POSITIVE SOLUTIONS OF A LOGISTIC EQUATION ON UNBOUNDED INTERVALS

LI MA AND XINGWANG XU

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ABSTRACT. In this paper, we study the existence of positive solutions of a one-parameter family of logistic equations on R_+ or on R. These equations are stationary versions of the Fisher equations and the KPP equations. We also study the blow-up region of a sequence of the solutions when the parameter approaches a critical value and the non-existence of positive solutions beyond the critical value. We use the direct method and the sub and super solution method.

1. INTRODUCTION

As pointed out in the book written by Buttazzo, Giaguinta, and Hildebrant [1], one-dimensional variational problems deserve special attention. In fact, these kind of problems have their own characters. Sometimes, higher-dimensional variational problems can be reduced to one-dimensional problems.

In this work, we study the existence of positive solutions of the following one-parameter family of logistic equations

$$(1)_a \qquad\qquad u'' + af(x)u - b(x)u^p = 0 \quad \text{in} \quad I,$$

where $I = (0, \infty)$ is the half real line R_+ or the whole real line R, $p > 1$, $a \in R$ is the parameter, $f(x)$ is a given C^1 function on I, which is positive somewhere, and $b(x)$ is a given non-negative smooth function on I. These equations are of interest in mathematical biology. Our equations appear also in Ginzburg-Landau theory of super-conductivity [21]. Such equations were also known as the stationary version of the Fisher equation and KPP equation [20], and they were considered by Kazdan-Warner [6], Ouyang [7], and de Pino [8] in compact manifolds of higher dimensions and Du and Huang [10] in compact domains in the euclidean space R^n. Remarkably, Ouyang [7] proved a conjecture posed by Kazdan and Warner in [6]. Such problems have also been considered by Li-Tam-Yang [12], Afrouzi-Brown [20] and Du-Ma [18]-[19] on R^n. For more background material, see [20]. However, to the authors best knowledge, there are very few results on $(1)_a$ on R_+ or on R.

These problems have a close relation with the principal eigenvalue problem

$$(2) \qquad\qquad u'' + \lambda f(x)u = 0 \quad \text{in} \quad R.$$

It is proved by Brown, Cosner, and Fleckinger [4] that if $\int_R f(x)dx > 0$, then there does not exist a positive principal eigenvalue of (2). This result is also true in dimension two. For more results about principal eigenvalue problems, we refer to the articles [5], [3] and [2]. So we may expect that the study of $(1)_a$ has some special properties in dimensions one and two. We will study logistic equations in dimension two in the near future.

For simplicity, we denote the left side of $(1)_a$ by the operator $T_a(u)$, and we state the results in the case where $I = R_+$. Since we will use the sub and super method to study $(1)_a$ in the case when $I = R$, the result will be stated later in this article. We now assume that $I = R_+$, and we say that $f(x)$ is in *class P* if $f(x)$ is positive somewhere and satisfies the condition that

$$max\{f(x), 0\} \leq p(x), \quad \text{on} \quad R_+,$$

where $p(x)$ is some positive function on R_+ with

$$\int_{R_+} xp(x)dx < \infty.$$

With this assumption we can obtain a compactness result, which is of the type of Rellich-Kondrakov imbedding for Sobolev spaces.

We define the zero set

$$M_0 = \{x \in R_+; b(x) = 0\},$$

which will play an important role in the existence problem and the blow-up problem (see [6]).

Assume that M_0 is a bounded open set in R. Let $\lambda_* := \lambda_1(M_0)$ and let λ_1 denote the principal eigenvalues of the operator $-u''$ with respect to the weighted quotient

$$Q(u, M) := \int_M |u'|^2 / \int_M f(x)u^2$$

where we assume $\int_M f(x)u^2 > 0$; $M = M_0$ and $M = R_+$ respectively. We write $\lambda_* = +\infty$ when M_0 is empty. Note that $\lambda_* \geq \lambda_1$.

We assume that f is positive somewhere outside M_0. So we have $\lambda_* > \lambda_1$. Then we will use the direct method to prove the existence of a positive solution of the equation $(1)_a$ when $a \in (\lambda_1, \lambda_*)$. The idea goes as follows. Define the space E as the completion of $C_0^1(R_+)$ with respect to the norm $\int_R |u'|^2$. Define the functional

$$J(u) = \frac{1}{2} \int_{R_+} (|u'|^2 - af(x)u^2) + \frac{1}{p+1} \int_{R_+} b(x)|u|^{p+1}$$

on E. $J(u) = +\infty$ if $\int_{R_+} b(x)|u|^{p+1} - af(x)u^2 = +\infty$. Note that by our inequality $(*)$ to be proved in the next section, the functional J is well-defined on E. It is easy to see that J is weakly lower semi-continuous on E. Assume $a \in (\lambda_1, \lambda_*)$. We then prove that the functional J is coercive on E and we get a minimizer of the functional J on E. We will show that this minimizer is a non-trivial positive solution of $(1)_a$.

We also study the blow-up region of a sequence of the solutions when the parameter approaches the critical value λ_* and the non-existence of positive solutions beyond the critical value. We use a variational method and a barrier method.

To state our result, we introduce some notations and concepts. We will write $||u||$ as the norm of $u \in H$ and $||u||_q$ as the norm of $u \in L^q(R_+)$ where $q > 1$. We will write

$$L^2(R_+, p) = L^2(R_+, pdx).$$

We will sometimes write $M := R_+$.

Our main result is

Theorem 1. *Given a positive bounded function $f(x)$ of class P and a non-trivial non-negative smooth function b on R_+, assume that f is positive somewhere outside the zero set M_0 of the function b. Then the following three assertions are true:*

(i) *When $a \in (\lambda_1, \lambda_*)$, there is a unique positive smooth solution $u_a \in H$ of $(1)_a$.*

(ii) *Assume $\lambda_* < +\infty$. When $a \to \lambda_*^-$, we have that $u_a \to +\infty$ on any compact set $K \subset\subset M_0$.*

(iii) *Assume $\lambda_* < +\infty$. When $a \geq \lambda_*$, no positive E-solutions of $(1)_a$ exist.*

We remark that we can extend the existence part of the result above to the following system:

$$W_j'' + aF_j(x)W_j - |W|^{p-1}W_j = 0 \quad \text{on} \quad R_+,$$

with $W_j(0) = 0$ for $j = 1, ..., N$. Here we write $W = (W_1, ..., W_N)$ and assume that all the coefficients $F_j(x)$ are dominated by functions in class (P). Because the formulation of the result is straightforward, we omit it.

The rest of this paper is organized as follows. In section two, we set up some inequalities and we prove a compactness result. We also discuss the existence of the principal eigenvalue and eigen-function problem (2) on R_+. Uniqueness and existence of positive E-solutions for $(1)_a$ will be proved in section three. In section four and in section five, blow-up behavior of a solution sequence and non-existence of positive solutions will be presented. In the final section, we study the existence and uniqueness of positive solutions of $(1)_a$ in the case when $I = R$.

2. COMPACTNESS LEMMATA

In this section we will set up some useful calculus results, which are easy to prove, but they are not easy to find in the literature (see [1] for more references). Hence we give full proofs. We point out that our compactness result is new.

2.1. The case when $I = R_+$. Assume that $u \in C_0^1(R_+)$ with

$$\int_{R_+} |u'|^2 dx \leq C^2$$

where C is a positive constant. Then, for any $x, y, \in R_+$, we have

$$|u(x) - u(y)| \leq \int_{[x,y]} |u'| dx.$$

Using the Cauchy-Schwartz inequality we find

$$|u(x) - u(y)| \leq C|x - y|^{1/2}.$$

In particular, since $u(0) = 0$, we have

$$|u(x)| \leq C|x|^{1/2}.$$

Given a positive function $p = p(x)$ on R_+ satisfying

$$\int_{R_+} p(x)x \, dx < +\infty,$$

we then have, for $R \geq 0$, that

$$(*) \qquad \int_{(R,+\infty)} p(x)u^2 \, dx \leq C \int_{(R,+\infty)} p(x)x \, dx.$$

Define E as the completion of $C_0^1(R_+)$ in the norm

$$|u| = \left(\int_{R_+} |u'|^2 \, dx \right)^{1/2}.$$

It is clear that the inequality $(*)$ is also true for every $u \in E$.

Let

$$L^2(R_+, p \, dx) = \{ u \in L^1_{loc}(R_+); \int_{R_+} pu^2 < +\infty \}.$$

Then we have the following compactness result.

Proposition 2. *The imbedding from E to $L^2(R_+, p \, dx)$ is bounded and compact.*

Proof. The boundedness of the imbedding is clear from the inequality $(*)$.

Now let $\{u_k\} \subset E$ with

$$\int_{R_+} |u_k'|^2 \, dx \leq C^2.$$

Then we have, for every k and any $x, y \in R_+$,

$$|u_k(x) - u_k(y)| \leq C|x - y|^{1/2}.$$

In particular, take $y = 0$ and we have

$$|u_k(x)| \leq C|x|^{1/2}.$$

Given $\epsilon > 0$. By inequality $(*)$ we can choose $R > 1$ sufficiently large such that for every k we have

$$\int_{[R,+\infty)} pu_k^2 \leq C \int_{[R,+\infty)} xp(x) \, dx < \epsilon/2.$$

By this and the Azela-Ascoli theorem we can find a subsequence, still denoted by $\{u_k\}$, such that it converges in $C_{loc}^0(R_+)$, and hence in $L^2(R_+, p \, dx)$. This completes the proof of compactness part of the theorem. $\qquad \square$

Note that we cannot handle the case when R_+ is replaced by the entire real line R^1 without further restriction. But we have another inequality, which will be introduced below. We remark that the inequality below cannot be used to prove a similar compactness result as above.

Using the compactness result, one readily gets the following result on the existence of the principal eigenvalue and function.

Theorem 3. *Assume the function f is positive somewhere on R_+, which is of the class (P). Then there exist a positive function V in E and a positive number μ such that*

$$V'' + \mu f(x)V = 0, \quad on \quad R_+.$$

2.2. The case when $I = R$. Assume that $p = p(x)$ is a positive function on the whole real line R^1. Now we define a Hilbert space E', which is the completion of the space $C_0^1(R^1)$ according to the Dirichlet integral. As before, we can define the space $L^2(R, pdx)$.

Definition. We say that a positive function p satisfies the condition (AB) if there are two positive constants A and B such that for any $x \in R$,

$$p(x) \le \frac{A}{(|x| + B)^2}.$$

Although the following result will not be used later in this work, we believe that it is a useful tool in the analysis of solutions of ordinary differential equations.

Proposition 4. *Assume that the positive function p on R satisfies the condition (AB). Then there is a positive constant C such that for every $u \in E'$, it holds that*

$$\int_R p(x)u^2 dx \le \frac{4A}{B}u(0)^2 + C\int_R |u'|^2 dx.$$

We should mention that this inequality is a version of Hardy's inequality.

Proof. Without loss of generality, we may assume $A = 1 = B$ and take $u \in C_0^1(R)$. Note that

$$\int_R \frac{1}{(|x| + 1)^2}u^2 dx = \int_0^\infty \frac{u^2}{(x + 1)^2}dx + + \int_{-\infty}^0 \frac{u^2}{(x - 1)^2}dx.$$

Then by integration by parts, we get

$$\int_0^\infty \frac{u^2}{(x + 1)^2}dx = u(0)^2 + \int_0^\infty \frac{2uu'}{(x + 1)}dx.$$

Similarly we have

$$\int_{-\infty}^0 \frac{u^2}{(x - 1)^2}dx = u(0)^2 + \int_{-\infty}^0 \frac{2uu'}{(x - 1)}dx.$$

Putting these two equalities together, we have

$$\int_R \frac{1}{(|x| + 1)^2}u^2 dx = 2u(0)^2 + \int_R \frac{2sgn(x)uu'}{(|x| + 1)}dx.$$

Note that by the Cauchy-Schwartz inequality, we find

$$\int_R \frac{2|u||u'|}{(|x| + 1)}dx \le \frac{1}{2}\int_R \frac{1}{(|x| + 1)^2}u^2 dx + 8\int_R |u'|^2 dx.$$

Hence we get

$$\int_R \frac{1}{(|x| + 1)^2}u^2 dx = 4u(0)^2 + 16\int_R |u'|^2 dx.$$

□

3. An apriori bound and a comparison lemma

In this section, $a \in R$ is fixed. We write $M = R_+$.

As in the proof of Lemma 2.1 of [18] we have the following comparison lemma:

Lemma 5. *Assume Ω is a bounded open subset in M. Suppose $u_1, u_2 \in C^2(\Omega)$ are positive in Ω, and they satisfy $T_a(u_1) \leq 0$ and $T_a(u_2) \geq 0$ on Ω and $u_1 \geq u_2$ on $\partial\Omega$. Then $u_1 \geq u_2$ on Ω.*

We now use this result to obtain the local existence of a boundary blow-up positive solution of $(1)_a$ on a bounded interval $G \subset M - M_0$. This result may be of independent interest (see [18]).

Proposition 6. *Assume $a \in R$. There is at least one positive regular solution of the equation*

$$u'' + af(x)u - b(x)u^p = 0 \quad in \quad G,$$

with the boundary condition $u = +\infty$ on ∂G.

Proof. Fix an integer $k \in \{1, 2, 3, ...\}$. As in the introduction we let

$$I(u) = \frac{1}{2}\int_G (|u'|^2 - af(x)u^2) + \frac{1}{p+1}\int_G b(x)|u|^{p+1}$$

be defined on

$$E = \{u \in H^1(G); u|_{\partial G} = k\}.$$

It is easy to see that I is coercive on E (see section three for a more difficult case) and there is a minimizer, which is denoted by u_k, of I on E.

By the comparison lemma above we have $u_k \leq u_{k+1}$, for $k = 1, 2, ...$. By a standard argument we can assume $u_k \to U_a$ in $C^2_{loc}(G)$. This U_a is what we wanted. $\qquad\square$

From the proof of the result above we can get the following:

Theorem 7. *Assume that M_0 is non-empty and the set $M - M_0$ admits a bounded interval exhaustion, i.e., there is a bounded open subset sequence $\{\Omega_j\}$ such that $M - M_0 = \bigcup \Omega_j$. Then for any $a \in R$, there is at least one positive smooth solution $\bar{U}_a$ of $T_a(u) = 0$ on $M - M_0$ with the boundary blow-up condition $\bar{U}_a = +\infty$ on M_0.*

We remark that when M_0 is empty and $a \leq 0$, there may be no positive solutions of $T_a(u) = 0$ on M at all (see [18]). One should also note that there is no contradiction between Theorem 1(iii) with our Theorem 5 because we require $T_a(u) = 0$ on the whole R_+ in Theorem 1(iii).

4. Existence and uniqueness of positive solutions

We will discuss the uniqueness of positive E-solutions of $(1)_a$ for $a \in R$. The proofs of the following results are similar to the corresponding ones in [18], but for completeness, we give full proofs.

Lemma 8. *For $a \in R$, there is at most one positive E-solution to $(1)_a$.*

Proof. Assume u_1 and u_2 are two positive E-solutions of $(1)_a$. We assume $u_1 \neq u_2$, i.e., the set $M_1 := \{x \in M; u_1(x) \neq u_2(x)\}$ has positive measure.

Let $\epsilon_1 > \epsilon_2 > 0$ and denote

$$v_i = (u_i + \epsilon_i)^{-1}[(u_2 + \epsilon_2)^2 - (u_1 + \epsilon_1)^2]^+, \quad i = 1, 2,$$

where $[.]^+$ is the positive part of $[.]$. Then each v_i is an H-limit of a sequence in H with compact support. Hence, we have for $i = 1, 2$,

$$- \int [u_i' v_i' - a f u_i v_i] = \int b(x) u_i^p v_i.$$

By these we get

$$- \int [u_2' v_2' - u_1' v_1'] = \int a f(x)[u_1 v_1 - u_2 v_2] + \int b(x)(u_2^p v_2 - u_1^p v_1).$$

Note that the left side is

$$- \int (|u_2' - \frac{u_2 + \epsilon_2}{u_1 + \epsilon_1} u_1'|^2 + |u_1' - \frac{u_1 + \epsilon_1}{u_2 + \epsilon_2} u_2'|^2),$$

which is non-positive and goes to

$$- \int (|u_2' - \frac{u_2}{u_1} u_1'|^2 + |u_1' - \frac{u_1}{u_2} u_2'|^2) \le 0.$$

The first term on the right goes to zero as $\epsilon_1 \to 0$. The second term on the right goes to

$$\int b(x)(u_2^{p-1} - u_1^{p-1})(u_2^2 - u_1^2),$$

which is non-negative by our assumption. Hence, $M_1 \subset M_0$. By all these we get that

$$\int_{M_0} (|u_2' - \frac{u_2}{u_1} u_1'|^2 + |u_1' - \frac{u_1}{u_2} u_2'|^2) = 0$$

and

$$u_2 = c u_1, \quad \text{on} \quad M_0$$

for some constant $c \neq 1$, a contradiction. $\qquad\qquad\square$

In the rest of this section we assume that $a \in (\lambda_1, \lambda_*)$. We now prove part (i) of Theorem 1:

Proof. Recall that we assume M_0 is bounded. It is clear that the functional J is weakly lower semi-continuous on E according to our compactness result. We claim that

$$J(u) \to +\infty \quad \text{as} \quad ||u|| \to +\infty.$$

Assume not. Then we have the sequence $\{u_n\} \subset H$ such that $J(u_n)$ is bounded above and $||u_n|| \to \infty$. Then we have that $d_n^2 := \int f(x) u_n^2 \to \infty$; for otherwise, $\{||u_n||\}$ is bounded.

Set

$$\bar{u}_n = u_n / d_n.$$

Then we have $\int f(x) \bar{u}_n^2 = 1$ and

$$\limsup \int_M |\bar{u}_n'|^2 - a + \frac{2}{(p+1)||u_n||_2^2} \int_M b u_n^{p+1} \le 0.$$

By this, we may assume that $||\bar{u}_n|| + \int_M bu_n^{p+1}$ is uniformly bounded. Hence, we may assume further that $\bar{u}_n \to \bar{u}$ weakly in E, almost everywhere, and strongly in $L^2(R_+, p)$. Then by Fatou's lemma, we get

$$\int_{R_+} b(x)\bar{u}^{p+1} = 0.$$

Hence, $\bar{u} = 0$ on the set $R_+ - M_0$. Therefore, we can use the strong convergence on $L^2(R_+, p)$ to get $\int_{M_0} f(x)\bar{u}^2 = 1$ and

$$\int_{M_0} |\bar{u}'|^2 \leq a.$$

By this and the definition of λ_* we have

$$\lambda_* \leq a.$$

This is a contradiction with our hypothesis at the beginning.

Therefore, we have a minimizer u_0, which is a non-negative function, of the functional J on E. To prove that u_0 is non-trivial, we take a large interval I_R with $M_0 \subset I_R$ such that the first eigenvalue $\lambda_1(I_R) - a \leq -\epsilon$ for some small $\epsilon > 0$. Let ϕ_1 be the eigenfunction of the eigenvalue $\lambda_1(I_R)$ of the weighted quotient $Q(u, I_R)$ and $\int_{I_R} f(x)\phi^2 = 1$. Then, for small $t > 0$,

$$J(t\phi_1) = \frac{\lambda_1(I_R) - a}{2}t^2 + \frac{t^{p+1}}{p+1}\int_{I_R} b\phi^{p+1} < 0.$$

Note that $J(0) = 0$. Hence u_0 is non-trivial and $u_0 > 0$ everywhere on R_+ by the strong maximum principle. The smoothness of u_0 follows from the standard argument.

For the uniqueness of positive solutions, we use Lemma 8 above. Hence the proof is complete. □

Remark. By uniqueness of positive solutions and the barrier method, we know immediately that, for $x \in R_+$, $a \to u_a(x)$ is increasing (for a proof, one may see [18]). This fact will be used later.

5. BLOW-UP OF THE SOLUTION SEQUENCE

In this section we assume $\lambda_* < +\infty$.

We study the behavior of the solution sequence when $a_n \to \lambda_*^-$. We assume $a_n > \lambda_1$. Set $u_n = u_{a_n}$.

We now prove Theorem 1(ii). First we note that

$$lim_{n\to+\infty}\int_{R_+} f(x)u_n^2 = +\infty.$$

Otherwise, we can find a minimizer, which is the limit of u_n, of J as in Theorem 1(i), and this is impossible.

As in the proof of Theorem 1(i), we set

$$\bar{u}_n = u_n/d_n.$$

Then we find $\bar{u} \in H$ with $\bar{u} = 0$ on $R_+ - M_0$. Note that

$$\bar{u}_n'' + a_n f(x)\bar{u}_n = 0$$

on M_0. We can use the standard estimate [10] to conclude the uniform convergence of $\bar{u}_n$ to $\bar{u}$ on K. This implies that $u_n(x) \to +\infty$ on K.

6. Non-existence of positive solutions

We prove in this section the third part of Theorem 1. That is, under the assumption $\lambda_* < +\infty$, there is no positive solution of $(1)_a$ when $a \geq \lambda_*$.

Assume that $a = \lambda_*$ and there is a positive solution, say u, of $(1)_a$. Let v be the λ_*-eigenfunction on M_0 which satisfies the equation

$$u'' + af(x)u = 0 \quad \text{on} \quad M_0$$

and the boundary condition $v = 0$ on ∂M_0. Set $\phi = v/u$. Then by direct computation we find

$$\phi'' + 2\phi'(logu)' = 0 \quad \text{on} \quad M_0$$

and $\phi = 0$ on ∂M_0. However, this gives an absurd result that $\phi = 0$ on M_0 by the standard maximum principle (see Theorem 3.1 in [10]). Therefore, there is no positive solution in this case.

In the case where $a > \lambda_*$, if there is a positive E-solution of $(1)_a$, say u_*, then using the barrier method and the uniqueness result we find that $u_a(x) \leq u_*(x)$ for all $x \in M$ and $a < \lambda_*$. Therefore, there is no blow-up on M_0 for any sequence of (u_a), which contradicts Theorem 1(ii). Hence, we get the conclusion of Theorem 1(iii).

We can give another proof for positive E-solutions when $a > \lambda(M_0)$. Take a smooth bounded domain G inside M_0 such that $\lambda_1(G) < a$. Let w_1 be the positive eigenfunction associated to $\lambda_1(G)$. Therefore, we can find a function $u_0 \in E$ such that the number

$$\mu = \int_G (u_0'w_1' - af(x)u_0w_1)$$

is positive (see also [3]). Otherwise, we have

$$\int_{R_+} (u'w_1' - af(x)uw_1) \leq 0$$

for every $u \in H$. Hence, using $-u$ as a test function, we have

$$\int_{R_+} (u'w_1' - af(x)uw_1) = 0$$

for every $u \in H$. Therefore,

$$-w_1'' - af(x)w_1 = 0 \quad \text{on} \quad R_+.$$

It is absurd with the definition of w_1.

Let

$$u_t = tw_1 + u_0.$$

Then we compute $J(u_t)$ and find

$$J(u_t) = (\lambda_1(G) - a)t^2/2 + \mu t + d$$

where

$$d = \frac{1}{p+1} \int_{R_+} bu_0^{p+1} + \frac{1}{2} \int_{R_+} (|u_0'|^2 - af(x)u_0^2).$$

Hence, $J(u_t) \to -\infty$ as $t \to -\infty$, which contradicts the coercivity of J. So we conclude the result when $a > \lambda_*$.

7. The real line case

In this section we discuss the equations $(1)_a$ on the real line R. We point out that the method used here cannot work for the case when $b(x) = 0$ somewhere.

We assume that f is a bounded integrable function on R, and $b(x)$ is positive and uniformly bounded away from zero, that is, there is a uniform constant $c > 0$ such that

$$(e) \qquad\qquad b(x) \geq c > 0 \quad \text{for} \quad x \in R.$$

For example, $b(x) = 2 + sin(x)$ satisfies condition (e).

We assume $a \in (\lambda_1, \lambda_*)$.

Set $I_k = (-k, k)$ for $k = 1, 2, ...$ Then as before, we define the functional

$$J_k(u) = \frac{1}{2} \int_{I_k} (|u'|^2 - af(x)u^2) + \frac{1}{p+1} \int_{I_k} b(x)|u|^{p+1}$$

on $E_k := H_0^1(I_k)$. Clearly, the functional is bounded below (see [1]) on E_k. We can easily get a minimizer $u_k \in E_k$ such that

$$J_k(u_k) = inf_{u \in E_k} J_k(u),$$

and therefore we get a family of positive point-wise increasing functions (u_k), which satisfy the equations $(1)_a$ with the Dirichlet boundary condition $u_k(k) = 0 = u_k(-k)$.

Using the maximum principle we can easily show that u_k is uniformly bounded on R. In fact, at the maximum point $x_0 \in I_k$, we have that $u''(x_0) \leq 0$. Hence, by using our equation $(1)_a$ we find

$$af(x_0)u(x_0) = b(x_0)u(x_0)^p - u''(x_0) \geq cu(x_0)^p$$

and $u(x_0) \leq [aD/c]^{1/(p-1)}$, where $D = sup_{x \in R} f_+(x)$. According to our assumption on p, we know that u_k is uniformly bounded in $L^2(R, pdx)$.

Choose $\xi \in C_0^1(I_1)$ and we have

$$J_k(u_k) \leq J_1(\xi)$$

for every $k = 1, 2, ...$ Using this bound and the uniform boundedness of the sequence (u_k) we obtain a uniform bound of $\int_R |u_k'|^2 dx$. In fact, multiplying both sides of equation (1) for u_k by u_k and integrating by parts, we know that the integrals

$$\int_R u_k^{p+1} dx + \int_R |u_k'|^2 dx$$

are uniformly bounded. Therefore, we can pass to a subsequence, still denoted by (u_k), which converges in $C_{loc}^2(R)$ to a positive solution U of $(1)_a$ on R. It is easy to see that

$$\int_R |U'|^2 dx + \int_R U^{p+1} dx < +\infty$$

and U is in $L^2(R, pdx)$. Hence by a standard estimate (see [18]), we have proven that this U is in E'. That is to say, $U(x) \to 0$ uniformly as $|x| \to \infty$. As before, we can prove the uniqueness of the E'-solution.

We point out that for the case when $a \in (0, \lambda_1]$, we can easily show by the maximum principle that there is no positive E'-solution of $(1)_a$.

In summary, we have

Theorem 9. *Assume f is a bounded, positive somewhere, integrable function on R. Assume that the positive function $b(x)$ satisfies (e) above. Then*

(i) *for $a \in (0, \lambda_1]$ there is no positive E'-solution of $(1)_a$;*
(ii) *for any $a \in (\lambda_1, +\infty)$ there is a positive E'-solution of $(1)_a$, and this solution is unique in E'.*

We remark that some extension of Theorem 9 to higher dimensions has been obtained by Du and Ma [18].

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DEPARTMENT OF MATHEMATICAL SCIENCES, TSINGHUA UNIVERSITY, BEIJING 100084, PEOPLE'S REPUBLIC OF CHINA
  E-mail address: `lma@math.tsinghua.edu.cn`

DEPARTMENT OF MATHEMATICS, THE NATIONAL UNIVERSITY OF SINGAPORE, 10 KENT RIDGE CRESCENT, SINGAPORE 119260
  E-mail address: `matxuxw@math.nus.edu.sg`