

## PARABOLIC SUBGROUPS OF VERSHIK-KEROV'S GROUP

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ABSTRACT. In this note we show that all parabolic subgroups of Vershik-Kerov's group  $GLB(R)$  (i.e. subgroups containing  $T(\infty, R)$ —the group of infinite dimensional upper triangular matrices) are net subgroups for a wide class of semilocal rings  $R$ .

### 1. INTRODUCTION

The classical result for finite dimensional general linear group over a field states that all “parabolic” subgroups, that is, containing the group of all upper triangular matrices, are “staircase groups” (see [1], p. 53, or [3] for more general result in the context of groups with  $BN$ -pair).

In [2] Borevich introduced a concept of a net of ideals  $\sigma$  and a net subgroup  $G(\sigma)$  (see definitions in the next section). Theorem 1 of [2] gives the following generalization:

**Theorem 1.1.** *Let  $R$  be a semilocal ring, in which 1 is a sum of two invertible elements. If  $H$  is a parabolic subgroup of  $GL_n(R)$ , then there exists a unique  $T$ -net  $\sigma = (\sigma_{ij})$  of two-sided ideals of  $R$ , such that  $H = G(\sigma)$ .*

In our paper we extend this result to one infinite dimensional linear group. For  $R$  an associative ring with 1 by  $GL(\infty, R)$  we denote the group of all column-finite invertible infinite matrices over  $R$  (indexed by positive integers  $\mathbb{N}$ ). Let  $T(\infty, R)$  denote a group of all infinite upper triangular matrices over  $R$  (with invertible elements on the main diagonal). We define  $GLB(R)$  as a subgroup of  $GL(\infty, R)$  of all matrices which have a finite number of nonzero entries below the main diagonal (clearly,  $T(\infty, R) < GLB(R) < GL(\infty, R)$ ). This group was considered in the case of finite field  $k$  by Vershik and Kerov [6] and has applications in representation theory.  $GLB(k)$  is infinite dimensional, locally compact, totally disconnected, and amenable in topological sense and unimodular group. The stable general linear group  $GL_\infty(k)$ , i.e. direct limit of  $GL_n(k)$  under natural embeddings  $g \rightarrow \text{diag}(g, 1)$ , is its dense subgroup and the quotient group of  $GLB(k)$  over the center is topologically simple.

In this paper we give a purely algebraic description of parabolic subgroups of  $GLB(R)$  (i.e. containing  $T(\infty, R)$ ). Our main result is the following theorem.

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**Theorem 1.2.** *Let  $R$  be a semilocal ring, in which 1 is a sum of two invertible elements. If  $H$  is a parabolic subgroup of  $GLB(R)$ , then there exists a unique  $T$ -net  $\sigma = (\sigma_{ij})$  of two-sided ideals of  $R$ , such that  $H = G(\sigma)$ .*

Using this theorem we can prove the “standard properties” (see [3], §2) of parabolic subgroups in  $GLB(R)$ .

**Theorem 1.3.** *If  $R$  is a semilocal ring, in which 1 is a sum of two invertible elements, then:*

- (i) *If  $P_1, P_2$  are two parabolic subgroups of  $GLB(R)$  and  $gP_1g^{-1} \subset P_2$  for some  $g \in GLB(R)$ , then  $g \in P_2$  and  $P_1 \subset P_2$ .*
- (ii) *Two different parabolic subgroups of  $GLB(R)$  are not conjugate.*
- (iii) *Every parabolic subgroup of  $GLB(R)$  is self-normalized.*

Let  $S_{\text{fin}}(\mathbb{N})$  denote the regular matrix representation of all permutations of positive integers  $\mathbb{N}$  with finite support. We have

**Theorem 1.4** (Bruhat Decomposition Theorem). *For any field  $K$ ,*

$$GLB(K) = T(\infty, K) \cdot S_{\text{fin}}(\mathbb{N}) \cdot T(\infty, K).$$

If  $K = \mathbb{C}$  (complex numbers), then Theorem 1.4 follows from [6]. In [6] the Hecke algebra of double cosets of  $GLB(\mathbb{C})$  over the Borel subgroup  $B = T(\infty, \mathbb{C})$  was introduced, and because this Hecke algebra is isomorphic to the group algebra  $\mathbb{C}(S_{\text{fin}}(\mathbb{N}))$ , we obtain the Bruhat decomposition.

## 2. PROOFS OF MAIN RESULTS

By  $e$  ( $e_n$ ) we denote the unit matrix in  $GL(\infty, R)$  ( $GL_n(R)$ ) and by  $e_{ij}$  a matrix with the only nontrivial element 1 in the  $i$ -th row and  $j$ -th column. We denote  $t_{ij}(\zeta) = e + \zeta e_{ij}$ ,  $\zeta \in R$ ,  $i, j \in \mathbb{N}$ ,  $d_i(\theta) = e + (\theta - 1)e_{ii}$ ,  $\theta$ -invertible, and  $[x, y] = xyx^{-1}y^{-1}$ .

**Definition 2.1.** A system  $\sigma = (\sigma_{ij})$  ( $i, j \in \mathbb{N}$ ) of two sided ideals  $\sigma_{ij}$  of  $R$  is called a *net* if

$$(\star) \quad \sigma_{ir} \cdot \sigma_{rj} \subseteq \sigma_{ij} \quad \text{for all } i, j, r \in \mathbb{N}.$$

We call  $\sigma$  a  $T$ -net if  $\sigma_{ij} = R$  for  $i \leq j$ . If the set of indexes is  $I = \{1, 2, \dots, n\}$  we have the finite nets of ideals in  $GL_n(R)$ .

Let the set  $M(\sigma)$  consist of all matrices  $a$ , such that  $a_{ij} \in \sigma_{ij}$ . If  $\sigma$  satisfies  $(\star)$ , then  $e + M(\sigma) = \{e + a : a \in M(\sigma)\}$  is closed under multiplication of matrices and by  $G(\sigma)$  we denote its maximal subgroup. Let  $G(m, \infty)$  denote the subgroup of  $GL(\infty, R)$  of all matrices  $a$  for which  $a_{ij} = 0$  for  $i > \max\{j, m\}$ . It is clear that  $GLB(R)$  is a direct limit of  $G(m, \infty)$  under natural embeddings.

*Proof of Theorem 1.2.* For  $H$ ,  $T(\infty, R) < H < GLB(R)$ , we define  $T$ -nets  $\sigma$  and  $\sigma(m)$  as follows:

$$\sigma_{ij} = \begin{cases} \{\zeta \in R : t_{ij}(\zeta) \in H\} & \text{for } i > j, \\ R & \text{for } i \leq j, \end{cases}$$

and

$$\sigma(m)_{ij} = \begin{cases} 0 & \text{if } i > \max\{m, j\}, \\ \sigma_{ij} & \text{otherwise.} \end{cases}$$

We put  $H(m) = H \cap G(m, \infty)$ . It is clear that  $H$  is a direct limit of  $H(m)$  and  $G(\sigma)$  is a direct limit of  $G(\sigma(m))$ . We now show that  $H(m) = G(\sigma(m))$ . If  $g \in H(m)$ , then  $g = \left( \begin{array}{c|c} g_1 & g_3 \\ \hline 0 & g_2 \end{array} \right)$  where  $g_1 \in GL_m(R)$ ,  $g_2 \in T(\infty, R)$ . Since  $H(m) \geq T(\infty, R)$ , multiplying  $g \in H(m)$  by matrices  $\left( \begin{array}{c|c} e_m & 0 \\ \hline 0 & g_2^{-1} \end{array} \right)$  and  $\left( \begin{array}{c|c} e_m & g_3 \\ \hline 0 & e \end{array} \right)$ , we see that  $g' = \left( \begin{array}{c|c} g_1 & 0 \\ \hline 0 & e \end{array} \right) \in H$ . We denote by  $\hat{H}(m)$  the subgroup of  $GL_m(R)$  generated by all such  $g_1$ . From Theorem 1.1 it follows that there exists a unique finite  $T$ -net  $\hat{\sigma}(m)$  of ideals of  $R$  such that  $\hat{H}(m) = G(\hat{\sigma}(m))$ . From the construction of  $\hat{\sigma}(m)$  and  $\sigma(m)$  we deduce the equality  $H(m) = G(\sigma(m))$  which implies  $H = G(\sigma)$ .

*Proof of Theorem 1.3.* We now prove (i). Then (ii) and (iii) follow easily from (i). If  $g \cdot G(\sigma) \cdot g^{-1} \subset G(\sigma')$ , then for some  $m$  we have  $g \in G(m, \infty)$ . So

$$\left( \begin{array}{c|c} g_1 & 0 \\ \hline 0 & e \end{array} \right) \cdot G(\sigma) \cdot \left( \begin{array}{c|c} g_1^{-1} & 0 \\ \hline 0 & e \end{array} \right) \subset G(\sigma')$$

or equivalently  $g_1 \cdot G(\hat{\sigma}(m)) \cdot g_1^{-1} \subset G(\hat{\sigma}'(m))$  in the group  $GL_m(R)$ . We show that  $g_1 \in G(\hat{\sigma}'(m))$  which implies  $g \in G(\sigma'(m))$ . From decomposition  $g_1 = uvdw$  where  $u, w$  are upper unitriangular,  $d$  is diagonal and  $v$  is lower unitriangular ([2], Thm. 1) it suffices to show that  $v \in G(\sigma'(m))$ . We have  $v = v_2 \cdot \dots \cdot v_m$ , where  $v_i = \prod_{j=1}^{i-1} t_{ij}(v_{ij})$ . We proceed by induction. Assume that for some  $r$ ,  $2 \leq r \leq m$ , we proved that  $v_k \in G(\sigma'(m))$ ,  $2 \leq k < r$ . Thus  $b \cdot G(\sigma(m)) \cdot b^{-1} \subset G(\sigma'(m))$ , where  $b = v_r \cdot \dots \cdot v_m$ . We have  $c = [d_s(\theta)^{-1}, b] \in G(\sigma')$  and hence  $c_{rs} = v_{rs}(\theta - 1) \in \sigma'_{rs}$ . This implies  $v_{rs} \in \sigma'_{rs}$  and  $v_r \in G(\sigma')$ .

*Proof of Theorem 1.4.* From [1], p. 45, for any field  $K$  we have  $GL_m(K) = T_m(K) \cdot S_m \cdot T_m(K)$ , where  $S_m$  is a regular matrix representation of symmetric group on  $m$  elements. It means that  $G(m, \infty) = T(\infty, K) \cdot S_m \cdot T(\infty, K)$  and since  $S_{\text{fin}}(\mathbb{N})$  is direct limit of  $S_m$  under natural embeddings Theorem 1.4 follows.

### 3. REMARKS

a) In a semilocal ring  $R$  the unit element 1 is a sum of two invertible elements if and only if every summand in the decomposition of a factor ring of  $R$  over a Jacobson radical is different from two elements field ([2], Thm. 3).

b) As was claimed in [5] it is possible to extend Theorem 1.1 to rings  $R$  such that  $R$  is additively generated by all invertible elements and 1 is a sum of two invertible elements. It means that our results are also valid in this case.

c) Under the same assumption on  $R$  as in the remark above, in [4] there is description of subgroups of  $T(\infty, R)$  containing  $D_{\text{fin}}(\infty, R)$  — the subgroup of finitary diagonal matrices. This result together with Theorem 1.2 give a description of two important intervals in the lattice of subgroups of  $GLB(R)$ .

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