_n) \geq \limsup_n f_n(x_n)$$

which together with (4.1) implies  $f_n(x_n) \rightarrow f(x)$ . By a similar argument we can prove that  $g_n(x_n) \rightarrow g(x)$ , so that  $(f_n + g_n)(x_n) \rightarrow (f + g)(x)$ .  $\square$

One could also use Theorem 1.4 in studying *epi-hypo-convergence* of saddle-functions as defined by Attouch and Wets [7]; see also Azé, Attouch and Wets [10]. However, this is more complicated than the study of epi-convergence and it is not clear whether Theorem 4.2 has a direct generalization to the saddle-function case.

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