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OPERATORS WHICH HAVE A CLOSED QUASI-NILPOTENT PART

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Abstract. We find several conditions for the quasi-nilpotent part of a bounded operator acting on a Banach space to be closed. Most of these conditions are established for semi-Fredholm operators or, more generally, for operators which admit a generalized Kato decomposition. For these operators the property of having a closed quasi-nilpotent part is related to the so-called single valued extension property.

1. The quasi-nilpotent part of an operator and the SVEP

The single valued extension property was first introduced by Dunford [5], [6] and has, successively, received a more systematic treatment in Dunford-Schwartz [7]. It also plays an important role in local spectral theory; see the monograph of Laursen and Neumann [14]. The following local version of this property has been studied in recent papers by Aiena and Monsalve [1], [2] and previously by Finch [8].

Definition 1.1. Let X be a complex Banach space and $T \in L(X)$. The operator T is said to have *the single valued extension property* at $\lambda_o \in \mathbb{C}$ (abbreviated SVEP at λ_o), if for every open disc $\mathbf{D}_{\lambda_o}$ centered at λ_o the only analytic function $f : \mathbf{D}_{\lambda_o} \to X$ which satisfies the equation $(\lambda I - T)f(\lambda) = 0$ for all $\lambda \in \mathbf{D}_{\lambda_o}$ is the function $f \equiv 0$.

An operator $T \in L(X)$ is said to have the SVEP if T has the SVEP at every point $\lambda \in \mathbb{C}$.

Let us consider the so-called *local resolvent set* $\rho_T(x)$ of T at the point $x \in X$, defined as the union of all open subsets $\mathcal{U}$ of $\mathbf{C}$ such that there exists an analytic function $f : \mathcal{U} \to X$ which satisfies $(\lambda I - T)f(\lambda) = x$ for all $\lambda \in \mathcal{U}$. The *local spectrum* $\sigma_T(x)$ of T at x is the set defined by $\sigma_T(x) := \mathbb{C} \setminus \rho_T(x)$. Obviously, $\sigma_T(x) \subseteq \sigma(T)$, where $\sigma(T)$ denotes the spectrum of T.

For every subset Ω of $\mathbf{C}$, the *analytic spectral subspace* of T associated with Ω is the set

$$X_T(\Omega) := \{x \in X : \sigma_T(x) \subseteq \Omega\}.$$

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It is easy to see from the definition that $X_T(\Omega)$ is a T-hyperinvariant linear subspace of X [14].

The SVEP, as well as the SVEP at a point $\lambda_o \in \mathbf{C}$, may be characterized in a very simple way.

Theorem 1.2. *Let $T \in L(X)$, X a Banach space. Then*:

(i) *T has the SVEP at λ_o if and only if* $\ker(\lambda_o I - T) \cap X_T(\emptyset) = \{0\}$*; see* [1, Theorem 1.9].

(ii) *T has the SVEP if and only if $X_T(\emptyset) = \{0\}$, and this is the case if and only if $X_T(\emptyset)$ is closed; see* [14, Proposition 1.2.16].

Definition 1.3. Let X be a Banach space and $T \in L(X)$. The *analytical core* of T is the set $K(T)$ of all $x \in X$ such that there exists a sequence $(u_n)_{n\in\mathbb{N}} \subset X$ and $\delta > 0$ for which $x = u_o$, $Tu_{n+1} = u_n$ and $\|u_n\| \le \delta^n \|x\|$, for every $n \in \mathbb{N} = \{0, 1, \cdots\}$.

It is easy to check that $K(T)$ is a linear, generally not closed, subspace of X. Furthermore, $T(K(T)) = K(T)$ and if F is a closed subspace of X for which $T(F) = F$, then $F \subseteq K(T)$; see [19, Proposition 2]. Note that if T is quasi-nilpotent, then $K(T) = \{0\}$; see [16, Remarque 1.1].

Definition 1.4. Let $T \in L(X)$, X a Banach space. The *quasi-nilpotent part* of T is the set

$$H_o(T) := \{x \in X : \lim_{n\to\infty} \|T^n x\|^{\frac{1}{n}} = 0\}.$$

Also $H_o(T)$ is a linear subspace of X, generally not closed. Furthermore, $\ker(T^m) \subseteq H_o(T)$ for every $m \in \mathbb{N}$, and T is quasi-nilpotent if and only if $H_o(T) = X$; see [20, Theorem 1.5].

The next result may be found in Vrbová [20], or Mbekhta [16]; see also [14, Propositions 3.3.7 and 3.3.13].

Theorem 1.5. *For a bounded operator $T \in L(X)$, X a Banach space, we have*:

(i) $K(\lambda_o I - T) = X_T(\mathbb{C} \setminus \{\lambda_o\})$.

(ii) $H_o(\lambda_o I - T) \subseteq X_T(\{\lambda_o\})$ *and the equality holds whenever T has the SVEP.*

In the sequel by $M^\perp$ we shall denote the *annihilator* of the subset $M \subseteq X$, and by ${}^\perp N$ the *pre-annihilator* of the subset $N \subseteq X^\star$.

Theorem 1.6. *For a bounded operator $T \in L(X)$, X a Banach space, the following implications hold:*

(i) *$H_o(\lambda_o I - T)$ closed $\Rightarrow H_o(\lambda_o I - T) \cap K(\lambda_o I - T) = \{0\} \Rightarrow T$ has the SVEP at λ_o.*

(ii) *$X = H(\lambda_o I - T) + K(\lambda_o I - T) \Rightarrow T^\star$ has the SVEP at λ_o.*

Proof. Without loss of generality, we may consider $\lambda_o = 0$.

(i) Assume that $H_o(T)$ is closed. Let $\tilde{T}$ denote the restriction of T to the Banach space $H_o(T)$. Clearly, $H_o(T) = H_o(\tilde{T})$, thus $\tilde{T}$ is quasi-nilpotent. Therefore $K(\tilde{T}) = \{0\}$. It is easy to see that $H_o(T) \cap K(T) = K(\tilde{T})$. This shows the first implication.

The second implication of (i) is an immediate consequence of Theorem 1.5. Indeed, we have

$$\ker(\lambda_o I - T) \cap X_T(\emptyset) \subseteq H_o(\lambda_o I - T) \cap K(\lambda_o I - T),$$

so, if the last intersection is $\{0\}$, then T has the SVEP at λ_o, by Theorem 1.2.

(ii) From [15, Proposition 1.8] we know that $H_o(T) \subseteq^{\perp} K(T^\star)$ and therefore $K(T^\star) \subseteq H_o(T)^\perp$. We also have $H_o(T^\star) \subseteq K(T)^\perp$. Indeed, let $\varphi \in H_o(T^\star)$ and consider an arbitrary element $x \in K(T)$. According to the definition of $K(T)$, there is a sequence $(u_n)_{n\in\mathbb{N}} \subset X$, and a $\delta > 0$, such that $u_o = x$, $Tu_{n+1} = u_n$ and $\|u_n\| \leq \delta^n\|x\|$ for every $n \in \mathbf{N}$. Clearly, $T^n u_n = x$ for every $n \in \mathbf{N}$. Consequently,

$$|\varphi(x)| = |\varphi(T^n u_n)| = |(T^{\star n}\varphi)(u_n)| \leq \|u_n\|\|T^{\star n}\varphi\| \leq \delta^n\|T^{\star n}\varphi\|,$$

and hence $|\varphi(x)|^{\frac{1}{n}} \leq \delta\|T^{\star n}\varphi\|^{\frac{1}{n}}$ for every $n \in \mathbf{N}$. The last term converges to 0 as $n \to \infty$, since $\varphi \in H_o(T^\star)$, and from this it follows that $\varphi(x) = 0$, i.e. $\varphi \in K(T)^\perp$. Finally, if $X = H_o(T) + K(T)$, then $\{0\} = H_o(T)^\perp \cap K(T)^\perp \supseteq H_o(T^\star) \cap K(T^\star)$. Thus, by part (i), $T^\star$ has the SVEP at 0. □

The two implications of part (i) of Theorem 1.6 were observed in [16, Proposition 1.10]. Later we shall prove that in the case of semi-Fredholm operators, or more generally in the case that $\lambda_o I - T$ admits a generalized Kato decomposition, the implications of Theorem 1.6 are actually equivalences.

Theorem 1.6 suggests in a very natural way the following definition:

Definition 1.7. A bounded operator $T \in L(X)$, X a Banach space, is said to have *property* (Q) if $H_o(\lambda I - T)$ is closed for every $\lambda \in \mathbf{C}$.

Recall that a bounded operator $T \in L(X)$, X a Banach space, is said to have *Dunford's property (C)*, shortly property (C), if the analytic subspace $X_T(\Omega)$ is closed for every closed subset $\Omega \subseteq \mathbb{C}$. From part (ii) of Theorem 1.2 it follows that if $T \in L(X)$ has property (C), then T has the SVEP.

An obvious consequence of part (ii) of Theorem 1.5 is that if T has property (C), then $H_o(\lambda I - T) = X_T(\{\lambda\})$ is closed for every $\lambda \in \mathbf{C}$, so that the following implications hold:

(1) T has property $(C) \Rightarrow T$ has property $(Q) \Rightarrow T$ has the SVEP.

Note that neither of the implications in (1) may be reversed in general. A first counterexample, which shows that the first implication is not reversed in general, may be found among the convolution operators of group algebras.

Recall that a Banach algebra A is said to be *semi-simple* if the radical rad $A = \{0\}$ [3]; A is said to be *semi-prime* if there is no non-zero two-sided ideal J for which $J^2 = \{0\}$. Note that A is semi-prime if and only if, for every $x \in A$, the identity $xAx = \{0\}$ implies that $x = 0$, and that a commutative algebra is semi-prime if and only if it contains no non-zero nilpotent elements. Clearly, any semi-simple Banach algebra is semi-prime. A map $T : A \to A$, A a Banach algebra, is said to be a *multiplier* if $(Tx)y = x(Ty)$ holds for all $x, y \in A$. Note that if T is a multiplier of a semi-prime commutative Banach algebra A, then $(Tx)y = x(Ty) = T(xy)$ for every $x, y \in A$; see the proof of [11, Theorem 1.1.1]. A very important example of a multiplier is given in the case that A is the semi-simple Banach algebra $L_1(G)$, the group algebra of a locally compact abelian group G with convolution as multiplication. Indeed, in this case to any complex Borel measure μ on G there corresponds a multiplier T_μ defined by

$$T_\mu(f) := \mu \star f \quad \text{for all } f \in L_1(G),$$

where

$$(\mu \star f)(t) := \int_G f(t-s)d\mu(s).$$

The classical Helson-Wendel Theorem shows that each multiplier is a convolution operator and the multiplier algebra of $A := L_1(G)$ may be identified with the measure algebra $M(G)$; see [11, Chapter 0].

Theorem 1.8. *Let T be a multiplier of a semi-simple Banach algebra A. Then*

$$H_o(T) = \ker T.$$

In particular, T has property (Q).

Proof. We know that $\ker T \subseteq H_o(T)$, so it remains to prove the inverse inclusion.

Suppose that $x \in H_o(T)$. By an easy inductive argument we have

$$(Ty)^n = (T^n y)y^{n-1} \text{ for every } y \in A \text{ and } n \in \mathbb{N}.$$

From this it follows that

$$\begin{aligned} \|(aTx)^n\| &= \|(Tax)^n\| = \|T^n(ax)(ax)^{n-1}\| \\ &\leq \|a\|\|T^n x\|\|(ax)^{n-1}\| \end{aligned}$$

for every $a \in A$, so the spectral radius of the element aTx satisfies

$$r(aTx) = \lim_{n\to\infty} \|(aTx)^n\|^{\frac{1}{n}} = 0$$

for every $a \in A$. This implies that $Tx \in \text{rad } A$; see [3, §25, Proposition 1]. Since A is semi-simple then $Tx = 0$ and therefore $H_o(T) \subseteq \ker T$.

The last assertion is clear, because $\lambda I - T$ is a multiplier of A for every $\lambda \in \mathbf{C}$. □

Clearly, if T is a quasi-nilpotent multiplier, then $\ker\ T = H_o(T) = A$, so $T = 0$ ([13]).

The following example shows that the assumption that A is semi-simple in Theorem 1.8 cannot replaced by the weaker assumption that A is semi-prime.

Example 1.9. Let $\omega := (\omega_n)_{n\in\mathbb{N}}$ be a sequence with the property that $0 < \omega_{m+n} \leq \omega_m\omega_n$ for all $m, n \in \mathbb{N}$. Let $\ell^1(\omega)$ denote the space of all complex sequences $x := (x_n)_{n\in\mathbb{N}}$ for which $\|x\|_\omega := \sum_{n=0}^{\infty} \omega_n|x_n| < \infty$. The space $\ell^1(\omega)$ equipped with convolution

$$(x \star y)_n := \sum_{j=0}^{n} x_{n-j}y_j \text{ for all } n \in \mathbb{N}$$

is a commutative unital Banach algebra. Let A_ω denote the maximal ideal of $\ell^1(\omega)$ given by

$$A_\omega := \{(x_n)_{n\in\mathbb{N}} \in \ell^1(\omega) : x_0 = 0\}.$$

The Banach algebra A_ω is an integral domain and hence semi-prime. Moreover, if the weight sequence ω satisfies the condition $\rho_\omega := \lim_{n\to\infty} \omega_n^{\frac{1}{n}} = 0$, then A_ω is a radical algebra ([14, Example 4.1.9]), i.e. coincides with its radical. Now, fix $0 \neq a \in A_\omega$ and let $T_a(x) := a \star x$, $x \in A_\omega$, denote the multiplication operator by the element a. From the estimate

$$\|T^n x\|^{\frac{1}{n}} = \|a^n \star x\|^{\frac{1}{n}} \leq \|a^n\|^{\frac{1}{n}} \|x\|^{\frac{1}{n}}$$

we see that T_a is quasi-nilpotent, thus $H_o(T_a) = A_\omega$. On the other hand, A_ω is an integral domain so that ker $T_a = \{0\}$.

Theorem 1.8 suggests the way of obtaining examples of operators which have property (Q) but not property (C). Indeed, there are convolution operators T_μ, $\mu \in M(G)$, on the group algebra $L_1(G)$ which do not enjoy property (C); see [14, Chapter 4].

The next example, which is obtained by a slight modification of Example 3.9 of [4], shows that also the second implication of (1) may be not reversed in general.

Example 1.10. Let $X := \ell_2 \oplus \ell_2 \cdots$ and define

$$T_n e_i := \begin{cases} e_{i+1} & \text{if } i = 1, \cdots, n, \\ \frac{e_{i+1}}{i-n} & \text{if } i > n. \end{cases}$$

Clearly,

$$\|T_n^{n+k}\| = \frac{1}{k!} \quad \text{and} \quad (\frac{1}{k!})^{\frac{1}{n+k}} \to 0 \text{ as } k \to \infty.$$

From this it follows that $\sigma(T_n) = \{0\}$. Moreover, T_n is injective and the point spectrum $\sigma_p(T_n)$ is empty. Now, let $T := T_1 \oplus \cdots \oplus T_n \oplus \cdots$. From $\|T_n\| = 1$, for every $n \in \mathbf{N}$, we obtain $\|T\| = 1$. From $\sigma_p(T_n) = \emptyset$ it also follows that $\sigma_p(T) = \emptyset$. Take $x = (x_n) \in X$ with $x_n := \frac{e_1}{n}$. We have

$$\|x\| = (\sum_{n=1}^{\infty} \frac{1}{n^2})^{\frac{1}{2}} < \infty,$$

thus $x \in X$. Moreover,

$$\|T^n x\|^{\frac{1}{n}} \geq \|T_n^n \frac{e_1}{n}\|^{\frac{1}{n}} = (\frac{1}{n})^{\frac{1}{n}}$$

and the last term does not converge to 0. Clearly, $x \notin H_o(T)$. Finally, $\ell_2 \oplus \ell_2 \cdots \oplus \ell_2 \oplus \{0\} \cdots \subset H_o(T)$, where the non-zero terms are n. Since $H_o(T)$ contains all sequences with only finitely many non-zero terms, it follows that $H_o(T)$ is dense in X. Since $H_o(T) \neq X$ then $H_o(T)$ is not closed, thus T has not property (Q). Note that the operator T has the SVEP, since $\sigma_p(T)$ is empty.

2. The case of semi-Fredholm operators

For every linear operator T on a vector space X, let us consider the increasing sequence of kernels ker T^n and the decreasing sequence of ranges $T^n(X)$.

Definition 2.1. Let T be a linear operator on a vector space X. The *generalized kernel* of T is defined by

$$\mathcal{N}(T) := \bigcup_{n \in \mathbb{N}} \ker T^n.$$

The *hyperrange* of T is defined by

$$T^\infty(X) := \bigcap_{n \in \mathbb{N}} T^n(X).$$

Clearly, $T^\infty(X)$ is a T-invariant subspace and it is easily seen that, if X is a Banach space, $K(T) \subseteq T^\infty(X)$. Moreover, for every $n \in \mathbf{N}$ we have

$$\ker (\lambda_o I - T)^n \subseteq \mathcal{N}(\lambda_o I - T) \subseteq H_o(\lambda_o I - T).$$

Recall that T is said to have *finite ascent* if $\mathcal{N}(T) = \ker T^k$ for some positive integer k. Clearly, in such a case there is a smallest positive integer $p = p(T)$ such that $\ker T^p = \ker T^{p+1}$. The positive integer p is called the *ascent* of T. Analogously, T is said to have *finite descent* if $T^\infty(X) = T^k(X)$ for some k. The smallest integer $q = q(T)$ such that $T^{q+1}(X) = T^q(X)$ is called the *descent* of T. It is possible to prove that if $p(T)$ and $q(T)$ are both finite, then $p(T) = q(T)$; see [9, Proposition 38.3].

Theorem 2.2 ([1])**.** *For a bounded operator T on a Banach space X the following implications hold:*

i) $p(\lambda_o I - T) < \infty \Rightarrow T$ *has SVEP at* λ_o.
ii) $q(\lambda_o I - T) < \infty \Rightarrow T^\star$ *has SVEP at* λ_o.

Hence each one of the two conditions $p(\lambda_o I - T) < \infty$ or $H_o(\lambda_o I - T)$ closed implies the SVEP at λ_o. In general these two conditions are not related. The operator T of Example 1.10 has ascent $p(T) = 0$ and quasi-nilpotent part $H_o(T)$ not closed. In the following example we find an operator T which has a closed quasi-nilpotent part but ascent $p(T) = \infty$.

Example 2.3. Let $T : \ell_2 \to \ell_2$ be defined by

$$Tx := (\frac{x_2}{2}, \cdots, \frac{x_n}{n}, \cdots), \text{ where } x = (x_1, \cdots, x_n, \cdots).$$

It is easily seen that $\|T^k\| = \frac{1}{(k+1)!}$ and from this it follows that T is quasi-nilpotent and therefore $H_o(T) = \ell_2$. Obviouly, $p(T) = \infty$.

Definition 2.4. An operator $T \in L(X)$, X a Banach space, is said to be *semi-regular* if $T(X)$ is closed and $\ker T \subseteq T^\infty(X)$.

An operator $T \in L(X)$ is said to admit a *generalized Kato decomposition*, abbreviated GKD, if there exists a pair of T-invariant closed subspaces (M, N) such that $X = M \oplus N$, the restriction $T\,|M$ is semi-regular and $T\,|N$ is quasi-nilpotent.

Remark 2.5. In the sequel we list some examples of operators which admit a GKD.

(i) Every semi-regular operator has the GKD $M = X$ and $N = \{0\}$. Note that if T is semi-regular, then $\overline{H_o(T)} = \overline{\mathcal{N}(T)}$; see [15, Proposition 2.10].

(ii) Every quasi-nilpotent operator has the GKD $M = \{0\}$ and $N = X$.

(iii) An important case is obtained if we assume in the definition above that $T\,|N$ is nilpotent. In this case T is said to be of *Kato type*; see [15]. Obviously, any semi-regular operator is of Kato type. Note that if T is of Kato type, then $T^\infty(X) = K(T)$ and $K(T)$ is closed; see [1, Lemma 2.4] or [2, Theorem 2.3 and Theorem 2.4].

(iv) Let

$$\Phi_+(X) := \{T \in L(X) : \ \dim \ker \ T < \infty, \ T(X) \text{ closed}\}$$

denote the class of all *upper semi-Fredholm* operators, and let

$$\Phi_-(X) := \{T \in L(X) : \ \text{codim} \ \ T(X) < \infty\}$$

denote the class of all *lower semi-Fredholm* operators. The class of all *semi-Fredholm operators* is defined as $\Phi_\pm(X) := \Phi_+(X) \cup \Phi_-(X)$ and the class of all *Fredholm operators* is defined as $\Phi(X) := \Phi_+(X) \cap \Phi_-(X)$. A well-known result of Kato [10, Theorem 4] establishes that every $T \in \Phi_\pm(X)$ is of Kato type. More precisely, T admits a GKD (M, N) with $T\,|N$ nilpotent and $\dim N < \infty$.

Recall that for $T \in \Phi_{\pm}(X)$ the *index* of T is defined by ind $T :=$ dim ker $T -$ codim $T(X)$. The index is an integer or $\pm\infty$.

Theorem 2.6. *Let $T \in L(X)$, X a Banach space, and assume that $\lambda_o I - T$ has a GKD (M, N). Then the following properties are equivalent:*

(i) *T has the SVEP at λ_o.*

(ii) $H_o(\lambda_o I - T) \cap K(\lambda_o I - T) = \{0\}$.

(iii) *$H_o(\lambda_o I - T)$ is closed.*

(iv) $H_o(\lambda_o I - T) = N$.

In particular, if $\lambda_o I - T$ is semi-regular the conditions (i)-(iv) *are equivalent to the following one.*

(v) $H_o(\lambda_o I - T) = \{0\}$.

Proof. Also here we only consider the case $\lambda_o = 0$. Clearly, (iv) $\Rightarrow$ (iii) and from Theorem 1.6 we know that (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i).

(i) $\Rightarrow$ (iv). First note that if T admits a GKD (M, N), then $H_o(T) = H_o(T\,|M) + H_o(T\,|N)$. The inclusion $H_o(T) \supseteq H_o(T\,|M) + H_o(T\,|N)$ is obvious. In order to show the opposite inclusion, consider an arbitrary element $x \in H_o(T)$ and let $x = u + v$, with $u \in M$ and $v \in N$. Since $T\,|N$ is quasi-nilpotent then $N = H_o(T\,|N) \subseteq H_o(T)$. Consequently, $u = x - v \in H_o(T) \cap M = H_o(T\,|M)$ and therefore $H_o(T) \subseteq H_o(T\,|M) + H_o(T\,|N)$. Hence $H_o(T) = H_o(T\,|M) + H_o(T\,|N) = H_o(T\,|M) + N$. Now, suppose that T has the SVEP 0. Clearly, the SVEP at a point is inherited by the restrictions to closed invariant subspaces, so $T\,|M$ has the SVEP at 0 and from the semi-regularity of $T\,|M$ it follows that $T\,|M$ is injective; see [1, Theorem 2.14]. From this we obtain

$$H_o(T\,|M) = \overline{\bigcup_{n=1}^{\infty} \ker\,(T\,|M)^n} = \{0\}$$

and therefore $H_o(T) = N$.

The final assertion is clear. □

Corollary 2.7. *Let $T \in L(X)$, X a Banach space, and assume that $\lambda_o I - T$ is of Kato type. Then the conditions* (i)-(iv) *of Theorem 2.6 are equivalent to the following one:*

(v′) $p(\lambda_o I - T) < \infty$.

In this case, if $p := p(\lambda_o I - T)$, then

$$H_o(\lambda_o I - T) = \mathcal{N}(\lambda_o I - T) = \ker\,(\lambda_o I - T)^p.$$

Proof. Assume $\lambda_o = 0$. We know that the inclusions $H_o(T) \supseteq \mathcal{N}(T) \supseteq \ker\, T^n$ hold for every $T \in L(X)$ and for every $n \in \mathbf{N}$. Let (M, N) be a GKD for T such that $(T|N)^k = 0$ for some $k \in \mathbf{N}$. Then $H_o(T) = N \subseteq \ker\, T^k$ and hence $H_o(T) = \mathcal{N}(T) = \ker\, T^k$. From this it follows that $p := p(T) \le k$ and therefore $\ker\, T^k = \ker\, T^p$. □

The next result shows that, under the assumption that $\lambda_o I - T$ is semi-Fredholm, another equivalent condition can be added to those given in Corollary 2.7.

Theorem 2.8. *Suppose that $\lambda_o I - T \in L(X)$ is a semi-Fredholm operator. Then the following statements are equivalent:*

(i) *$H_o(\lambda_o I - T)$ is closed.*

(ii) *$H_o(\lambda_o I - T)$ is finite dimensional.*

Proof. We have only to prove the implication (i) $\Rightarrow$ (ii). This immediately follows from the mentioned Kato decomposition of a semi-Fredholm operator: if (M, N) is a GKD for $\lambda_o I - T$ such that $\lambda_o I - T\ |N$ is nilpotent and dim $N < \infty$, then $H_o(\lambda_o I - T) = N$, by Theorem 2.6. □

The preceding result was obtained in [12, Theorem 2] under the assumption that $\lambda_o I - T$ is a Fredholm operator.

Theorem 2.9. *Suppose that $\lambda_o I - T \in L(X)$ is of Kato type. Then the following statements are equivalent:*

(i) *$T^\star$ has the SVEP at λ_o.*
(ii) *$q := q(\lambda_o I - T) < \infty$.*
(iii) *$X = H_o(\lambda_o I - T) + K(\lambda_o I - T)$.*

Moreover, if any of the equivalent conditions (i)-(iii) *holds, then*

$$(\lambda_o I - T)^\infty(X) = K(\lambda_o I - T) = (\lambda_o I - T)^q(X).$$

Proof. Assume that $\lambda_o = 0$. The equivalence (i) $\Leftrightarrow$ (ii) has been proved in [2, Theorem 2.6]. The implication (iii) $\Rightarrow$ (i) has been proved in Theorem 1.6.

(ii) $\Rightarrow$ (iii). Assume that $q := q(T) < \infty$. Since T is of Kato type then $K(T) = T^\infty(X) = T^q(X)$. Moreover, $X = \ker T^q + T^n(X)$ for every $n \in \mathbf{N}$ (see [9, Proposition 38.2]), and therefore $X = H_o(T) + T^\infty(X)$. □

Theorem 2.10. *Suppose that $\lambda_o I - T \in L(X)$ is a semi-Fredholm operator. Then the following statements are equivalent:*

(i) *$T^\star$ has the SVEP at λ_o.*
(ii) *$K(\lambda_o I - T)$ is finite codimensional.*

Proof. Also here we assume that $\lambda_o = 0$.

(i) $\Rightarrow$ (ii). From Fredholm theory we know that $T^\star$ is also a semi-Fredholm operator and ind $T^\star = -$ ind T. Now, if $T^\star$ has the SVEP at 0, then ind $T^\star \le 0$ (see [1, Corollary 2.7]) and therefore ind $T \ge 0$. From this it follows that T is a lower semi-Fredholm and consequently also T^q is lower semi-Fredholm., i.e. $T^q(X) = T^\infty(X) = K(T)$ is finite codimensional.

(ii) $\Rightarrow$ (i). Since $K(T) = T^\infty(X)$, condition (ii) means that $T^\infty(X)$ is of finite codimension. But from this it is immediate that $q(T) < \infty$, so that $T^\star$ has the SVEP at 0, by Theorem 2.9. □

It should be noted that if T is semi-Fredholm, then $T^\infty(X)$ coincides with the so-called algebraic core of T, i.e. the greatest subspace M of X for which $T(M) = M$; see for instance [1, Theorem 2.3].

Corollary 2.11. *Assume that $\lambda_o I - T \in L(X)$ is a semi-Fredholm operator. Then the following statements are equivalent:*

(i) *T and $T^\star$ have the SVEP at λ_o.*
(ii) *$X = H_o(\lambda_o I - T) \oplus K(\lambda_o I - T)$.*
(iii) *$H_o(\lambda_o I - T)$ is closed and $K(\lambda_o I - T)$ is finite-codimensional.*
(iv) *λ_o is a pole of $(\lambda I - T)^{-1}$, or equivalently $p(\lambda_o I - T) = q(\lambda_o I - T) < \infty$.*
(v) *The spectrum does not cluster at λ_o.*

In particular, if any of the equivalent conditions (i)-(v) *holds and $p := p(\lambda_o I - T) = q(\lambda_o I - T)$, then*

$$H_o(\lambda_o I - T) = \mathcal{N}(\lambda_o I - T) = \ker(\lambda_o I - T)^p$$

and

$$K(\lambda_o I - T) = (\lambda_o I - T)^\infty(X) = (\lambda_o I - T)^p(X).$$

Proof. The equivalences of (i), (ii), (iii), and (iv) are obtained by combining all the results established in this section. The implication (iv) $\Rightarrow$ (v) is obvious. The implication (v) $\Rightarrow$ (i) is an immediate consequence of the fact that both T and $T^\star$ have the SVEP at every point of the resolvent, as well as at every isolated point of the spectrum. ☐

Note that rather similar results to those of Corollary 2.11 have been established by Mbekhta [15, Théorème 1.6] and Schmoeger [19], in the case that λ_o is an isolated point of the spectrum.

Remark 2.12. Recall that for every $T \in L(X)$ the *semi-Fredholm region* is defined to be

$$\Sigma(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is semi-Fredholm}\}.$$

It is well-known that $\Sigma(T)$ is an open set and hence it may be decomposed in connected disjoint open nonempty components. Suppose that T has SVEP at some $\lambda_o \in \Omega$, Ω a component of $\Sigma(T)$. Then, by Theorem 2.6 and Corollary 2.7

$$\begin{aligned} \{0\} &= H_o(\lambda_o I - T) \cap K(\lambda_o I - T) = \mathcal{N}(\lambda_o I - T) \cap (\lambda_o I - T)^\infty(X) \\ &= \overline{\mathcal{N}(\lambda_o I - T)} \cap (\lambda_o I - T)^\infty(X)\ , \end{aligned}$$

and from the constancy of the map $\lambda \in \Omega \to \overline{\mathcal{N}(\lambda I - T)} \cap (\lambda I - T)^\infty(X)$ (see [18, Theorem 4.2]) we conclude that $\mathcal{N}(\lambda I - T) \cap (\lambda I - T)^\infty(X) = \{0\}$ for every point $\lambda \in \Omega$. From [2, Theorem 1.10] it follows that T has the SVEP at every point $\lambda \in \Omega$. Moreover, the set

$$\Gamma := \{\lambda \in \Omega : \text{jump } \ (\lambda I - T) \neq 0\}$$

is countable ([18]) and is equal to the set of all $\lambda \in \Omega$ such that $\lambda I - T$ is not semi-regular; see [21, Proposition 2.2]. From Theorem 2.6 then $H_o(\lambda I - T) = \{0\}$ for every $\lambda \in \Omega \setminus \Gamma$, while the remaining points $\lambda \in \Gamma$ are eigenvalues with ascent $p := p(\lambda I - T) < \infty$, $H_o(\lambda I - T) = \ker\, (\lambda I - T)^p$ and $0 < \dim\, H_o(\lambda I - T) < \infty$, by Corollary 2.7 and Theorem 2.8. In particular this situation occurs for every component of the semi-Fredholm region of an operator which has the SVEP.

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