PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 130, Number 9, Pages 2725–2735
S 0002-9939(02)06631-5
Article electronically published on April 11, 2002

A NEW FLOW ON STARLIKE CURVES IN $\mathbb{R}^3$

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(Communicated by Wolfgang Ziller)

Abstract. In this note we find a new evolution equation for starlike curves in $\mathbb{R}^3$. We study the evolution of the subaffine curvature and subaffine torsion under the flow and show that it is completely integrable. The solutions to the evolution which move without changing affine shape are subaffine elastic curves. We integrate the subaffine elastica by quadratures.

1. Introduction

This study is stimulated by the elastic curve and the vortex filament problem which describes a curve $\gamma[t](s) = \gamma(s,t)$ evolving in three-dimensional space $\mathbb{R}^3$. The *localized induction equation* (LIE) $\gamma_t = \gamma_s \times \gamma_{ss} = \kappa B$ is an idealized model of the evolution of the centerline of a thin vortex tube in an inviscid incompressible fluid ([1]). The subscript t represents differentiation with respect to time, the subscript s represents arc length along the evolving curve $\gamma[t]$, and κ is the curvature of the curve. Hasimoto ([8]) showed that LIE induces an evolution on the *complex curvature* $\psi = \kappa e^{i\int^s \tau du}$ (where τ is the torsion of the curve γ) governed by the *Nonlinear Schrödinger equation* (NLS) $\psi_t = i(\psi_{ss} + \frac{1}{2}|\psi|^2\psi)$. NLS is a well-known example of a *completely integrable system*, possessing infinitely many conserved quantities and soliton solutions. (See [10], [19].)

An *elastic curve* is a critical point for the functional $\int \kappa^2 ds$ defined on an appropriate class of regular space curves of fixed arc length ([3], [13], [14]). Elastic curves, and more generally *elastic rod centerlines*, evolve by Euclidean congruences under LIE; their complex curvature functions are solitons for NLS ([16], [11]).

We will examine a new evolution for curves X in $\mathbb{R}^3$ given by $X_t = -k_s X + kX_s$. Here s is the *subaffine arc length* and k is the *subaffine curvature*, defined in section 2. This flow induces an evolution for the curvature function k and the *subaffine torsion* τ. Define a *subaffine elastica* to be a critical point for the functional $\int k^2 ds$ defined on an appropriate class of regular space curves of fixed subaffine arc length. We will show that results analogous to those described above hold for this flow.

Our main results are

Theorem 1. *The evolution of k and τ under the flow generated by $-k_sX + kX_s$ is completely integrable. In particular, the evolution of k is governed by the Korteweg - de Vries equation, a completely integrable equation.*

Received by the editors February 4, 2000.
2000 *Mathematics Subject Classification.* Primary 53A04; Secondary 53A15.

Theorem 2. *The subaffine elastica gives the congruence solutions of the flow generated by $-k_sX + kX_s$, that is, solutions which evolve by symmetries of the special linear group in $\mathbb{R}^3$.*

In ([20]), Ulrich Pinkall has studied an evolution for starlike curves in $\mathbb{R}^2$ which also gives rise to the KdV flow and shows that there is a symplectic structure on the space of curves with respect to which the flow is generated by the Hamiltonian of total centro-affine curvature. The geometric connection between the two flows is not obvious, due to the fact that starlike planar curves are degenerate when viewed as starlike curves in $\mathbb{R}^3$. However, a starlike planar curve can be viewed as the tangent indicatrix of an affine planar curve (in the plane $z = 1$), which in turn can be viewed as a starlike curve with vanishing subaffine torsion in $\mathbb{R}^3$.

The following is an outline of the rest of the paper. In section 2 we give the definitions and some basic concepts in subaffine (centro-affine) geometry. In section 3 we illustrate the basic variation formula and the properties of the flow. In section 4 we study the subaffine elastica and give the complete solution by using Killing fields and the classification of the conjugacy classes in $sl(3,\mathbb{R})$.

A more thorough treatment of affine and subaffine elastic curves can be found in the first author's thesis ([9]).

2. Definition and variation formula

We consider curves in the affine space $\mathbb{R}^3$ ([18], [21]). Let (x, y, z) be a standard coordinate system of the affine space $\mathbb{R}^3$ and let $X : I \to \mathbb{R}^3$ be a smooth curve. We write it as a column vector

$$X(u) = (x(u), y(u), z(u))^T$$

where $x(u), y(u)$ and $z(u)$ are smooth functions of u defined on a certain interval I. We identify the tangent space of $\mathbb{R}^3$ with $\mathbb{R}^3$ and write the tangent vector as $X'(u) = \frac{dX}{du} = \begin{pmatrix} x'(u) \\ y'(u) \\ z'(u) \end{pmatrix}$ or simply X'. Similarly $(X'(u)\ X''(u)\ X^{(3)}(u))$ is the 3×3 matrix

$$\begin{pmatrix} x' & x'' & x^{(3)} \\ y' & y'' & y^{(3)} \\ z' & z'' & z^{(3)} \end{pmatrix}.$$

Definition 1. A curve $X : I \to \mathbb{R}^3$ is *starlike* if

$$|X(u)\ X'(u)\ X''(u)| = \det(X(u)\ X'(u)\ X''(u)) \neq 0$$

for any $u \in I$. A parameter s is called a subaffine arclength parameter of the starlike curve X if

$$A(s) = (X(s)\ X'(s)\ X''(s)) \in SL(3,\mathbb{R}),$$

that is, $|X(s)\ X'(s)\ X''(s)| = 1$, for any $s \in I$.

Definition 2. Let s be a subaffine arclength parameter of a curve $X : I = [0, L] \to \mathbb{R}^3$, where L is the *subaffine arclength* of X. We define the *subaffine curvature* and *subaffine torsion* of the curve by

$$k(s) = \left|X(s) \quad X''(s) \quad X^{(3)}(s)\right|, \quad \tau(s) = \left|X'(s) \quad X''(s) \quad X^{(3)}(s)\right|.$$

Differentiating $|X(s)\ X'(s)\ X''(s)| = 1$, we obtain $|X(s)\ X'(s)\ X^{(3)}(s)| = 0$. Hence $X(s), X'(s), X^{(3)}(s)$ are linearly dependent and from the definition of $k(s), \tau(s)$ we have

$$X^{(3)}(s) = \tau(s)X(s) - k(s)X'(s).$$

If $X(u)$ is a starlike curve, we may re-parametrize X by subaffine arclength, using the formula

$$s(u) = \int_{u_0}^{u} |X(u)\ X'(u)\ X''(u)|^{\frac{1}{3}} du.$$

This shows that a starlike curve $X(u)$ admits a subaffine arclength parameter uniquely up to a constant, and the starlike property is invariant under the action of $SL(3, \mathbb{R})$ on $\mathbb{R}^3$.

Remark 1. If $\sigma(s)$ is a non-degenerate curve in $\mathbb{R}^3$ parametrized by *affine* arclength, then its tangent indicatrix $X(s) = \sigma'(s)$ is a starlike curve parametrized by subaffine arclength. Furthermore, the subaffine curvature and torsion of X are the affine curvature and torsion of σ. See, e.g., [5], [6] for properties of affine curves and flows in $\mathbb{R}^3$.

We denote by ∇ the usual affine connection of $\mathbb{R}^3$. This is a torsion-free and flat connection. The letter X will also denote a variation $X = X_w(u) = X(w, u) : (-\varepsilon, \varepsilon) \times I \to \mathbb{R}^3$ with $X(0, u) = X(u)$. Associated with such a variation is the variational vector field $W = W(u) = (\partial X/\partial w)(0, u) = dX(\partial/\partial w)(0, u)$ along the curve $X(u)$. We will also write $W = W(w, u) = dX(\partial/\partial w)(w, u)$, $X'(u) = X'(w, u) = dX(\partial/\partial u)(w, u)$. We know

$$[X'(w, u), W(w, u)] = [dX(\frac{\partial}{\partial u}), dX(\frac{\partial}{\partial w})] = dX([\frac{\partial}{\partial u}, \frac{\partial}{\partial w}]) = 0.$$

Let s be the subaffine arclength parameter of the curve $X_w(u)$ and write $X(s)$, $k(w, s)$ for the corresponding reparametrizations and $s \in [0, L(w)]$, where $L(w)$ is the subaffine arclength of $X_w(s)$. We may assume that $u = s$ is the subaffine arclength parameter of $X(0, u)$ and then $I = [0, L]$.

By straightforward computation, we obtain the following lemma.

Lemma 1. *Using the above notation, we have the following formulas:*

1. $[X'(u), W(u)] = 0$.
2. $W(ds/du) = -g\,ds/du$, *where*
$$g = -\frac{1}{3}|X(u)\ X'(u)\ X''(u)|^{-1} W(|X(u)\ X'(u)\ X''(u)|).$$
3. $[W, X'(s)] = gX'(s)$.
4. $W[k] = \frac{\partial k}{\partial w} = |W\ X''(s)\ X^{(3)}(s)| + |X(s)\ \nabla^2_{X'(s)}W\ X^{(3)}(s)|$
$+|X(s)\ X''(s)\ \nabla^3_{X'(s)}W| + 5gk - g_{ss}$.
5. $W[\tau] = |W'\ X''(s)\ X^{(3)}(s)| + |X'(s)\ W''\ X^{(3)}(s)| + |X'(s)\ X''(s)\ W^{(3)}| + 6g\tau$.

Note that we do *not* assume that the curves $X_w(u)$ are parametrized by arclength for $w \neq 0$. Thus, in the third part of the lemma above, even though $X'(u) = X'(0, u) = X'(0, s) = X'(s) = \frac{dX}{ds}$, the Lie bracket of $W(s)$ with $X(s)$ should not be taken to be 0. Instead, one should write $\frac{dX}{ds}(w, u) = \frac{du}{ds}X'(w, u)$ before computing the bracket, then evaluate at $w = 0$, where $\frac{du}{ds} \equiv 1$. Said differently, the Lie bracket $[\frac{\partial}{\partial s}, \frac{\partial}{\partial w}]$ does not necessarily vanish.

3. The evolution equation

We will consider the flow generated by the vector field $-k_s X + kX_s$ on starlike curves in 3-dimensional affine space. Here k is the subaffine curvature. Let $X : J \times I \longrightarrow \mathbb{R}^3$, $(t,s) \longmapsto X(t,s)$ be a smooth surface. Here I, J are given intervals; we may assume $0 \in J$. For a fixed t, s is the subaffine arclength parameter of the curve $s \longmapsto X(t,s)$. We will consider the evolution system

$$\frac{\partial X}{\partial t} = W$$

where W is the vector field $-\partial k(t,s)/\partial s\ X(t,s) + k(t,s)\partial X(t,s)/\partial s$. We consider W as a variational vector field on the starlike curve $X_t(s) = X(t,s)$. By using Lemma 1, we have the variation formulae for the subaffine curvature k and the subaffine torsion τ

$$W(k) = 2k^{(3)} + 3kk', \tag{1}$$

$$W(\tau) = k\tau' + 3k'\tau - k^{(4)} - kk''. \tag{2}$$

Here we use $k^{(n)}$ to represent the n-th partial derivative of $k(t,s)$ with respect to the subaffine arclength parameter s.

We write the variation formulae of k and τ as two standard partial differential equations

$$\frac{\partial k}{\partial t} = 3k\frac{\partial k}{\partial s} + 2\frac{\partial^3 k}{\partial s^3}, \tag{3}$$

$$\frac{\partial \tau}{\partial t} = k\frac{\partial \tau}{\partial s} + 3\frac{\partial k}{\partial s}\tau - \frac{\partial^4 k}{\partial s^4} - k\frac{\partial^2 k}{\partial s^2}. \tag{4}$$

The first equation is a form of the Korteweg-de Vries (or KdV) equation. For an initial condition, we can solve it by using the inverse scattering method. For the second equation, we use the solution of the first equation. At this time, k is a function of t and s. We have the solution

$$\tau = \frac{1}{k^3}[c(t + \int \frac{ds}{k}) + \int (k^{(4)} + kk'')k^2 ds].$$

Here c can be determined by the initial condition. We have proved:

Theorem 1. *The evolution of k and τ under the flow generated by $-k_s X + kX_s$ is completely integrable. In particular, the evolution of k is governed by the Korteweg - de Vries equation, a completely integrable equation.*

The curve evolution possesses conservation laws; that is, there is a sequence of functionals of the form $I_i = \int f_i ds$ which are time-independent. The first several functionals are

$$
\begin{aligned}
f_0 =& 1,\\
f_1 =& k,\\
f_2 =& k^2,\\
f_3 =& k^3 - 2(k')^2,\\
f_4 =& k^4 - 8k(k')^2 + \frac{16}{5}(k'')^2,\\
f_5 =& k^5 - 20k^2(k')^2 + 16k(k'')^2 - \frac{32}{7}(k^{(3)})^2,\\
f_6 =& k^6 - 40k^3(k')^2 - \frac{40}{3}(k')^4 + 48k^2(k'')^2 + \frac{640}{21}(k'')^3 - \frac{192}{7}k(k^{(3)})^2\\
&+ \frac{128}{21}(k^{(4)})^2,\\
f_7 =& k^7 - 70k^4(k')^2 - \frac{280}{3}k(k')^4 + 112k^3(k'')^2 224(k')^2(k'')^2 + \frac{640}{3}k(k'')^3\\
&- 96k^2(k^{(3)})^2 - \frac{640}{3}k''(k^{(3)})^2 + 128k(k^{(4)})^2 - \frac{256}{33}(k^{(5)})^2,\\
&\cdots .
\end{aligned}
$$

The corresponding phenomenon for LIE is described in ([11]), where the sequence of conserved integrals is derived by a recursion operator. We found the integrals above by an *ad hoc* method. It would be interesting to determine a recursion operator in the present situation. See ([12]) for further discussion of geometric invariants of integrable flows.

4. The subaffine elastica

We consider the "energy" functional defined on a class of starlike curves in $\mathbb{R}^3$.

$$\int_0^{L(w)} (k^2+\lambda)ds = \int_0^L (|X(w,s)\ X''\ X^{(3)}|^2+\lambda)|X(w,u)\ X'(w,u)\ X''(w,u)|^{\frac{1}{3}}du.$$

Here λ is a Lagrange multiplier, s is the subaffine arclength parameter of the curve $X_w(u) = X(w,u)$. We give $X(w,u)$ a boundary condition such that $W(0,0) = W(0,L) = 0$, $\nabla_{X'}W(0,0) = \nabla_{X'}W(0,L) = 0$, $\nabla^2_{X'}W(0,0) = \nabla^2_{X'}W(0,L) = 0$, $\nabla^3_{X'}W(0,0) = \nabla^3_{X'}W(0,L) = 0$. We have the variation formula

$$\frac{d}{dw}\int_0^{L(w)}(k^2+\lambda)ds|_{w=0}$$

$$= \frac{1}{3}\int_0^L [2(k^{(4)} + 7kk'' + 9k'\tau + 6(k')^2 + 3k\tau' + 3k^3 - \lambda k)|W\ X\ X'|$$

$$-3(3k^2 - \lambda + 4k'')|W\ X'\ X''|]ds.$$

The Euler-Lagrange equations are

$$k^{(4)} + 7kk'' + 6(k')^2 + 9\tau k' + 3\tau' k + 3k^3 - \lambda k = 0, \tag{5}$$

$$4k'' + 3k^2 - \lambda = 0. \tag{6}$$

Definition 3. A starlike curve in $\mathbb{R}^3$ with the subaffine arclength parameter s is called a *subaffine elastica* with respect to the subaffine curvature k if it satisfies the above Euler-Lagrange equations.

For k constant, we know $k = \pm\sqrt{\lambda/3}, \tau = 0$, where $\lambda \neq 0$. Now we may assume k is not constant. We can obtain the following first integral of the Euler-Lagrange equations:

$$2(k')^2 + k^3 - \lambda k = c_1, \tag{7}$$

$$\tau(s) = -\frac{1}{2}k' + \frac{c_2}{k^3}. \tag{8}$$

We can integrate the first equation directly. For the stable case, i.e., the case in which k is periodic in s, we can express it by using Jacobi elliptic functions. In this case, $4\lambda^3 - 27c_1^2 > 0$, and $x^3 - \lambda x - c_1 = 0$ has three real distinct solutions $\alpha_1 < \alpha_2 < \alpha_3$. The solution can be written in terms of the Jacobi elliptic sine function as

$$k(s) = \alpha_3(1 - q^2 sn^2(\frac{d}{\sqrt{2}}s, p)) \tag{9}$$

where $p^2 = (\alpha_3 - \alpha_2)/(\alpha_3 - \alpha_1), q^2 = (\alpha_3 - \alpha_2)/\alpha_3$ and $d = \sqrt{\alpha_3 q^2/4p^2} = \sqrt{\alpha_3 - \alpha_1}/2$. (For the properties of the Jacobi elliptic functions, see [4], [13].) Since the structure equation

$$X^{(3)}(s) = \tau(s)X(s) - k(s)X'(s)$$

is a linear differential equation of $X(s)$, it has global solutions. For the constant case, we can solve it directly. We will only consider the nonconstant curvature case and solve it by using Killing fields and normal forms under conjugation in $sl(3, \mathbb{R})$.

Definition 4. Let $X(u)$ be a starlike curve in $\mathbb{R}^3$. We call a vector field W *Killing along* $X(u)$ if it annihilates ds/du, k and τ. (This is the exact analogue of the definition in the Euclidean case; see [14], p. 514.)

We represent W as $W = f_1(s)X(s) + f_2(s)X'(s) + f_3(s)X''(s)$, for appropriately chosen functions $f_i(s)$. Then from Lemma 1, we have

$$3f_1 + 3f_2' + f_3'' - 2kf_3 = 0,$$

$$-3f_1'' + 3kf_1 - f_2^{(3)} + 5kf_2' + k'f_2 + 4kf_3'' - 3(\tau - k')f_3' + (k'' - 2\tau' - 2k^2)f_3 = 0,$$

$$f_1^{(3)} + kf_1' + 3\tau f_1 + 6\tau f_2' + \tau' f_2 + 4\tau f_3'' + 3\tau' f_3' + (\tau'' - 2k\tau)f_3 = 0.$$

One can verify by direct computation that $-k'X + kX'$ is Killing along any subaffine elastica.

Theorem 2. *The subaffine elastica gives the congruence solutions of the flow generated by $-k'X + kX'$, that is, solutions which evolve by symmetries of the special linear group in $\mathbb{R}^3$.*

The proof follows immediately from the fact that, exactly as in the Euclidean case, there are fundamental existence and uniqueness theorems for affine curves and for subaffine curves. (See, e.g., [21], pp. 5 - 16.)

Definition 5. A vector field W on $\mathbb{R}^3$ is *Killing with respect to* $SL(3, \mathbb{R})$ if its flow generates a one-parameter subgroup of $SL(3, \mathbb{R})$.

The above equations of f_1, f_2, f_3 can be written as a system of first order linear equations of $f_3', f_3, f_2'', f_2', f_2, f_1'', f_1', f_1$ in the following form:

$$(f_3', f_3, f_2'', f_2', f_2, f_1'', f_1', f_1)' = (f_3', f_3, f_2'', f_2', f_2, f_1'', f_1', f_1)A.$$

Here A is an 8×8 matrix. Therefore the dimension of the solution space is 8. This dimension agrees with the dimension of the group $SL(3, \mathbb{R})$. Therefore, it follows that a vector field which is Killing along a subaffine elastica X can be extended to a Killing field on $\mathbb{R}^3$, a field which is invariant under the action of a one-parameter group in $SL(3, \mathbb{R})$. We denote it by $\tilde{W}$. By choosing the appropriate coordinate system on $\mathbb{R}^3$, we may bring this field into standard form. The basic tool for doing this is the classification of elements of the Lie algebra by normal forms.

We denote the Lie algebra of $SL(3, \mathbb{R})$ by $sl(3, \mathbb{R})$. For any $M \in sl(3, \mathbb{R})$, let $\lambda_1, \lambda_2, \lambda_3$ be its eigenvalues. At least one of them is real and $\lambda_1 + \lambda_2 + \lambda_3 = 0$. We classify it into the following cases:

1. All roots are zero and the kernel has dimension 1.
2. Two equal nonzero roots: $\lambda_1 = \lambda_2 = a, \lambda_3 = -2a$, the dimension of invariant subspace corresponding to a is 2.
3. Three nonzero distinct real roots: $\lambda_1 = a, \lambda_2 = b, \lambda_3 = -a - b$.
4. $\lambda_1 = \lambda_2 = a, \lambda_3 = -2a$, diagonalizable case.
5. All roots are real and one is zero: $\lambda_1 = 0, \lambda_2 = a, \lambda_3 = -a, a \neq 0$.
6. $\lambda_1 = -2a, \lambda_2 = a + bi, \lambda_3 = a - bi, b \neq 0$.
7. All roots are zero and the kernel has dimension 2.

Subaffine elastic curves are determined by the two parameters c_1 and c_2. We divide the (c_1, c_2) plane into four pieces by using the classification of the conjugacy class of $\tilde{W}$ in the Lie algebra $sl(3, \mathbb{R})$.

The Killing field $\tilde{W}$ is one of the Killing fields generated by the 1-parameter subgroup corresponding to one of the seven cases above. We study it case by case.

Case 1. The normal form for $M \in sl(3, \mathbb{R})$ is

$$M \sim \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

We can choose a coordinate system (x, y, z) such that the Killing field generated by the 1-parameter Lie subgroup corresponding to M is

$$W = x\frac{\partial}{\partial y} + y\frac{\partial}{\partial z}.$$

The equation $W = -k'X + kX'$ is a system of first order equations in x, y, z:

$$\begin{cases} kx' - k'x = 0, \\ ky' - k'y = x, \\ kz' - k'z = y. \end{cases}$$

The solution is the subaffine elastica equation

$$\begin{cases} x(s) = d_1 k(s), \\ y(s) = k(s)(\int \frac{d_1}{k} ds + d_2), \\ z(s) = k(s)(\int (\int \frac{d_1}{k} ds + d_2)/k ds + d_3). \end{cases}$$

Here $d_1 = 1$, d_2 and d_3 are constants.

If we now calculate the curvature and torsion of this solution, we arrive at the equations

$$k(s) = \frac{1}{2k^2(s)}(2k^3 + c_1), \qquad \tau(s) = \frac{1}{2k^3(s)}(c_1 - k^3)k'.$$

The first equation implies $\mathbf{c}_1 = 0$. Comparing the second equation with equation (8) shows that $\mathbf{c}_2 = 0$.

Case 2. We can represent $M \in sl(3, \mathbb{R})$ as

$$M \sim \begin{pmatrix} -2a & 0 & 0 \\ 0 & a & 0 \\ 0 & 1 & a \end{pmatrix}.$$

We can choose a coordinate system (x, y, z) such that the Killing field generated by the 1-parameter Lie subgroup corresponding to M is

$$W = -2ax\frac{\partial}{\partial x} + ay\frac{\partial}{\partial y} + (y + az)\frac{\partial}{\partial z}.$$

This leads to the equations

$$\begin{cases} x(s) = d_1 k(s) e^{-\int \frac{2a}{k} ds}, \\ y(s) = d_2 k(s) e^{\int \frac{a}{k} ds}, \\ z(s) = k(s) e^{\int \frac{a}{k} ds}(d_3 + d_2 \int \frac{1}{k} ds). \end{cases}$$

Here d_1, d_2, d_3 are constants and satisfy $9a^2 d_1 d_2^2 = 1$.

Computing curvature and torsion leads to the conditions

$$\begin{cases} c_1 = 6a^2, \\ c_2 = -2a^3. \end{cases}$$

This implies $\mathbf{c}_1 \neq 0$, $\mathbf{c}_2 \neq 0$ and $\mathbf{c}_1^3 - 54c_2^2 = 0$.

Cases 3, 4, 5. We choose the element $M \in sl(3, \mathbb{R})$ as

$$M \sim \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & -a-b \end{pmatrix}.$$

We can choose a coordinate system (x, y, z) such that the Killing field generated by the 1-parameter Lie subgroup corresponding to M is

$$W = ax\frac{\partial}{\partial x} + by\frac{\partial}{\partial y} - (a + b)z\frac{\partial}{\partial z}.$$

The solution is the subaffine elastica equation

$$\begin{cases} x(s) = d_1 k(s) e^{\int \frac{a}{k} ds}, \\ y(s) = d_2 k(s) e^{\int \frac{b}{k} ds}, \\ z(s) = d_3 k(s) e^{-\int \frac{(a+b)}{k} ds}. \end{cases}$$

Here d_1, d_2, d_3 are constants and satisfy the condition

$$-(2a^3 + 3a^2b - 3ab^2 - 2b^3)d_1d_2d_3 = 1.$$

Note that this implies $a \neq b$, so case 4 cannot occur. Calculation reveals that

$$\begin{cases} a^2 + ab + b^2 = \frac{1}{2}c_1, \\ -ab(a+b) = c_2, \end{cases}$$

which implies $\mathbf{c}_1^3 - 54c_2^2 > 0$.

Case 6. We pick an element $M \in sl(3, \mathbb{R})$ as

$$M \sim \begin{pmatrix} -2a & 0 & 0 \\ 0 & 0 & a^2 + b^2 \\ 0 & -1 & 2a \end{pmatrix}.$$

We can choose a coordinate system (x, y, z) such that the Killing field generated by the 1-parameter Lie subgroup corresponding to M is

$$W = -2ax\frac{\partial}{\partial x} + (a^2 + b^2)z\frac{\partial}{\partial y} + (-y + 2az)\frac{\partial}{\partial z}.$$

Along the subaffine elastica X, we have a linear system

$$\begin{cases} kx' - k'x = -2ax, \\ ky' - k'y = (a^2 + b^2)z, \\ kz' - k'z = -y + 2az. \end{cases}$$

From the first equation, we find

$$x(s) = d_1k(s)e^{\int \frac{-2a}{k}ds}.$$

Introducing two new variables $U = y/k, V = z/k$, the other two equations become

$$kU' = (a^2 + b^2)V, \tag{10}$$

$$kV' = -U + 2aV. \tag{11}$$

Combining these two equations, we obtain the following second order equations:

$$k(kV')' - 2akV' + (a^2 + b^2)V = 0,$$

$$k(kU')' - 2akU' + (a^2 + b^2)U = 0.$$

We need only consider the first equation. By Setting $r = kV'/V$, we obtain the Riccati equation

$$kr' + r^2 - 2ar + (a^2 + b^2) = 0.$$

Integrating this equation ([2]), we have

$$r(s) = a + b\tan(-\int \frac{b}{k}ds + d_2).$$

Back to the curve equation, we get the solution

$$\begin{cases} x(s) = d_1k(s)e^{\int \frac{-2a}{k}ds}, \\ y(s) = d_3k(2a - r)e^{\int \frac{r}{k}ds}, \\ z(s) = d_3ke^{\int \frac{r}{k}ds}. \end{cases}$$

Here d_1, d_2, d_3 are constants and satisfy the condition

$$-(9a^2+b^2)d_1d_2^2(r^2-2ar+a^2+b^2)e^{2\int(r-a)/kds}=1.$$

Again we solve for the two parameters

$$\begin{cases} c_1 = 6a^2-2b^2, \\ c_2 = -2a(a^2+b^2). \end{cases}$$

In this case we have $\mathbf{c}_1^3 - 54c_2^2 < 0$.

Case 7. This case turns out not to occur.

When we look at the regions on the plane (c_1, c_2) corresponding to these cases, we see that the origin $(0,0)$ represents case 1, the curve $c_1^3 - 54c_2^2 = 0$ minus the origin represents case 2, the region to the left of the curve $c_1^3 - 54c_2^2 = 0$ represents case 6, the positive c_1 axis represents case 5 and the remaining region represents case 3. This gives the bifurcation diagram in the (c_1, c_2) plane.

We have proven the theorem:

Theorem 3. *The subaffine elastica of $\mathbb{R}^3$ is integrable by quadratures.*

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