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GEOMETRIC CHARACTERIZATIONS OF SOME CLASSES OF OPERATORS IN C*-ALGEBRAS AND VON NEUMANN ALGEBRAS

CHARLES AKEMANN AND NIK WEAVER

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Dedicated to Richard V. Kadison on his 75th birthday

Abstract. We present geometric characterizations of the partial isometries, unitaries, and invertible operators in C*-algebras and von Neumann algebras.

Let $\mathcal{A}$ be a unital C*-algebra. A result due to Roger Smith, given as Lemma 4.5 of [BEZ], allows one in some sense to characterize the unitaries in $\mathcal{A}$ in terms of norms of 2×1 matrices over $\mathcal{A}$. This led David Blecher to ask us whether the unitaries can be recovered from the Banach space structure of $\mathcal{A}$ alone, without recourse to the product and the adjoint operations.

One cannot expect to recover the entire C*-algebra structure of $\mathcal{A}$ from its Banach space structure. Indeed, the identity map is always an isometry from $\mathcal{A}$ onto its opposite algebra, but there exist C*-algebras $\mathcal{A}$ for which $\mathcal{A}$ and $\mathcal{A}^{op}$ are not $*$-isomorphic. (See, e.g., [P].) Thus, linearly isometric C*-algebras need not be $*$-isomorphic.

However, Kadison proved in [K, Theorem 7] that any surjective linear isometry between unitary C*-algebras can be written as a Jordan $*$-isomomorphism followed by left multiplication by a unitary in the range. Paterson and Sinclair [PS] extended this result to nonunital C*-algebras with the unitary coming from the multiplier algebra of the range algebra. Kadison notes in [K] that any surjective linear isometry between unital C*-algebras takes unitaries to unitaries. Thus, the unitaries of a C*-algebra are a Banach space invariant, which answers Blecher's question but does not give a usable criterion for determining which elements are unitary. By Kadison's result, one can describe the set of unitaries of $\mathcal{A}$ as the orbit of the identity (or any other unitary) under the group of isometries of $\mathcal{A}$ onto itself. We asked, "Can we get a direct characterization of unitaries without starting with a unitary?"

We found that we could give a fairly simple characterization of the unitaries of $\mathcal{A}$ by working in the dual Banach space $\mathcal{A}^*$. Since only Banach space structure is used, it is a "geometric" characterization. This led to the question of whether other standard classes of operators can be similarly characterized. We were able to do this for invertible operators and for partial isometries.

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Results of this type are not possible for other standard classes of operators. For example, since left multiplication by any unitary of $\mathcal{A}$ is an isometry from $\mathcal{A}$ onto itself, it follows that in general the classes of self-adjoint, normal, and projection operators in $\mathcal{A}$ are not stable under isometries of $\mathcal{A}$. (These classes are not stable under left multiplication by unitaries, in general.) Thus, there can be no geometric characterizations of these classes in the above sense. In contrast, the class of isometries in $\mathcal{A}$ is stable under multiplication by any unitary; however, the identity map from $\mathcal{A}$ to $\mathcal{A}^{op}$, which also preserves Banach space structure, takes isometries in $\mathcal{A}$ to co-isometries in $\mathcal{A}^{op}$, so here too a geometric characterization is impossible.

We proceed to our results. We treat partial isometries first, as our technique in this case is slightly more elementary.

Theorem 1. *Let $\mathcal{A}$ be a C*-algebra and let $x \in \mathcal{A}$, $\|x\| = 1$. Then x is a partial isometry if and only if*

$$\{y \in \mathcal{A} : \text{ there exists } a > 0 \text{ with } \|x + ay\| = \|x - ay\| = 1\}$$
$$= \{y \in \mathcal{A} : \|x + by\| = \max(1, \|by\|) \text{ for all } b \in \mathbf{C}\}.$$

Proof. Let $X_1 = \{y \in \mathcal{A} : \text{there exists } a > 0 \text{ with } \|x + ay\| = \|x - ay\| = 1\}$ and $X_2 = \{y \in \mathcal{A} : \|x + by\| = \max(1, \|by\|) \text{ for all } b \in \mathbf{C}\}$. Suppose $y \in X_2$, $y \neq 0$. Then taking $b = \pm\|y\|^{-1}$, the definition of X_2 yields $\|x + ay\| = \|x - ay\| = 1$ for $a = \|y\|^{-1}$. This shows that $X_2 \subset X_1$.

For the reverse containment, let $\mathcal{A}$ be faithfully represented on a Hilbert space H and suppose x is a partial isometry in $\mathcal{A}$. Let $p = x^*x$ and $q = xx^*$ be the right and left support projections of x, and let $y \in X_1$. For any unit vector $\xi \in \mathrm{ran}(p)$ we have $\|x(\xi)\| = 1$; so if $y(\xi) \neq 0$, then

$$\max(\|x(\xi) + ay(\xi)\|, \|x(\xi) - ay(\xi)\|) > 1$$

for all $a > 0$ (because the unit ball of any Hilbert space is strictly convex). This contradicts the assumption $y \in X_1$, so we must have $y(\xi) = 0$. Thus, $yp = 0$, or equivalently, $y = y(1 - p)$. Applying the same argument to x^* and y^* yields $y = (1 - q)y$, so we have $y = (1 - q)y(1 - p)$. Since $x = qxp$ by [KR, 6.1.1], it follows that $x^*y = 0 = y^*x$. Hence

$$\|x + by\|^2 = \|(x + by)^*(x + by)\| = \|x^*x + |b|^2y^*y\|$$
$$= \|px^*xp + |b|^2(1 - p)y^*y(1 - p)\|.$$

Thus

$$\|x + by\| = \max(\|x\|, \|by\|) = \max(1, \|by\|)$$

for all $b \in \mathbf{C}$, and we have shown $X_1 \subset X_2$. So if x is a partial isometry, then $X_1 = X_2$.

Now suppose x is not a partial isometry. We shall construct $y \in X_1 \setminus X_2$. Retain the faithful representation of $\mathcal{A}$ on H and let $x = |x|u$ be the polar decomposition of x [KR, 6.1.2]. (The absolute value $|x|$ of x lies in $\mathcal{A}$, but partial isometry u will generally lie in the von Neumann algebra $B(H)$ of all bounded linear operators on H.) Then $0 \leq |x| \leq 1$ and $|x|$ is not a projection. Let t be a point in the spectrum of $|x|$ which is neither 0 nor 1 and let $f : [0, 1] \to [0, 1]$ be a continuous function such that $f(0) = f(1) = 0$, $f(t) > 0$, and $f(s) \leq s^{-1} - 1$ for all $s \in (0, 1)$. Then define $y = f(|x|)x$. Observe that $y \in \mathcal{A}$. Also, since u is unitary,

$$\|x \pm y\| = \|(1 \pm f(|x|))|x|u\| = \|(1 \pm f(|x|))|x|\|.$$

By routine spectral theory arguments [T, pp. 17-21], using the facts that $1 \in \mathrm{spec}(|x|)$ and $f(1) = 0$, we get $0 \leq (1 \pm f(|x|))|x| \leq 1$, so $\|x \pm y\| \leq 1$, hence $1 \in \mathrm{spec}((1 \pm f(|x|))|x|)$. Thus $\|x \pm y\| = 1$ and so $y \in X_1$.

However, $y \notin X_2$. To see this let $b = \|y\|^{-1} = \|f(|x|)|x|\|^{-1} = \|f(s)s\|_\infty^{-1}$ (again using [T, pp. 17-21], where $\|\cdot\|_\infty$ denotes the supremum norm over $s \in [0,1]$). Then

$$\|x + by\| = \|(1 + bf(|x|))|x|\| = \|(1 + bf(s))s\|_\infty > 1$$

since $\|bf(s)s\|_\infty = 1$. This contradicts the equation $\|x + by\| = \max(1, \|by\|)$, so that $y \notin X_2$. So if x is not a partial isometry, then $X_1 \neq X_2$. □

It might be worth noting that $x \in \mathcal{A}$, $\|x\| = 1$, is an extreme point of ball($\mathcal{A}$) (the unit ball of $\mathcal{A}$) if and only if $X_1 = \{0\}$. The forward direction is an immediate consequence of the definition of extreme points. For the reverse direction, suppose x is not an extreme point, and find a nonzero $y \in \mathcal{A}$ such that $x \pm y \in$ ball($\mathcal{A}$). Then

$$1 = \|x\| \leq \frac{1}{2}(\|x + y\| + \|x - y\|) \leq 1,$$

so that $\|x \pm y\| = 1$, and hence $y \in X_1$.

Now if $\mathcal{A}$ is a C*-algebra and $x \in \mathcal{A}$, $\|x\| = 1$, write $S_x = \{f \in \mathcal{A}^* : f(x) = \|f\| = 1\}$. The notation comes from the fact that, if x is the unit of $\mathcal{A}$, then S_x is the set of states. If $\mathcal{M}$ is a von Neumann algebra, it is more natural and convenient to work in the predual $\mathcal{M}_*$; thus for $x \in \mathcal{M}, \|x\| = 1$, we write $S^x = \{f \in M_* : f(x) = \|f\| = 1\}$.

Theorem 2. *Let $\mathcal{A}$ be a C*-algebra and let $x \in \mathcal{A}$, $\|x\| = 1$. Then x is unitary if and only if S_x spans $\mathcal{A}^*$.*

Proof. Suppose x is unitary (and hence $\mathcal{A}$ is unital) and consider the map $T : \mathcal{A} \to \mathcal{A}$ given by $T(y) = xy$. This map is a bijective isometry, and hence so is the adjoint map $T^* : \mathcal{A}^* \to \mathcal{A}^*$. We have $T^*f(y) = f(xy)$ for all $y \in \mathcal{A}$ and $f \in \mathcal{A}^*$. It follows that $T^*(S_x) = S_1$, where 1 is the unit of $\mathcal{A}$ and S_1 is the set of states on $\mathcal{A}$. Since the states span $\mathcal{A}^*$ [T, p. 120], and T^* is a linear isomorphism, it follows that S_x spans $\mathcal{A}^*$.

Conversely, suppose S_x spans $\mathcal{A}^*$. This implies that x must be an extreme point of ball($\mathcal{A}$) by the following argument. If x is not extreme, then there exists a nonzero $y \in \mathcal{A}$ such that $x \pm y \in$ ball($\mathcal{A}$), and then every $f \in S_x$ must satisfy $|f(x \pm y)| \leq 1$, which together with $f(x) = 1$ implies $f(y) = 0$. So S_x cannot span $\mathcal{A}^*$, contradicting our hypothesis. Thus x is an extreme point as claimed, and therefore $\mathcal{A}$ is unital by [S, Theorem 1.6.1]. By [K, Theorem 1] the extreme points of the unit ball of a unital C*-algebra are partial isometries, so we conclude that x is a partial isometry.

Suppose that x is not unitary. Then either x^*x or xx^* is not the identity, and (since the cases are almost identical) we may assume WLOG that $p = 1 - x^*x \neq 0$. We claim that any $f \in S_x$ satisfies $f(p) = 0$, which implies that S_x does not span $\mathcal{A}^*$. Thus, let $f \in S_x$ and suppose $f(p) \neq 0$. Fix $|a| = 1$ such that $r = f(ap) > 0$, and for t real consider the element $x + atp$. Using the C*-norm condition and the facts that p is a projection (hence has norm 1), that x and p have orthogonal right supports, and that $\|x\| = |a| = 1$, we have

$$\|x + atp\|^2 = \|x^*x + |t|^2 p\| \leq 1 + t^2.$$

By definition of S_x and the choice of a, we also have

$$|f(x + atp)|^2 = (1 + rt)^2 = 1 + 2rt + r^2t^2.$$

Since $\|f\| = 1$, we get $|f(x + atp)| \leq \|x + atp\|$, and this implies

$$1 + 2rt + r^2t^2 \leq 1 + t^2$$

for all real t, which is falsified by $t = r/(1 - r^2)$ (unless $r = 1$, when it is falsified by $t = 1$). This contradiction establishes that $f(p) = 0$, as claimed. □

Theorem 3. *Let $\mathcal{M}$ be a von Neumann algebra and let $x \in \mathcal{M}$, $\|x\| = 1$. Then x is unitary if and only if S^x spans $\mathcal{M}_*$.*

Proof. The proof is the same as the proof of Theorem 2 except that we work in the predual of $\mathcal{M}$ instead of the dual of $\mathcal{A}$. □

The characterization of invertible elements requires polar decomposition, hence it is natural to work in a von Neumann algebra $\mathcal{M}$ at first. We shall use the characterization of unitaries in $\mathcal{M}$ given by Theorem 3 in our characterization of invertible elements of $\mathcal{M}$.

Theorem 4. *Let $\mathcal{M}$ be a von Neumann algebra. Then $x \in \mathcal{M}$ is invertible if and only if there exists a unitary $u \in \mathcal{M}$ and an $\epsilon > 0$ such that $f(x) \geq \epsilon$ for all $f \in S^u$.*

Proof. Suppose x is invertible in $\mathcal{M}$. Let $x = |x|u$ be its polar decomposition [KR, p. 401]. Since x is invertible and u is a partial isometry, u must be unitary. Thus $|x| = xu^*$, so $|x|$ is positive and invertible. By spectral theory there is an $\epsilon > 0$ such that $|x| \geq \epsilon \cdot 1$. Now for any $f \in S^u$ the map $y \mapsto f(yu)$ is a state in $\mathcal{M}_*$ [S, Props. 1.5.2 and 1.7.8], so $f(x) = f(|x|u) \geq \epsilon$. This proves the forward direction.

Conversely, suppose there is a unitary $u \in \mathcal{M}_*$ and an $\epsilon > 0$ such that $f(x) \geq \epsilon$ for all $f \in S^u$. For any state $g \in \mathcal{M}_*$ the map $y \mapsto g(yu^*)$ belongs to S^u [S, Props. 1.5.2 and 1.7.8], so by hypothesis $g(xu^*) \geq \epsilon$. Thus $xu^* \geq \epsilon$ by spectral theory, which implies that xu^* is invertible, hence x is invertible. □

The characterization of invertible elements in a C*-algebra $\mathcal{A}$ requires that we have handy a von Neumann algebra so that we can take polar decompositions. This is intrinsically arranged by embedding $\mathcal{A}$ in its double dual $\mathcal{A}^{**}$, which has structure as a von Neumann algebra, and the canonical embedding of $\mathcal{A}$ into $\mathcal{A}^{**}$ is a $*$-isomorphism onto a weak* dense C*-subalgebra [T, p. 122]. Since the predual of $\mathcal{A}^{**}$ is, of course, $\mathcal{A}^*$, the characterization of unitaries in $\mathcal{A}^{**}$ given by Theorem 3 is intrinsic to $\mathcal{A}$. The next corollary is stated in somewhat greater generality.

Corollary 5. *Let $\mathcal{A}$ be a weak* dense C*-subalgebra of the von Neumann algebra $\mathcal{M}$, and let $x \in \mathcal{A}$. Then x is invertible in $\mathcal{A}$ if and only if x is invertible in $\mathcal{M}$ if and only if there exists a unitary $u \in \mathcal{M}$ and an $\epsilon > 0$ such that $f(x) \geq \epsilon$ for all $f \in S^u$.*

Proof. This follows immediately from Theorem 4 using the weak* continuity of multiplication in $\mathcal{M}$ [S, Prop. 1.7.8]. □

Now let's add a little more information and see where it leads. Suppose that we know $\mathcal{A}$ is a unital C*-algebra and we know which element is the unit of $\mathcal{A}$. As shown in [L, proof of Theorem 21], an element x of $\mathcal{A}$ is self-adjoint if and only if $||1 + i\alpha x|| \leq 1 + o(\alpha), \alpha \in \mathbf{R}, \alpha \to 0$. If the unit is assumed to be known, Lumer's criterion is geometric. We can also characterize self-adjointness via the condition $f(x) \in \mathbf{R}$ for all $f \in S_1$. Moreover, the involution in $\mathcal{A}$ can be recovered from the fact that $(x+iy)^* = x-iy$ when x and y are self-adjoint. In the same spirit we note the following characterizations of positive operators and projections as a sample of the additional power obtained by identifying the unit element.

Proposition 6. *Let $\mathcal{A}$ be a C*-algebra with identified unit element 1 and let $x \in \mathcal{A}$. The following are equivalent.*

1. *x is positive.*
2. *$f(x) \geq 0$ for all $f \in S_1$.*
3. *x is self-adjoint and $\| \|x\| \cdot 1 - x\| \leq \|x\|$.*

Proposition 7. *Let $\mathcal{A}$ be a C*-algebra with identified unit element 1 and let $x \in \mathcal{A}$. The following are equivalent.*

1. *x is a projection.*
2. *x is a partial isometry and $x \geq 0$.*
3. *There is a self-adjoint unitary v in $\mathcal{A}$ such that $x = (1+v)/2$.*

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Department of Mathematics, University of California, Santa Barbara, California 93106

E-mail address: akemann@math.ucsb.edu

Department of Mathematics, Washington University, St. Louis, Missouri 63130

E-mail address: nweaver@math.wustl.edu