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A COVERING THEOREM WITH APPLICATIONS

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ABSTRACT. We prove a Covering Theorem that allows us to prove modified norm inequalities for general maximal operators. We will also give applications to convergence of a sequence of linear operators and the differentiation of the integral.

1. INTRODUCTION

Let $\mu \geq 0, \nu \geq 0$ be two Borel measures on $\mathbb{R}^n$. For $f : \mathbb{R}^n \to \mathbb{R}$ and $E \subset \mathbb{R}^n$ let $A_{E;\mu,\nu} f = 0$ if $\mu(E) = 0$, and $A_{E;\mu,\nu} f = \dfrac{1}{\mu(E)} \displaystyle\int_E f d\nu$ if $\mu(E) > 0$. The Hardy-Littlewood maximal function with respect to μ, ν is

$$M_{\mu,\nu} f(x) = \sup A_{Q;\mu,\nu}(|f|), \tag{1}$$

where the sup is extended over all cubes $Q \subset \mathbb{R}^n$ containing x. The important weak-type inequality

$$\mu\{x : M_{\mu,\nu} f(x) > y\} \leq \frac{c}{y} \int_{\mathbb{R}^n} |f| d\nu \tag{2}$$

holds if $n = 1$, and, if $n > 1$, it holds if in (1) $x \in Q$ means x is the center of Q [4, p.44]. However (2) is not true in general even if $d\mu = d\nu = u(x)dx$—in which case we write $M_u = M_{\mu,\nu}$—and the reader is referred to [1], [2], [3] for examples.

In this note we show that a modification of (2) is valid: if $u \geq 0$ is in $L^1(\mathbb{R}^n)$ and $d\mu = u(x)dx$, then (2) holds for all $y > 0$ for which the left side of (2) is $\geq \epsilon > 0$ with a constant $c = c_{\epsilon,u}$. This will follow from a Covering Theorem which will be taken up in section 2. In section 3 we apply the Covering Theorem to get substitute norm inequalities for $M_{\mu,\nu}$, and in section 4 we give an application to convergence of a sequence of linear operators $T_j f$: we will show that the weak-type inequality for $T_* f(x) = \sup |T_j f(x)|$ in the hypothesis of the theorems for convergence of $\{T_j f(x)\}$ (see [5, p.60]) can be replaced by a condition which allows cases where T_* is not weak-type. In the final section 5 we examine some problems in the differentiation of the integral.

2. COVERING THEOREM

This section is devoted to the Covering Theorem which will be needed in the proof of our main results. If $u : \mathbb{R}^n \to \mathbb{R}_+$ we use the notation $u(E) = \int_E u$ for the u-measure of E.

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Theorem 1. *Let* $u \geq 0$ *be in* $L^1(\mathbb{R}^n)$ *and let* $\epsilon > 0$ *be given. Then there exists* $0 < c_{\epsilon,u} < \infty$ *such that for every* $E \subset \mathbb{R}^n$ *with* $u(E) \geq \epsilon$ *the following holds: if* $\sigma = \{Q_\alpha\}$ *is a cover of* E *by cubes* Q_α, $E \subset \bigcup Q_\alpha$, *then there exists* $\{Q_j\} \subset \sigma, Q_i \cap Q_j = \emptyset, i \neq j$, *such that*

$$u(E) \leq c_{\epsilon,u} \sum u(Q_j).$$

Proof. Deny! Then there exists $\epsilon_0 > 0$ such that for every $k \in \mathbb{N}$ there is $E_k \subset \mathbb{R}^n$ with $u(E_k) \geq \epsilon_0$ and there exists $\sigma_k = \{Q_\alpha^k\}$ with $E_k \subset \bigcup Q_\alpha^k$ such that for every disjoint collection $\{Q_j\} \subset \sigma_k$ we have

$$\sum u(Q_j) \leq 2^{-k} u(E_k).$$

We may adjoin to σ_k all cubes Q with $u(Q) = 0$.

Now let $\Sigma_s = \bigcup_{k \geq s} \sigma_k$ and let $A_s = \bigcup_{k \geq s} E_k$. Then $A_s \subset \bigcup\{Q : Q \in \Sigma_s\}$. We claim: if $\{Q_j\}$ is a disjoint subcollection of Σ_s, then

$$\sum u(Q_j) \leq \frac{1}{2^{s-1}} u(A_s). \tag{3}$$

To see this note that

$$\sum u(Q_j) \leq \sum_{k \geq s} \sum_{Q_j \in \sigma_k} u(Q_j)$$
$$\leq \sum_{k \geq s} \frac{1}{2^k} u(E_k) \leq u(A_s) \sum_{k \geq s} \frac{1}{2^k} = \frac{1}{2^{s-1}} u(A_s).$$

Now let $A = \bigcap_{s \geq 1} A_s = \limsup E_k$. Since $u(E_k) \geq \epsilon_0$ we see that $u(A) \geq \epsilon_0$. We claim: for each $s_0 \geq 1$, Σ_{s_0} covers A in the sense of Vitali. We have to show that for each $x \in A$ there is $\{Q_i\} \subset \Sigma_{s_0}$ such that $x \in \bigcap Q_i$ and $|Q_i| \to 0$. Since $x \in A$, $x \in \bigcap_{k_j \geq s_0} E_{k_j}$ and hence there is $Q_j \in \sigma_{k_j}$ with $x \in Q_j$. By (3)

$$u(Q_j) \leq \frac{1}{2^{k_j - 1}} u(A_{k_j}) \leq \frac{1}{2^{k_j - 1}} u(\mathbb{R}^n).$$

Since $u(\mathbb{R}^n) < \infty$, $u(Q_j) \to 0$. We can select a subsequence $\{Q_{j_i}\}$ so that $|Q_{j_i}| \to |Q|$ and $\chi_{Q_{j_i}} \to \chi_Q$ where Q is a (perhaps degenerate) cube with $x \in Q$. Since $u \in L^1(\mathbb{R}^n)$, we see that $u(Q_{j_i}) \to u(Q)$. Hence $u(Q) = 0$. If $|Q| = 0$, we are done and if $|Q| > 0$, we are also done, since every subcube of Q is in Σ_{s_0}.

Next we choose $s_0 \geq 3$ such that $u(A_{s_0}) \leq 2u(A)$. This can be done since $0 < u(A) < \infty$ and $u(A_s) \to u(A)$. Since Σ_{s_0} covers A in the sense of Vitali, we have a disjoint collection $\{Q_j\} \subset \Sigma_{s_0}$ such that $|A \setminus \bigcup Q_j| = 0$. Hence $u(A) \leq \sum u(Q_j)$. From (3)

$$u(A_{s_0}) \leq 2u(A) \leq 2 \sum u(Q_j) \leq \frac{1}{2^{s_0 - 2}} u(A_{s_0}) \leq \frac{1}{2} u(A_{s_0}).$$

This is a contradiction since $\epsilon_0 \leq u(A_{s_0}) < \infty$. The proof is now complete.

Remark. We cannot replace u by an arbitrary measure μ with $\mu(\mathbb{R}^n) < \infty$ since $|A \setminus \bigcup Q_j| = 0$ need not imply that $\mu(A) \leq \sum \mu(Q_j)$. Further, we shall see below that ϵ has to be positive and that $u \in L^1(\mathbb{R}^n)$ cannot be weakened to $u \in L^1_{loc}(\mathbb{R}^n)$.

3. Maximal operators

The Covering Theorem implies certain norm inequalities for $M_{\mu,\nu}$ which will be the content of the next three theorems.

Theorem 2. *Let* $d\mu = u(x)dx$ *with* $u \geq 0$ *and in* $L^1(\mathbb{R}^n)$*, and let* $\epsilon > 0$ *be given. Then there exists* $0 < c_{\epsilon,u} < \infty$ *such that the following holds: if for* $f : \mathbb{R}^n \to \mathbb{R}$, $E_y(f) = \{x : M_{\mu,\nu}f(x) > y\}$*, then for all* $y > 0$ *for which* $u(E_y(f)) \geq \epsilon$,

$$\mu(E_y(f)) \leq \frac{c_{\epsilon,u}}{y}\int_{\mathbb{R}^n}|f|d\nu.$$

Proof. Cover $E_y(f)$ by cubes Q for which $\mu(Q) < \frac{1}{y}\int_Q |f|d\nu$. By Theorem 1 we have a disjoint collection of cubes $\{Q_j\}$ in this cover with the property that $u(E_y(f)) \leq c_{\epsilon,u}\sum u(Q_j)$. From this the weak-type inequality follows.

Remarks. Theorem 2 allows us to show that Theorem 1 is sharp.

(i) Theorem 1 is not true if $\epsilon = 0$. Let $u(x) = e^{-|x|^2/2}$, $x \in \mathbb{R}^2$, and let $d\mu = d\nu = u(x)dx$. Then [3] M_u does not satisfy

$$u\{x : M_u f(x) > y\} \leq \frac{c}{y}\int_{\mathbb{R}^n}|f|u.$$

Theorem 2 would contradict this if Theorem 1 were true for $\epsilon = 0$.

(ii) Theorem 1 is not true if $u \in L^1(\mathbb{R}^n)$ is replaced by $u \in L^1_{loc}(\mathbb{R}^n)$. Let $u(x) = e^{|x|}, x \in \mathbb{R}^2$, and again let $d\mu = d\nu = u(x)dx$. If δ_0 is the Dirac delta measure at 0, then [2]

$$\frac{1/y}{u\{x : M_u\delta_0(x) > y\}} \to 0 \text{ as } y \searrow 0.$$

If Theorem 1 were true for $u \in L^1_{loc}(\mathbb{R}^n)$, then Theorem 2 would imply that this fraction is bounded below. Another example where the weak-type inequality for the maximal operator M_u fails can be found in [1].

Theorem 3. *Let* $u \geq 0$ *be in* $L^1(\mathbb{R}^n)$, $1 < p < \infty$*, and let* $\epsilon > 0$ *be given. Then there exists* $0 < c_{\epsilon,p} < \infty$ *such that*

$$\int_{\mathbb{R}^n}\min\{M_u f(x), r\}^p u \leq c_{\epsilon,p}\int_{\mathbb{R}^n}|f|^p u$$

for all $r > 0$ *for which* $u\{x : M_u f(x) > r\} \geq \epsilon$.

Proof. First note that

$$\int_{\mathbb{R}^n}\min\{M_u f(x), r\}^p u = p\int_0^r y^{p-1}u\{x : M_u f(x) > y\}dy.$$

Fix $0 < y < r$ and let $f_y(x) = 0$ if $|f(x)| \leq y/2$, and $f_y(x) = f(x)$ if $|f(x)| > y/2$. This splits $f = f_y + f^y$ from which $u\{x : M_u f(x) > y\} \leq u\{x : M_u f_y(x) > y/2\}$ and this is by Theorem 2 $\leq \frac{c_\epsilon}{y}\int_{\mathbb{R}^n}|f_y|u = \frac{c_\epsilon}{y}\int_{\{x:|f(x)|>y/2\}}|f|u$. Substitute this into the above integral, and interchange the order of integration to obtain the norm inequality.

Remark. Even though M_u need not be weak-type $(1,1)$, differentiation of the integral with respect to u is still possible since

$$\frac{1}{u(Q)}\int_Q fu = \frac{|Q|}{u(Q)}\frac{1}{|Q|}\int_Q fu \rightarrow \frac{f(x)u(x)}{u(x)} = f(x)$$

for u-a.e. x as $Q \rightarrow x$, i.e., $x \in Q$ and $\delta(Q)$ =diam$(Q) \rightarrow 0$. In particular, $M_u f(x) \geq f(x)$ for u-a.e. x. This will be needed below in Theorem 4.

Theorems 2 and 3 can be restated as a Lusin-type approximation to norm inequalities for M_u. We may assume that our functions f are non-negative.

Theorem 4. *Let $u \geq 0$ be in $L^1(\mathbb{R}^n)$, $1 \leq p < \infty$, and let $\epsilon > 0$ be given. Then there exists $0 < c_{\epsilon,p} < \infty$ such that for every $f : \mathbb{R}^n \rightarrow \mathbb{R}_+$ there is $f_\epsilon : \mathbb{R}^n \rightarrow \mathbb{R}_+$ with the following properties:*

$$u\{x : f(x) \neq f_\epsilon(x)\} \leq \epsilon,$$
$$p = 1 : \quad u\{x : M_u f_\epsilon(x) > y\} \leq \frac{c_{\epsilon,1}}{y}\int_{\mathbb{R}^n} f_\epsilon u,$$
$$1 < p < \infty : \quad \int_{\mathbb{R}^n} M_u f_\epsilon^p u \leq c_{\epsilon,p}\int_{\mathbb{R}^n} f_\epsilon^p u,$$
$$f_\epsilon \nearrow f \textit{ as } \epsilon \searrow 0 \textit{ for } u\textit{-a.e. } x.$$

Proof. We may assume that $f(x) < \infty$ u-a.e. x and that $u\{x : f(x) > 0\} > 0$ (otherwise let $f_\epsilon = f$). Finally, we may assume that $0 < \epsilon < u\{x : f(x) > 0\}$. Let $A_r = \{x : f(x) > r\}$.

As a function of r, $u(A_r)$ is non-increasing and right-continuous. Let $r_\epsilon = \inf\{r : u(A_r) \leq \epsilon\}$. Then $0 < r_\epsilon < \infty$ and $u(A_{r_\epsilon}) \leq \epsilon$ as well as $u(A_r) \geq \epsilon$, $r < r_\epsilon$. Define

$$f_\epsilon(x) = r_\epsilon \chi_{A_{r_\epsilon}}(x) + f(x)\chi_{\mathbb{R}^n \setminus A_{r_\epsilon}}(x).$$

Then $u\{x : f(x) \neq f_\epsilon(x)\} \leq \epsilon$ and $f_\epsilon \nearrow f$ for u-a.e. x as $\epsilon \searrow 0$. Since $M_u f_\epsilon(x) \geq f_\epsilon(x)$ for u-a.e. x, we see that $u\{x : M_u f_\epsilon(x) > r\} \geq \epsilon$, $r < r_\epsilon$. Also note that $M_u f_\epsilon$ is bounded above by r_ϵ. Since $u\{x : M_u f_\epsilon(x) > y\}$ is either $\geq \epsilon$ or 0, Theorem 2 proves the weak-type inequality. Further, since $\min\{M_u f_\epsilon(x), r\} \nearrow M_u f_\epsilon(x)$ as $r \nearrow r_\epsilon$, Theorem 3 establishes the strong-type inequality.

4. Convergence

For our second application, let $f \rightarrow T_j f$, $j = 1, 2, \cdots$, be a sequence of linear operators. We wish to investigate when $\{T_j f(x)\}$ converges. Our overall setup is as follows: let $\mu \geq 0, \nu \geq 0$ be Borel measures finite on compact sets, and let $u(x)dx$ be the absolutely continuous part in the Lebesgue decomposition of μ with respect to dx. Below we need the maximal function $M_{\mu,\nu}$ as defined in the introduction.

Theorem 5. *Assume that $T_* f(x) = \sup_j |T_j f(x)| \leq c M_{\mu,\nu} f(x)$, where c is independent of $x \in \mathbb{R}^n$ and of $f : \mathbb{R}^n \rightarrow \mathbb{R}$. If $\{T_j g(x)\}$ converges for u-a.e. x and every $g \in D$, D a dense subset of $L^1(\nu)$, then the same holds for every $f \in L^1(\nu)$.*

Remark. This is well-known if T_* satisfies a weak-type inequality (see [5, p.60]). The point here is that such a weak-type inequality for T_* may not hold. We shall see in section 5 that in the conclusion u cannot be replaced by μ.

Proof. Fix $f \in L^1(\nu)$. We will first show that $T_*f(x) < \infty, u-$a.e. x. Let $E = \{x : T_*f(x) = \infty\}$ and assume that $u(E) > \epsilon > 0$. Fix N so that $u_N(E) > \epsilon$, where $u_N = u\chi_{B_N}, B_N = \{x : |x| \le N\}$. For $0 < y < \infty$ let $E_y = \{x : T_*f(x) > y\} \subset \{x : M_{\mu,\nu}f(x) > y/c\}$. Hence $E_y \cap B_N \subset \bigcup\{Q : \mu(Q) < (c/y)\int_Q |f|d\nu\}$. By Theorem 1 there is a constant $0 < c_{\epsilon,N} < \infty$ and a disjoint subcollection $\{Q_j\}$ of this cover such that

$$\epsilon < u_N(E) \le c_{\epsilon,N}\sum u_N(Q_j) \le c_{\epsilon,N}\sum \mu(Q_j) \le \frac{c \cdot c_{\epsilon,N}}{y}\int_{\mathbb{R}^n}|f|d\nu.$$

This is impossible as $y \to \infty$.

Now let $E = \{x : \limsup T_jf(x) - \liminf T_jf(x) > 0\}$. Then $E = \bigcup E_i$, where $E_i = \{x : \limsup T_jf(x) - \liminf T_jf(x) > 1/i\}$. We have to show that $u(E_i) = 0$. Fix i and assume that $u(E_i) > \epsilon > 0$. If $u_N = u\chi_{B_N}$, fix N so that $u_N(E_i) \ge \epsilon$. If $g \in D$, then

$$\begin{aligned} u_N(E_i) &= u_N\{x : \limsup T_j(f-g)(x) - \liminf T_j(f-g)(x) > 1/i\} \\ &\le u_N\{x : T_*(f-g)(x) > 1/(2i)\} \le u_N\{x : M_{\mu,\nu}(f-g)(x) > 1/(2ci)\} \\ &\equiv u_N(K). \end{aligned}$$

Since $u_N(K) \ge \epsilon$ and $K \subset \bigcup\{Q : \mu(Q) < 2ci\int_Q |f-g|d\nu\}$, we can apply Theorem 1 and get a constant $0 < c_{\epsilon,N} < \infty$ and a disjoint subcollection $\{Q_j\}$ of this cover such that

$$u_N(K) \le c_{\epsilon,N}\sum u_N(Q_j) \le c_{\epsilon,N}\sum \mu(Q_j) \le c_{\epsilon,N}2ci\int_{\mathbb{R}^n}|f-g|d\nu.$$

Since $u_N(K) \ge \epsilon$, this contradicts that D is dense in $L^1(\nu)$.

Remark. There are important special cases where the natural necessary condition for convergence of $\{T_jf(x)\}$ for a.e. x—namely $T_*f(x) < \infty$ for a.e. x—is also a sufficient condition. For example, if $T_jf(x) = f \star \mu_j(x)$, where $\{\mu_j\}$ is a sequence of finite Borel measures on $\mathbb{R}^n$ supported in a common compact set, and if $\{T_jg(x)\}$ converges for a.e. x and every $g \in C_c(\mathbb{R}^n)$, then the same is true for functions in $L^p(\mathbb{R}^n)$, $1 \le p < \infty$, if and only if $|\{x : T_*f(x) < \infty\}| > 0$ for every $f \in L^p(\mathbb{R}^n)$. By [4, p.441] the assumption that $T_*f(x) < \infty$ on a set of positive measure for every $f \in L^p(\mathbb{R}^n)$ implies that

$$|\{x : T_*f(x) > y\}| \le \frac{c}{y^p}||f||_p^p,$$

and this in conjunction with convergence on $C_c(\mathbb{R}^n)$ gives the desired conclusion.

5. Lebesgue points and differentiation

Let $\mu \ge 0, \nu \ge 0$ be Borel measures finite on compact sets. Associate with each $x \in \mathbb{R}^n$ a sequence of sets $\{E_j^x\}$ "converging" to x in the sense that for each $\epsilon > 0$ there is j_ϵ such that $E_j^x \subset \{y : |y - x| \le \epsilon\}$ for $j \ge j_\epsilon$. Note that x need not belong to E_j^x. We say that x is a (μ,ν)-Lebesgue point of $f : \mathbb{R}^n \to \mathbb{R}$ with respect to $\{E_j^x\}$ if

$$\lim_{j\to\infty} L_{E_j^x;\mu,\nu}f = 0$$

where

$$L_{E_j^x;\mu,\nu} f = \begin{cases} 0, & \mu(E_j^x) = 0, \\ \frac{1}{\mu(E_j^x)} \int_{E_j^x} |f(t) - f(x)| d\nu(t), & \mu(E_j^x) > 0. \end{cases}$$

We wish to examine the problem of when x is a (μ,ν)-Lebesgue point of f with respect to $\{E_j^x\}$ and the related problem of the differentiation of the integral, i.e., the existence of $\lim_{j\to\infty} A_{E_j^x;\mu,\nu} f$, where the average $A_{E_j^x;\mu,\nu} f$ is as in the Introduction. We also need

$$A_{E_j^x;\mu,\nu} = \begin{cases} 0, & \mu(E_j^x) = 0, \\ \frac{\nu(E_j^x)}{\mu(E_j^x)}, & \mu(E_j^x) > 0, \end{cases}$$

the above average for $f \equiv 1$. It is immediate that the existence of a (μ,ν)-Lebesgue point x for f with respect to $\{E_j^x\}$ implies that

$$A_{E_j^x,\mu,\nu} f - f(x) A_{E_j^x;\mu,\nu} \to 0,$$

as $j \to \infty$, and thus differentiation of the integral is only possible if $\{A_{E_j^x;\mu,\nu}\}$ converges. We will give examples at the end of this section.

Let, as in section 4, $u(x)dx$ be the absolutely continuous part in the Lebesgue decomposition of μ with respect to dx. Then $u \in L^1_{loc}(\mathbb{R}^n)$. We also need the Hardy-Littlewood maximal function $M_{\mu,\nu} f$ as defined in the Introduction.

Theorem 6. *Assume that*

$$\sup_j |A_{E_j^x;\mu,\nu} f| \le c M_{\mu,\nu} f(x), \quad x \in \mathbb{R}^n,$$

where c is independent of x and f. If $A(x) = \limsup_{j\to\infty} A_{E_j^x;\mu,\nu}$, then $A(x) < \infty$ for u-a.e. x, and if, in addition, the measure $d\lambda = A(x)u(x)dx$ is finite on compact sets, then

$$\lim_{j\to\infty} L_{E_j^x;\mu,\nu} f = 0$$

for u-a.e. x and every $f \in L^1_{loc}(\nu + \lambda)$.

Proof. Below B_N is the ball with center 0 and radius N. First we show that $A(x) < \infty$ for u-a.e. x. Let $E = \{x : A(x) = \infty\}$ and let $E_y = \{x : A(x) > y\}$. Assume that $u(E) > \epsilon > 0$. Fix N so that $u_N(E) > \epsilon$, where $u_N = u\chi_{B_N}$. Then $E_y \cap B_N \subset \{x \in B_{N+1} : M_{\mu,\nu} f(x) > y/c\}$, where $f = \chi_{B_{N+1}}$. Hence $\epsilon < u_N(E_y) \le u_N\{x : M_{\mu,\nu} f(x) > y/c\} \equiv u_N(A)$. Since $A \subset \bigcup\{Q : \mu(Q) < (c/y)\int_Q f d\nu\}$, we have by Theorem 1 a constant $0 < c_{\epsilon,N} < \infty$ and a disjoint subcollection $\{Q_j\}$ of this cover such that

$$\begin{aligned} \epsilon < u_N(E_y) &\le c_{\epsilon,N} \sum u_N(Q_j) \le c_{\epsilon,N} \sum \mu(Q_j) \\ &\le \frac{c_{\epsilon,N} c}{y} \nu(B_{N+1}). \end{aligned}$$

This is impossible when $y \to \infty$.

Let $g \in C_c(\mathbb{R}^n)$. Then a straightforward calculation shows that at every x at which $A(x) < \infty$,

$$\lim_{j\to\infty} L_{E_j^x;\mu,\nu} g = 0.$$

To prove our general statement, we first fix $f \in L^1(\nu + \lambda)$. Write the set $E = \{x : \limsup_{j\to\infty} L_{E_j^x;\mu,\nu} f > 0\}$ as $\bigcup E_i$, where E_i is defined as E except

> 0 is replaced by $> 1/i$. We have to show that $u(E_i) = 0$. Fix i and assume that $u(E_i) > \epsilon > 0$. Now choose N so that $u_N(E_i) > \epsilon$, where $u_N = u\chi_{B_N}$. If $g \in C_c(\mathbb{R}^n)$, then

$$\begin{aligned}\epsilon &< u_N\{x \in B_N : \limsup_{j\to\infty} L_{E_j^x;\mu,\nu}(f-g) > 1/i\}\\ &\leq u_N\{x : M_{\mu,\nu}(f-g)(x) > 1/2ic\} + u_N\{x : |f(x)-g(x)|A(x) > 1/2i\}.\end{aligned}$$

Then either

$$(*) \qquad \epsilon/2 < u_N\{x : M_{\mu,\nu}(f-g)(x) > 1/2ic\} \equiv u_N(A_1)$$

or

$$(**) \qquad \epsilon/2 < u_N\{x : |f(x)-g(x)|A(x) > 1/2i\} \equiv u_N(A_2).$$

If $(*)$ holds, then, since $A_1 \subset \bigcup\{Q : \mu(Q) < 2ic\int_Q |f-g|d\nu\}$, we have by Theorem 1 a constant $0 < c_{\epsilon,N} < \infty$ and a disjoint subcollection $\{Q_j\}$ of this cover such that

$$\begin{aligned}\epsilon/2 < u_N(A_1) &\leq c_{\epsilon,N}\sum u_N(Q_j)\\ &\leq c_{\epsilon,N}\sum \mu(Q_j) \leq 2ic_{\epsilon,N}c\int_{\mathbb{R}^n}|f-g|d\nu.\end{aligned}$$

If $(**)$ occurs, then

$$\epsilon/2 < 2i\int_{\mathbb{R}^n}|f-g|Audx \leq 2i\int_{\mathbb{R}^n}|f-g|d\lambda.$$

Thus in either case

$$\epsilon/2 \leq C\int_{\mathbb{R}^n}|f-g|d(\nu+\lambda)$$

with C independent of $g \in C_c(\mathbb{R}^n)$. This contradicts that $C_c(\mathbb{R}^n)$ is dense in $L^1(\nu+\lambda)$. If $f \in L^1_{loc}(\nu+\lambda)$, apply the above argument to $f_N = f\chi_{B_N}$ and let $N \to \infty$.

Remarks. (1) If $\lim_{j\to\infty} A_{E_j^x;\mu,\nu} = A(x)$ for u-a.e. x, then from Theorem 6 we have differentiation of the integral, i.e.,

$$\lim_{j\to\infty} A_{E_j^x;\mu,\nu}f = f(x)A(x)$$

for u-a.e. x and every $f \in L^1_{loc}(\nu+\lambda)$, if the overall hypothesis $\sup_j |A_{E_j^x;\mu,\nu}f| \leq cM_{\mu,\nu}f(x)$ is satisfied.

We have already observed that in our set-up the existence of Lebesgue points of f relative to μ,ν need not imply the differentiability of the integral at those points. Here is an example. Let C be a compact nowhere dense subset of $\mathbb{R}$ of positive measure. Let $d\mu = dx$ and $d\nu = \chi_C(x)dx$. Associate with each $x \in \mathbb{R}$ a sequence of intervals $\{I_j^x\}_{j\geq 1}$ converging to x in the following way: if $x \notin C$, let $I_j^x \subset \mathbb{R}\backslash C$, and if $x \in C$, first select a sequence of intervals $\{J_j^x\}$ in $\mathbb{R}\backslash C$ converging to x and then define $I_{2j+1}^x = J_j^x$, $I_{2j}^x = [x-1/j, x+1/j]$. The overall hypothesis $\sup_j |A_{I_j^x;\mu,\nu}f| \leq cM_{\mu,\nu}f(x)$ holds with $c = 1$ since the only time $A_{I_j^x;\mu,\nu}f \neq 0$ is when $x \in I_j^x$. Also $\limsup A_{I_j^x;\mu,\nu} = 1$ at every point of density x of C. Hence $A(x) = 1$ for a.e. $x \in C$, and $A(x) = 0$ for $x \notin C$. Thus by Theorem 6, a.e. x is a (μ,ν)-Lebesgue point for f with respect to $\{I_j^x\}$ for every $f \in L^1(\nu)$.

However, if $f \neq 0$ on C and $f \in L^1(\nu)$, then for $x \in C$

$$\frac{1}{|I^x_{2j+1}|}\int_{I^x_{2j+1}} f d\nu = 0, \quad \frac{1}{|I^x_{2j}|}\int_{I^x_{2j}} f d\nu \to f(x), \text{ a.e. } x.$$

(2) The convergence of $A_{E^x_j;\mu,\nu}$ as $j \to \infty$ follows if each E^x_j is a cube Q^x_j containing x. In fact, if $v(x)dx$ is the absolutely continuous part in the Lebesgue decomposition of ν with respect to dx, then

$$A_{Q;\mu,\nu} \to \frac{v(x)}{u(x)}$$

as $Q \to x$ for u-a.e. x. Recall that $Q \to x$ means that $x \in Q$ and $\delta(Q) \to 0$.

To see this let $\sigma \geq 0$ be a Borel measure finite on compact sets, and let $w(x)dx$ be the absolutely continuous part in the Lebesgue decomposition of σ with respect to dx. Then $w \in L^1_{loc}(\mathbb{R}^n)$. If $\sigma_s \perp dx$ is the singular part, then

$$\frac{\sigma(Q)}{|Q|} = \frac{w(Q)}{|Q|} + \frac{\sigma_s(Q)}{|Q|},$$

and $w(Q)/|Q| \to w(x)$ as $Q \to x$, a.e. x. To handle σ_s, note that if $x \in Q$, then there is a cube Q_x centered at x such that $|Q_x| \leq c_n|Q|$ and $Q \subset Q_x$. Hence $\sigma_s(Q)/|Q| \leq c_n\sigma_s(Q_x)/|Q_x|$. Now the Besicovitch Covering Theorem is available and from this it follows that $\lim \sigma_s(Q_x)/|Q_x| \to 0$ as $Q_x \to x$ for a.e. x. For details see [6, p.188].

In the general case, since $u(x) > 0$ for u-a.e. x, for a.e. such x

$$\frac{\nu(Q)}{\mu(Q)} = \frac{\nu(Q)/|Q|}{\mu(Q)/|Q|} \to \frac{v(x)}{u(x)}$$

as $Q \to x$. In Theorem 6, λ can be taken to be ν since $d\lambda = [v(x)/u(x)]u(x)dx \leq d\nu$.

(3) An example similar to the one in (1) shows that the overall hypothesis

$$\sup_j |A_{E^x_j;\mu,\nu} f| \leq cM_{\mu,\nu}f(x)$$

cannot be omitted. Let C be a nowhere dense compact subset of $\mathbb{R}$ of positive measure and let $\{J_k\}$ be the sequence of complementary components of $\mathbb{R} \setminus C$. For each $x \notin C$ let $\{I^x_j\}$ be a sequence of intervals converging to x with $x \in I^x_j$. If $x \in C$ let $\{I^x_j\} \subset \{J_k\}$ converging to x. The measures will be $\mu = \nu = dx$. Let x_0 be a point of C, and let $f_{j_0} = \chi_{I^{x_0}_{j_0}}$. Then $\sup_j |A_{I^{x_0}_j;\mu,\nu} f_{j_0}| = 1$. If, say, $I^{x_0}_{j_0} = [\alpha, \beta]$ is to the right of x_0 and $I = [x_0, \beta]$, then

$$Mf_{j_0}(x_0) = \frac{|I^{x_0}_{j_0}|}{|I|}.$$

Hence

$$\sup_j |A_{I^{x_0}_j;\mu,\nu} f_{j_0}| = \frac{|I|}{|I^{x_0}_{j_0}|} Mf_{j_0}(x_0).$$

If x_0 is a point of density of C, then $\dfrac{|I|}{|I^{x_0}_{j_0}|} \to \infty$ as $j_0 \to \infty$. Also note that at every $x \in C$

$$\frac{1}{|I^x_j|}\int_{I^x_j} |f(t) - f(x)|dt = 1,$$

if $f = \chi_C$.

(4) If $x \in Q$ means that x is the center of Q, then $A_{Q;\mu,\nu}$ converges as $Q \to x$ for μ-a.e. x [6, p.189]. This is stronger than u - a.e. x. The question then arises as to whether in the non-centered approach to x in (2) above, u can be replaced by μ. This is not so, as the following example shows.

Let L be the line $y = x$ in $\mathbb{R}^2$. Let ν be Lebesgue measure on $\mathbb{R}^2$, and define for $E \subset \mathbb{R}^2$, $\mu(E) = m_1(E \cap L)$, where m_1 is one-dimensional Lebesgue measure on L. Then for every $p \in L$, $\liminf \nu(Q)/\mu(Q) = 0$, $\limsup \nu(Q)/\mu(Q) = \infty$ as $Q \to p$, and $\mu(L) > 0$.

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