

## ANTI-WICK QUANTIZATION WITH SYMBOLS IN $L^p$ SPACES

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**ABSTRACT.** We give a classification of pseudo-differential operators with anti-Wick symbols belonging to  $L^p$  spaces: if  $p = 1$  the corresponding operator belongs to trace classes; if  $1 \leq p \leq 2$  we get Hilbert-Schmidt operators; finally, if  $p < \infty$ , the operator is compact. This classification cannot be improved, as shown by some examples.

### INTRODUCTION: WEYL AND ANTI-WICK QUANTIZATION

A common feature of all types of quantization proposed in the literature is that they establish a correspondence between self-adjoint operators and classical observables, i.e. real functions on the phase space. In general in the frame of pseudo-differential calculus the correspondence between symbols and operators does not fulfill this requirement, that is, the operator fails to be self-adjoint when the symbol is real valued. However this is true for the the Weyl and the anti-Wick symbols, as is shown for instance in [10]. If on the phase space  $\mathbb{R}_x^n \times \mathbb{R}_\xi^n = \mathbb{R}_z^{2n}$ ,  $z = (x, \xi)$ , we consider the Shubin classes

$$(0.1) \quad \Gamma_\rho^m = \{a(z) \in C^\infty(\mathbb{R}^{2n}) : |\partial_z^\gamma a(z)| \leq C_\gamma \langle z \rangle^{m-\rho|\gamma|}\}$$

with  $m \in \mathbb{R}$ ,  $\rho \in (0, 1]$ ,  $\langle z \rangle = \sqrt{1 + |z|^2}$ , then a satisfactory pseudo-differential calculus, both in the case of Weyl and anti-Wick operators, has been developed, see for instance [10].

M.W. Wong analysed in [13] the case of Weyl quantization with symbols in  $L^p(\mathbb{R}^{2n})$  and gave conditions for the operators to be continuous, compact and Hilbert-Schmidt. In this work we also consider symbols belonging to  $L^p(\mathbb{R}^{2n})$  and study the behaviour of the corresponding anti-Wick operators. We obtain a classification of the operators according to the  $L^p(\mathbb{R}^{2n})$  space of the symbols.

To be more precise, given a function  $a(z)$ ,  $z = (x, \xi) \in \mathbb{R}_x^n \times \mathbb{R}_\xi^n = \mathbb{R}_z^{2n}$ , consider the Weyl operator:

$$Op_w[a]u(x) = \int_{\mathbb{R}^{2n}} e^{i(x-y)\xi} a\left(\frac{x+y}{2}, \xi\right) u(y) dy d\xi, \quad d\xi = (2\pi)^{-n} d\xi,$$

as well as the anti-Wick operator:

$$Op_{aw}[a]u(x) = \int_{\mathbb{R}^{2n}} a(z) P_z u dz,$$

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where  $P_z u(x) = (u, \Phi_z) \Phi_z(x)$  are orthogonal projections in  $L^2(\mathbb{R}^n)$  on the functions  $\Phi_z(x) = \pi^{-\frac{n}{4}} e^{ix\eta} e^{-\frac{|x-\eta|^2}{2}}$  depending on the parameter  $z = (y, \eta) \in \mathbb{R}^{2n}$ . If  $a(z)$  belongs to a Shubin class  $\Gamma_\rho^m$ , then both operators are continuous maps from  $S(\mathbb{R}^n)$  to itself, extending to continuous maps on the duals  $S'(\mathbb{R}^n)$ . The adjoint operator is in both cases the operator associated with  $\bar{a}(z)$ , the complex conjugate of the symbol  $a(z)$ , so that, as we mentioned before, real symbols correspond to self-adjoint operators. Moreover the anti-Wick operator satisfies the additional property:  $a(z) \geq 0$  ( $a(z) > 0$ ) for all  $z \in \mathbb{R}^{2n}$  implies  $Op_{aw}[a] \geq 0$  ( $Op_{aw}[a] > 0$ ) and, if  $a(z) > 0$  and elliptic, the operator is a bijection of  $S(\mathbb{R}^n)$  extending to a bijection of  $S'(\mathbb{R}^n)$ ; see for reference [2], [3]. The relation between Weyl and anti-Wick operators is given by the equality  $Op_{aw}[a] = Op_w[(2\pi)^{-n} a * \sigma]$ ; that means every anti-Wick operator is also a Weyl operator with Weyl symbol given by the convolution between its anti-Wick symbol and the Gaussian function  $\sigma(z) = 2^n e^{-|z|^2}$ . The converse is true only modulo regularizing operators; that is, every Weyl operator can be written as the sum of an anti-Wick operator plus a regularizing operator, for details see [10], [2]. If  $a(z)$  is in the Shubin class  $\Gamma_\rho^m$ , then the convolution  $(a * \sigma)(z)$  is also in the same class, so that in this case the properties of Weyl and anti-Wick operators are very similar. But, if we consider  $a \in L^p(\mathbb{R}^n)$ , then in general we only have  $a * \sigma \in \Gamma_0^0$ , that is, we are in the “limit” case  $\rho = 0$  of Shubin classes (see in the next section the second proof of Theorem 1.2 and Lemma 1.1); however all the continuity properties mentioned above remain valid in view of the Calderon-Vaillancourt theorem; see [8], [4], [5]. We obtain in this case an interesting class of compact operators on  $L^2(\mathbb{R}^n)$ . Operators of this type, called also *localization operators*, were introduced by Daubechies as filters in signal analysis (see [6]); they were studied by Wong in [13], where the continuity of the map  $a(z) \in L^p(\mathbb{R}^{2n}) \rightarrow Op_{aw}[a] \in B(L^2)$  is obtained by means of interpolation methods.

We follow here a direct approach, and in section 2 we present two different proofs of the continuity of this map, which we consider of interest for their particular simplicity; as a corollary we obtain the compactness property as in [13]. For Weyl symbols, on the contrary, we remark that the boundedness of the map  $b(z) \in L^p(\mathbb{R}^{2n}) \rightarrow Op_w[b] \in B(L^2)$  fails if  $p > 2$ ; see [11]. In section 3 we study Hilbert-Schmidt and trace class properties of these operators, pointing out the following differences between Weyl and anti-Wick symbols. An operator is Hilbert-Schmidt if and only if its Weyl symbol is in  $L^2$ , while we show that for anti-Wick symbols in  $L^p$  with  $p \in [1, 2]$  one has always Hilbert-Schmidt operators; for  $2 \leq p \leq \infty$  one can find operators with anti-Wick symbol in  $L^p$  which are not Hilbert-Schmidt; a sufficient condition for an operator to be trace class is that its anti-Wick symbol belongs to  $L^1$ , while the Weyl symbol needs to be in  $L^1$  together with all its derivatives; see for instance Robert [9]. For  $p > 1$ , there exist operators with anti-Wick symbol in  $L^p$  which are not trace class.

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## 1. THEOREMS ON BOUNDEDNESS AND COMPACTNESS

In this section we first give a direct proof of the continuity of the map  $a \rightarrow Op_{aw}[a]$ , for  $a \in L^p(\mathbb{R}^{2n})$ . Second, we prove the same result as a consequence of the Calderon-Vaillancourt theorem. Finally, we show the compactness of  $Op_{aw}[a]$ .

The following lemma, used in Theorem 1.2, clarifies the relation between  $L^p$  anti-Wick and Weyl symbols in the  $\Gamma_\rho^m$  classes.

**Lemma 1.1.** *Let  $k \geq 0$  and  $\langle z \rangle^k a(z) \in L^p(\mathbb{R}^{2n}), 1 \leq p < +\infty$ . Then, for any multi-index  $\alpha$  there exist positive functions  $\Psi_\alpha(z)$  converging to zero as  $|z| \rightarrow \infty$  such that  $|\langle z \rangle^k \partial_z^\alpha (a * \sigma)(z)| \leq \|\langle z \rangle^k a(z)\|_{L^p}^{1/2} \Psi_\alpha(z)$ . In particular,  $a * \sigma \in \Gamma_0^{-k}$ .*

*Proof.* Using Peetre’s inequality  $\langle z \rangle^k \leq 2^k \langle w \rangle^k \langle z - w \rangle^k$ , we have, for suitable  $M > 0$ ,

$$\begin{aligned} & \langle z \rangle^k |\partial_z^\alpha (a * \sigma)(z)| \\ & \leq 2^k \int \langle z - w \rangle^{-M} |a(w)| \langle w \rangle^k \langle z - w \rangle^{M+k} \partial_z^\alpha \sigma(z - w) dw \\ & \leq 2^k \|\langle z - w \rangle^{-M}\|_{L^q} \left( \int |a(w)|^p \langle w \rangle^{pk} \langle z - w \rangle^{p(M+k)} \left( \partial_z^\alpha \sigma(z - w) \right)^p dw \right)^{\frac{1}{p}} \\ & \leq 2^k \|\langle z - w \rangle^{-M}\|_{L^q} \|a(w)\langle w \rangle^k\|_{L^p}^{1/2} \\ & \quad \times \left( \int |a(w)|^p \langle w \rangle^{pk} \langle z - w \rangle^{2p(M+k)} \left( \partial_z^\alpha \sigma(z - w) \right)^{2p} dw \right)^{\frac{1}{2p}}, \end{aligned}$$

where  $\frac{1}{q} + \frac{1}{p} = 1$  and we have used Hölder’s inequality. An application of the dominated convergence theorem shows that the last integral tends to zero as  $|z| \rightarrow +\infty$ . □

**Theorem 1.2.** *Let  $a \in L^p(\mathbb{R}^{2n}), 1 \leq p \leq \infty$ . Then the linear map  $T : a \in L^p(\mathbb{R}^{2n}) \rightarrow Op_{aw}[a] \in B(L^2(\mathbb{R}^n))$  is continuous.*

*First proof.* Letting  $u \in S(\mathbb{R}^n)$ , we consider two different estimates of  $|(u, \Phi_z)|$ :

$$\begin{aligned} (1.1) \quad |(u, \Phi_z)| &= \pi^{-n/4} \left| \int_{\mathbb{R}^n} e^{-i\eta x} e^{-\frac{|x-y|^2}{2}} u(x) dx \right| \\ &\leq \pi^{-n/4} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{2}} |u(x)| dx = (\Phi_0 * |u|)(y), \end{aligned}$$

with  $\Phi_0(x) = \pi^{-n/4} e^{-\frac{|x|^2}{2}}$ ;

$$\begin{aligned} (1.2) \quad |(u, \Phi_z)| &= \left| \mathcal{F}_{x \rightarrow \eta} [(\tau_y \Phi_0)u](\eta) \right| = \left| \mathcal{F}_{x \rightarrow \eta} [(\tau_y \Phi_0)(\eta)] * \widehat{u}(\eta) \right| \\ &= (2\pi)^{n/2} \left| \Phi_0(\eta) * \widehat{u}(\eta) \right| \leq (2\pi)^{n/2} (\Phi_0 * |\widehat{u}|)(\eta), \end{aligned}$$

where  $\mathcal{F}$  denotes the Fourier transform and  $\tau_y$  the translation by  $y$ .

Using (1.1) and (1.2), we can estimate the  $L^q$  norm of the orthogonal projection  $P_z u$  onto the vector  $\Phi_z$  uniformly with respect to  $z$ :

$$\begin{aligned} (1.3) \quad \|(u, \Phi_z)\Phi_z(x)\|_{L^q} &= \pi^{-\frac{n}{4}} \left( \int |(u, \Phi_z)|^q e^{-q\frac{|x-y|^2}{2}} dz \right)^{\frac{1}{q}} \\ &\leq (2\pi)^{n/4} \left( \int |(\Phi_0 * |u|)(y)|^{q/2} |(\Phi_0 * |\widehat{u}|)(\eta)|^{q/2} dy d\eta \right)^{\frac{1}{q}} \\ &= (2\pi)^{n/4} \|\sqrt{\Phi_0 * |u|}\|_{L^q_y} \|\sqrt{\Phi_0 * |\widehat{u}|}\|_{L^q_\eta} := K(u). \end{aligned}$$

Now, let  $\{a_n\}_{n \in \mathbb{N}}$  be a sequence of functions converging to zero in  $L^p(\mathbb{R}^{2n})$  as  $n \rightarrow \infty$ , and such that  $A_n := Ta_n$  tends to an operator  $B$  in  $B(L^2(\mathbb{R}^n))$ ; we will prove that  $B = 0$ , so that the closed graph theorem gives the continuity of the map  $T$ .

Using (1.3) and Hölder’s inequality, we get, for  $\frac{1}{p} + \frac{1}{q} = 1$ ,

$$\begin{aligned} |A_n u(x)| &= \left| \int_{\mathbb{R}^{2n}} a_n(z) P_z u \, dz \right| \leq \int |a_n(z)| |(u, \Phi_z) \Phi_z(x)| \, dz \\ &\leq \|a_n\|_p \|(u, \Phi_z) \Phi_z(x)\|_q \leq K(u) \|a_n\|_p. \end{aligned}$$

Therefore the sequence of functions  $A_n u(x)$  converges to the null function uniformly with respect to  $x$ .

If  $u, v \in S(\mathbb{R}^n)$ , we get

$$|(v, A_n u)| \leq \int_{\mathbb{R}^n} |v(x)| |A_n u(x)| \, dx \leq \|v\|_{L^1} \|A_n u\|_{L^\infty};$$

hence  $(v, A_n u) \rightarrow 0$  as  $n \rightarrow \infty$ . For  $u, v \in S(\mathbb{R}^n)$  we have

$$|(v, (A_n - B)u)| \leq \|v\|_{L^2} \|u\|_{L^2} \|A_n - B\|_{B(L^2)};$$

since  $\|A_n - B\|_{B(L^2)} \rightarrow 0$  as  $n \rightarrow \infty$ , it follows  $B = 0$ . □

*Second proof.* In  $\Gamma_0^0$  we consider the topology induced by the seminorms:

$$\sup_{\substack{z \in \mathbb{R}^{2n} \\ |\alpha| \leq k}} |D_z^\alpha b(z)|, \quad k \in \mathbb{N}, \alpha \in \mathbb{Z}_+^n.$$

From the Calderon-Vaillancourt theorem (see [8], [4], [5]) we get the continuity of the map  $b \in \Gamma_0^0 \rightarrow Op_w[b] \in L^2(\mathbb{R}^n)$  with the norm estimate

$$(1.4) \quad \|Op_w[b]\| \leq C \sup_{\substack{z \in \mathbb{R}^{2n} \\ |\alpha| \leq 2n+1}} |D_z^\alpha b(z)|$$

for suitable  $C > 0$ .

Using Lemma 1.1 with  $k = 0$ , we have the estimate  $|D_z^\alpha b(z)| \leq C \|a\|_{L^p}$  (which holds also for  $p = +\infty$ , with a slight modification of the argument of Lemma 1.1); therefore from (1.4) it follows that  $\|Op_{aw}[a]\| \leq C \|a\|_{L^p}$ , for suitable constants  $C > 0$ . □

If  $b \in L^p(\mathbb{R}^{2n})$  has compact support and for any  $\phi \in S(\mathbb{R}^{2n})$  we have  $\phi * b \in S(\mathbb{R}^{2n})$ , then the following lemma holds:

**Lemma 1.3.** *Let  $b \in L^p(\mathbb{R}^{2n}), 1 \leq p \leq +\infty$ . If  $\text{supp } b$  is compact, then  $B = Op_{aw}[b]$  is a regularizing operator, i.e. an operator with Weyl symbol in  $S(\mathbb{R}^{2n})$ .*

**Corollary 1.4.** *Let  $a \in L^p(\mathbb{R}^{2n}), 1 \leq p < +\infty$ . Then the operator  $Op_{aw}[a] : L^2(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^n)$  is compact.*

*Proof.* Let  $\chi_M(z)$  the characteristic function of the set  $\{z \in \mathbb{R}^{2n} : |z| \leq M\}$ . Then from Lemma 1.3 the operator  $A_M = Op_{aw}[a\chi_M]$  is compact. By Theorem 1.2,  $\|A - A_M\|_{B(L^2)} \leq C \|a - a\chi_M\|_{L^p} \rightarrow 0$  as  $M \rightarrow +\infty$ ; hence  $A$  is compact. □

More generally we can consider anti-Wick symbols in the sum of  $L^p$  spaces, setting by definition  $Op_{aw}[a + b] = Op_{aw}[a] + Op_{aw}[b]$  if  $a \in L^{p_1}$  and  $b \in L^{p_2}$ ,  $1 \leq p_1, p_2 \leq +\infty$ . Then the following result holds:

**Corollary 1.5.** *Let  $p \in [1, +\infty)$  and  $a(z) \in L^p(\mathbb{R}^{2n})$ , with  $a(z) \geq 0$ . Then, for every  $\epsilon > 0$ ,  $Op_{aw}[a + \epsilon]$  is a bicontinuous bijection on  $L^2(\mathbb{R}^{2n})$ .*

*Proof.* Since  $a(z) + \epsilon > 0$ , we have  $Op_{aw}[a + \epsilon] > 0$ ; so it is injective on  $L^2(\mathbb{R}^{2n})$ . As  $Op_{aw}[a]$  is compact,  $Op_{aw}[a + \epsilon] = Op_{aw}[a] + \epsilon I$  is a Fredholm operator of index zero, so it is also surjective. The continuity of  $Op_{aw}[a + \epsilon]^{-1}$  follows from the inverse mapping theorem.  $\square$

We notice that, extending to  $\Gamma_0^0$  the argument in [10], Chapter 4, the compactness of Weyl operators also follows from Lemma 1.1.

## 2. HILBERT-SCHMIDT AND TRACE CLASS OPERATORS

Let  $S_2(L^2(\mathbb{R}^n))$  be the space of Hilbert-Schmidt operators and  $S_1(L^2(\mathbb{R}^n))$  the space of trace class operators on  $L^2(\mathbb{R}^n)$ . We give sufficient conditions for operators with anti-Wick symbols in  $L^p(\mathbb{R}^{2n})$  to belong to these classes. Our results are the following.

**Theorem 2.1.** *Let  $a \in L^p(\mathbb{R}^{2n})$ ,  $p \in [1, 2]$ . Then  $A = Op_{aw}[a] \in S_2(L^2(\mathbb{R}^n))$  and  $\|A\|_2 = \|b\|_{L^2(\mathbb{R}^{2n})}$ , where  $b = (2\pi)^{-n} a * \sigma \in L^2(\mathbb{R}^{2n})$  is the Weyl symbol of  $A$ .*

*Proof.* Using the same arguments as in [10], we have also for symbols  $b$  in the limit class  $\Gamma_0^0$  that  $b \in L^2(\mathbb{R}^{2n})$  if and only if  $Op_w[b] \in S_2(L^2(\mathbb{R}^n))$ , with  $\|A\|_2 = \|b\|_{L^2(\mathbb{R}^{2n})}$ . Using Young's inequality  $\|b\|_2 \leq (2\pi)^{-n} \|a\|_p \|\sigma\|_q$ ,  $p \geq 1$ ,  $q \geq 1$ ,  $\frac{1}{p} = \frac{3}{2} - \frac{1}{q}$ , we then have  $b \in L^2(\mathbb{R}^{2n})$  under the condition  $p \in [1, 2]$ .  $\square$

The previous result cannot be improved, as shown by the following:

**Theorem 2.2.** *For all  $p > 2$ , there exist  $a \in L^p(\mathbb{R}^{2n})$  such that  $Op_{aw}[a] \notin S_2(L^2(\mathbb{R}^n))$ .*

*Proof.* Assume, to the contrary, that there exists  $p > 2$  such that  $Op_{aw}[a] \in S_2(L^2(\mathbb{R}^n))$  for all  $a \in L^p$ . Then, for all  $a \in L^p$ , the convolution  $b = (2\pi)^{-n} a * \sigma$  should belong to  $L^2(\mathbb{R}^{2n})$ , since every symbol  $b \in \Gamma_0^0$  satisfies  $b \in L^2 \Leftrightarrow Op_w[b] \in S_2(L^2(\mathbb{R}^n))$ . Then the map  $a \in L^p \rightarrow a * \sigma \in L^2$ , ( $p > 2$ ) is linear and continuous; moreover it commutes with translations, so by a property of the convolution transform it would be the null operator (see [12]), a contradiction.  $\square$

Now we give an explicit example:

**Example 2.3.** For all  $p > 2$  we determine anti-Wick symbols  $a \in L^p(\mathbb{R}^{2n})$  such that  $Op_w(a * \sigma) \notin S_2(L^2(\mathbb{R}^n))$ . Using again the property  $Op_w(a * \sigma) \in S_2(L^2(\mathbb{R}^n)) \Leftrightarrow a * \sigma \in L^2(\mathbb{R}^{2n})$ , we search for an  $a$  such that  $|a * \sigma|^2 \notin L^1(\mathbb{R}^{2n})$ .

Let  $\langle z \rangle^{-s} = (1 + |z|^2)^{-s/2}$ ; since  $\langle z \rangle^{-s} > 0$ , it follows that

$$|\langle z \rangle^{-s} * \sigma(z)|^2 = 2^n \left| \int_{\mathbb{R}^{2n}} e^{-|t|^2} \langle z - t \rangle^{-s} dt \right|^2 = 2^n \left( \int_{\mathbb{R}^{2n}} e^{-|t|^2} \langle z - t \rangle^{-s} dt \right)^2.$$

We want to determine  $s$  such that

$$\left( \int_{\mathbb{R}^{2n}} e^{-|t|^2} \langle z - t \rangle^{-s} dt \right)^2 \geq \frac{C}{(1 + |z|^2)^n} \notin L^1(\mathbb{R}^{2n}),$$

i.e.  $\int_{\mathbb{R}^{2n}} e^{-|t|^2} \langle z - t \rangle^{-s} \langle z \rangle^n dt \geq C$ . Using Peetre’s inequality, we get

$$\begin{aligned} & \int (1 + |z|^2)^{(n-s)/2} (1 + |z|^2)^{s/2} e^{-|t|^2} (1 + |z - t|^2)^{-s/2} dt \\ & \geq 2^{-s} \int (1 + |z|^2)^{(n-s)/2} (1 + |t|^2)^{-s/2} e^{-|t|^2} dt \\ & = C'_s (1 + |z|^2)^{(n-s)/2} > C \end{aligned}$$

if  $s \leq n$ . On the other hand, since  $\langle z \rangle^{-s} \in L^p(\mathbb{R}^{2n})$  we get  $s > \frac{2n}{p}$ . Therefore,  $a(z) = \langle z \rangle^{-s}$  is the example we have searched for, under the condition  $\frac{2n}{p} < s \leq n$ , which we can satisfy for some  $s$  if  $p > 2$ .

Now we state our result about trace class operators.

**Theorem 2.4.** *Let  $a \in L^1(\mathbb{R}^{2n})$ . Then  $Op_{aw}[a] \in S_1(L^2(\mathbb{R}^n))$  and*

$$(2.1) \quad Tr(Op_{aw}[a]) = (2\pi)^{-2n} \int a(y, \eta) \sigma(x - y, \xi - \eta) dx dy d\xi d\eta.$$

*Proof.* Let  $b = (2\pi)^{-n} a * \sigma$ . By Young’s inequality  $\|b\|_1 \leq (2\pi)^{-n} \|a\|_1 \|\sigma\|_1$ , and, for all  $\alpha \in \mathbb{Z}_+^n$ ,  $\|\partial^\alpha b\|_1 \leq (2\pi)^{-n} \|a\|_1 \|\partial^\alpha \sigma\|_1$ . Then  $b \in L^1$  and  $\partial^\alpha b \in L^1, \forall \alpha \in \mathbb{Z}_+^n$ , and we have that  $Op_w[b] \in S_1(L^2(\mathbb{R}^n))$  and trace’s formula (2.1) holds (see Theorem II-53 of [9]). □

We notice that a similar result under different hypothesis is in Berezin [1]. We also remark that, with a minor modification of the same argument, Theorem 2.4 also holds for complex Borel measures as well as for distributions with compact support.

*Remark 2.5.* Let  $a \in L^p(\mathbb{R}^{2n}), p > 1$ . Then  $Op_{aw}[a]$  is not, in general, a trace class operator. An easy example can be constructed from the harmonic oscillator (for simplicity in dimension 1)  $H = -\frac{d^2}{dx^2} + |x|^2$ . Denote by  $G_\rho^m$  the space of pseudo-differential operators with Weyl symbols in  $\Gamma_\rho^m$ , and consider the operator  $A = H + I \in G_1^2$ ; its Weyl symbol is  $h_0(x, \xi) = \xi^2 + x^2 + 1 = |z|^2 + 1$ , with  $z = (x, \xi)$ . The operator  $A^{-1} = (H + I)^{-1} \in G_1^{-2}$  has Weyl symbol  $\sigma_w(A^{-1}) = (|z|^2 + 1)^{-1} + r(z)$ , with  $r(z) \in \Gamma_1^{-3}$ .

Then  $\sigma_w(A^{-1}) \in L^p(\mathbb{R}^2)$  for  $p > 1$ , but  $\sigma_w(A^{-1}) \notin L^1(\mathbb{R}^2)$ , and, therefore,  $A^{-1} \in S_2(L^2(\mathbb{R}))$ , but  $A^{-1} \notin S_1(L^2(\mathbb{R}))$  because  $\|A^{-1}\|_{tr} = \sum_{j \in \mathbb{N}} \frac{1}{2j+2} = +\infty$ .

Every  $K \in G_\rho^m$  has an anti-Wick symbol, if we consider it modulo regularizing operators (see, for instance, [2], [10]). Hence there exists  $\tilde{A}^{-1} \in G_1^2$ , such that  $\tilde{A}^{-1} - A^{-1} \in G^{-\infty}$ , and its anti-Wick symbol is  $\tilde{a}(z) = (1 + |z|^2)^{-1} + r(z) + \tilde{r}(z)$ , with  $\tilde{r}(z) \in \Gamma_1^{-4}$ .

Then,  $\tilde{a}(z) = (1 + |z|^2)^{-1} + r(z)$ , with  $r(z) \in \Gamma_1^{-3}$ . Therefore  $\tilde{A}^{-1} \notin S_1(L^2(\mathbb{R}))$ , and its anti-Wick symbol satisfies  $\tilde{a}(z) \notin L^1(\mathbb{R}^2)$  but  $\tilde{a}(z) \in L^p(\mathbb{R}^2), \forall p > 1$ .

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