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G-COINCIDENCES FOR MAPS OF HOMOTOPY SPHERES INTO CW-COMPLEXES

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ABSTRACT. Let G be a finite group acting freely in a CW-complex Σ^m which is a homotopy m-dimensional sphere and let $f : \Sigma^m \to Y$ be a map of Σ^m to a finite k-dimensional CW-complex Y. We show that if $m \geq |G|k$, then f has an (H, G)-coincidence for some nontrivial subgroup H of G.

1. INTRODUCTION

The classical Borsuk-Ulam Theorem has been generalized in many directions; see, for example, [14], [4, Ch. 2, §34] and [10]. More recently (see [7], [8], [9], and [10]), free actions of the cyclic group $\mathbf{Z}_r$ on S^m and certain types of coincidences under maps $f : S^m \to Y$, where Y is a finite-dimensional polyhedron, were studied. It was shown that under certain dimension hypothesis there is at least one (p, r)-coincidence (see below for the definition). In the present work we study a similar problem for an arbitary finite group G which acts freely in a homotopy sphere. It turns out that results similar to those of [7], [8] and [9] remain true in this more general situation. There are many examples of finite groups, besides the cyclic ones, which act freely in homotopy spheres. Suppose that G is a finite group which acts freely in some homotopy sphere Σ^m of dimension m. By [3, Ch. XVI, §9] such groups have periodic cohomology. Further, it was proved in [13, Theorem A, page 267] that if s is the period, then G acts freely on a finite simplicial complex which has the type of homotopy of a sphere of dimension $ds - 1$, where d is the greatest common divisor of $|G|$ and $\phi(|G|)$, $\phi(|G|)$ being the Euler function. Thus each group which has periodic cohomology will provide an example of a free action on a homotopy sphere which is a finite CW-complex. If we do not require the complex to be finite, such groups can act in a complex of the homotopy type of a sphere of dimension $s - 1$ [13, Proposition 4.4, page 277]. There also exist several different actions of G in a fixed homotopy sphere. Such actions can be classified by the number of homotopy types of the orbit spaces. For more details see [6].

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If Σ is a free G-space and H is a subgroup of G, then H acts on the right on each orbit Gx of G as follows: if $y \in Gx$ and $y = gx, g \in G$, then $hy := ghx$. Following [7] and [9], the concept of G-coincidence can be generalized as follows.

Definition. Suppose that Σ is a G-space and H is a subgroup of G. We say that a map $f : \Sigma \to Y$ has a (H, G)-coincidence if there is a point $x \in \Sigma$ such that f sends every orbit of the action of H on the G-orbit of x to a single point.

Of course, if H is the trivial subgroup, then every point of Σ is an (H, G)-coincidence. If $H = G$ this is the usual definition of coincidence. If $G = \mathbf{Z}_r$ and $h = \mathbf{Z}_p$, then an (H, G)-coincidence is a (p, r)-coincidence in the sense of [9]. The purpose of this paper is to prove the following theorem.

Theorem. *Suppose that G is a finite group acting freely on a CW-complex Σ^m which is a homotopy m-dimensional sphere and suppose that $f : \Sigma^m \to Y$ is a map. Suppose one of the following conditions holds:*

a) *Σ^m is a finite m-dimensional CW-complex and Y a k-dimensional CW-complex.*

b) *Y is a finite k-dimensional CW-complex.*

Then, if $m \geq |G|k$, there is a nontrivial subgroup $H \subset G$ and an (H, G)-coincidence for f.

If $G = \mathbf{Z}_2$ this result is known and it is true for every m. In this case the coincidence problem has been completely solved in [8] and [9] (compare also [11]). Furthermore, such a result is proved in [7] and [9] for cyclic groups $\mathbf{Z}_q$ acting on S^{2n+1}. We will restrict ourselves to finite G-CW-complexes which have the homotopy type of an odd-dimensional sphere S^{2n+1}.

This paper is divided into two sections besides the Introduction. In section 2 we prove some results about G and the orbit space of an action of G in a homotopy sphere. In section 3 we prove the main result. It would be interesting to know if an (H, G)-coincidence exists for every nontrivial cyclic subgroup $H \subset G$ of a prime order. Our main result asserts only the existence of such an (H, G)-coincidence for some subgroup $H \subset G$.

2. Background

From now on we will assume that m is odd, $m = 2n + 1$, Σ^{2n+1} is a $(2n + 1)$-dimensional CW-complex (not necessarily finite) which has the homotopy type of the sphere S^{2n+1}, and G is a finite group which acts freely in Σ^{2n+1}. We denote by $B(G)$ the classifying space of G.

Lemma 1. *Let p be a prime which divides the order of the group G. Then the cohomology $H^i(B(G), \mathbf{Z}_p)$ contains $\mathbf{Z}_p$ as a summand for $i = 2n + 1,\ 2n + 2$.*

Proof. By [3, Ch. XII, §11], $H^{2n+2}(G, \mathbf{Z}) \approx \mathbf{Z}_{|G|}$, the cyclic group of order $|G|$. Since G is a finite group, either the $\mathbf{Z}$-cohomology or the $\mathbf{Z}$-homology of $B(G)$ has only torsion in dimensions greater than zero. By the universal coefficient formula, $\mathbf{Z}_{|G|} \cong H^{2n+2}(B(G), \mathbf{Z}) =$ free part of $(H_{2n+2}(B(G), \mathbf{Z})) \oplus$ torsion $(H_{2n+1}(B(G), \mathbf{Z})) = H_{2n+1}(B(G), \mathbf{Z})$. By the universal coefficient theorem in dimensions $2n + 1$ and $2n + 2$ and the fact that p divides $|G|$, $H^{2n+2}(B(G), \mathbf{Z}_p)$ has a direct summand which is $Ext(\mathbf{Z}_{|G|}, \mathbf{Z}_p) \cong \mathbf{Z}_p$, and $H^{2n+1}(B(G), \mathbf{Z}_p)$ has a direct summand which is $Hom(\mathbf{Z}_{|G|}, \mathbf{Z}_p) = \mathbf{Z}_p$. Hence the result follows. □

Let $\pi : \Sigma^{2n+1} \to \Sigma^{2n+1}/G$ be the quotient map and let $c : \Sigma^{2n+1}/G \to B(G)$ be a classifying map for the G-principal bundle $\pi : \Sigma^{2n+1} \to \Sigma^{2n+1}/G$.

Lemma 2. *Let p be a prime which divides $|G|$. Then the homomorphism $c^* : H^{2n+1}(B(G), \mathbf{Z}_p) \to H^{2n+1}(\Sigma^{2n+1}/G, \mathbf{Z}_p)$ is nontrivial.*

Proof. Since $H^{2n+1}(\Sigma^{2n+1}/G, \mathbf{Z}_p) \neq 0$, it suffices to show that $c^* : H^{2n+1}(B(G), \mathbf{Z}_p) \to H^{2n+1}(\Sigma^{2n+1}/G, \mathbf{Z}_p)$ is surjective.

Consider the G-universal bundle $E(G) \to B(G)$. By [5, Ch. III, §4, page 208] we have the bundle $\Sigma^{2n+1} \to E(G) \times_G \Sigma^{2n+1} \xrightarrow{\rho} B(G)$ with base space $B(G)$ and fiber Σ^{2n+1} (see also [1, Ch. IV, §6, Lemma 6.2, page 146]). Since G is a finite group and acts freely on Σ^{2n+1}, then $E(G) \times_G \Sigma^{2n+1}$ is homotopy equivalent to Σ^{2n+1}/G. If $t : \Sigma^{2n+1}/G \to E(G) \times_G \Sigma^{2n+1}$ is a homotopy equivalence, $\rho t : \Sigma^{2n+1}/G \to B(G)$ also classifies the G-principal bundle $\pi : \Sigma^{2n+1} \to \Sigma^{2n+1}/G$, and so it is homotopic to c. Hence it suffices to show that

$$\rho^* : H^{2n+1}(B(G), \mathbf{Z}_p) \to H^{2n+1}(E(G) \times_G \Sigma^{2n+1}, \mathbf{Z}_p)$$

is surjective. To do this, consider the generalized Gysin cohomology sequence associated to the fibration $\rho : E(G) \times_G \Sigma^{2n+1} \to B(G)$ (see [12, Ch. 9, §5, Theorem 2]):

$$H^{2n+1}(B(G), \mathbf{Z}_p) \xrightarrow{\rho^*} H^{2n+1}(E(G) \times_G \Sigma^{2n+1}, \mathbf{Z}_p) \to H^0(B(G), \mathbf{Z}_p)$$
$$\xrightarrow{\Psi} H^{2n+2}(B(G), \mathbf{Z}_p) \xrightarrow{\rho^*} H^{2n+2}(E(G) \times_G \Sigma^{2n+1}, \mathbf{Z}_p).$$

Since $H^{2n+2}(E(G) \times_G \Sigma^{2n+1}, \mathbf{Z}_p) = 0$, Ψ is surjective; but $H^0(B(G), \mathbf{Z}_p) \cong \mathbf{Z}_p$, and by Lemma 2.1 $H^{2n+2}(B(G), \mathbf{Z}_p) \neq 0$, hence the only possibility is that $H^{2n+2}(B(G), \mathbf{Z}_p) = \mathbf{Z}_p$, which implies that Ψ is an isomorphism. Thus ρ^* is surjective and the fact is proved. □

3. Proof of the main result

Let $G = \{g_1, ..., g_r\}$ be a fixed enumeration of elements of G, where r is the order of G. We construct a map $G \times Y^r \to Y^r$, where $Y^r = Y \times ... \times Y$ is the r-fold product, as follows. For each $g \in G$ and $(y_1, ..., y_r) \in Y^r$, let $g.(y_1, ..., y_r) = (y_{\sigma_g(1)}, ..., y_{\sigma_g(r)})$, where the permutation σ_g is defined by $g_i g = g_{\sigma_g(i)}$. It is straightforward to verify that the above map is a left G-action on Y^r. For a subgroup $H \subset G$, let $(Y^r)^H$ be the fixed point set of H and $F = \bigcup_H (Y^r)^H$, where H runs over all nontrivial subgroups of G. Let $Y_0^{(r)} := Y^r - F$; it is precisely the part of Y^r where the G-action is free. If X is any space with a G-action, then a map $f : X \to Y$ induces an equivariant map $\phi : X \to Y^r$, $\phi(x) = (f(g_1 x), ..., f(g_r x))$.

Suppose that $f : \Sigma^{2n+1} \to Y$ has no (H, G)-coincidence points for any nontrivial subgroup $H \subset G$. Then $\phi(\Sigma^{2n+1}) \subset Y_0^{(r)}$, so ϕ factors through $Y_0^{(r)}$. Let $\phi_0 : \Sigma^{2n+1} \to Y_0^{(r)}$ be this factorization; it is an equivariant map.

Let $\gamma : Y_0^{(r)} \to Y_0^{(r)}/G$ be the quotient map and $\overline{\phi}_0 : \Sigma^{2n+1}/G \to Y_0^{(r)}/G$ be the map induced by ϕ_0. Let $c_Y : Y_0^{(r)}/G \to B(G)$ be a classifying map for the G-principal bundle $\gamma : Y_0^{(r)} \to Y_0^{(r)}/G$. Then $c = c_Y \overline{\phi}_0 : \Sigma^{2n+1}/G \to B(G)$ is a classifying map for the G-principal bundle $\pi : \Sigma^{2n+1} \to \Sigma^{2n+1}/G$ of the previous section.

First, if $2n+1 > rk$, then the topological dimension of $Y_0^{(r)}/G$ is less than $2n+1$, hence $c_Y^* : H^{2n+1}(B(G), \mathbf{Z}_p) \to H^{2n+1}(Y_0^{(r)}/G, \mathbf{Z}_p)$ is zero and $c^* = \overline{\phi}_0^* c_Y^* = 0$ which contradicts Lemma 2.2. Thus we can assume that $2n+1 = rk$. We will again obtain a contradiction to Lemma 2.2 by showing that $\overline{\phi}_0^* : H^{2n+1}(Y_0^{(r)}/G, \mathbf{Z}_p) \to H^{2n+1}(\Sigma^{2n+1}/G, \mathbf{Z}_p)$ is zero. First, if Σ^{2n+1} is a finite CW-complex, then $f(\Sigma^{2n+1})$ is compact and so it is contained in a finite subcomplex of Y. Thus without loss of generality we can assume that Y is a finite CW-complex.

The map $i^* : H^{2n+1}(Y^r, \mathbf{Z}_p) \to H^{2n+1}(Y_0^{(r)}, \mathbf{Z}_p)$ is a part of the cohomology sequence of the pair $(Y^r, Y_0^{(r)})$ and $H^{2n+1}(Y^r, Y_0^{(r)}) = 0$, so i^* is surjective. On the other hand, by the Künneth formula, any class of $H^{2n+1}(Y^r, \mathbf{Z}_p)$ is a cup product of classes of lower dimension, and $H^s(\Sigma^{2n+1}, \mathbf{Z}_p) = 0$ for $0 < s < 2n+1$ which implies that $\phi^* : H^{2n+1}(Y^r, \mathbf{Z}_p) \to H^{2n+1}(\Sigma^{2n+1}, \mathbf{Z}_p)$ is zero. Since $\phi_0 i = \phi$, it follows that $\phi_0^* : H^{2n+1}(Y_0^{(r)}, \mathbf{Z}_p) \to H^{2n+1}(\Sigma^{2n+1}, \mathbf{Z}_p)$ is zero. Now since $\pi : \Sigma^{2n+1} \to \Sigma^{2n+1}/G$ and $\sigma : Y_0^{(r)} \to Y_0^{(r)}/G$ are covering projections, there are transfer homomorphisms $\tau : H^{2n+1}(\Sigma^{2n+1}, \mathbf{Z}_p) \to H^{2n+1}(\Sigma^{2n+1}/G, \mathbf{Z}_p)$ and $\tau_0 : H^{2n+1}(Y_0^{(r)}, \mathbf{Z}_p) \to H^{2n+1}(Y_0^{(r)}/G, \mathbf{Z}_p)$ [2, Chapter III, Section 2, page 118]. Further we have the following commutative rectangle:

$$\begin{array}{ccccc} H^{2n+1}(\Sigma^{2n+1}, \mathbf{Z}_p) & \xrightarrow{\tau} & H^{2n+1}(\Sigma^{2n+1}/G, \mathbf{Z}_p) & & \\ \uparrow{\scriptstyle \phi_0^*} & & \uparrow{\scriptstyle \overline{\phi}_0^*} & & \\ H^{2n+1}(Y_0^{(r)}, \mathbf{Z}_p) & \xrightarrow{\tau_0} & H^{2n+1}(Y_0^{(r)}/G, \mathbf{Z}_p) & \longrightarrow & 0 \end{array}$$

and $\tau\phi_0^* = \overline{\phi}_0^*\tau_0$ implies that $\overline{\phi}_0^*\tau_0 : H^{2n+1}(Y_0^{(r)}, \mathbf{Z}_p) \to H^{2n+1}(\Sigma^{2n+1}/G, \mathbf{Z}_p)$ is zero.

Since $Y_0^{(r)}$ is a $(2n+1)$-dimensional CW-complex, the transfer homomorphism $\tau_0 : H^{2n+1}(Y_0^{(r)}, \mathbf{Z}_p) \to H^{2n+1}(Y_0^{(r)}/G, \mathbf{Z}_p)$ is surjective. This implies that $(\overline{\phi}_0)^* : H^{2n+1}(Y_0^{(r)}/G, \mathbf{Z}_p) \to H^{2n+1}(\Sigma^{2n+1}/G, \mathbf{Z}_p)$ is the zero homomorphism, and the result follows.

Remark. The result of the Theorem is the best possible if all finite groups are considered. In fact, for each $k > 1$ and $m < 2k$, an example of a k-dimensional polyhedron Y^k and a map $f : S^r \to Y^k$ without antipodal coincidences was constructed in [8] (compare also [11]). It is an interesting question whether a similar example can be constructed for $G = \mathbf{Z}_r$, $r > 2$.

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