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WOODIN CARDINALS, SHELAH CARDINALS, AND THE MITCHELL-STEEL CORE MODEL

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ABSTRACT. Theorem 4 is a characterization of Woodin cardinals in terms of Skolem hulls and Mostowski collapses. We define *weakly hyper-Woodin cardinals* and *hyper-Woodin cardinals*. Theorem 5 is a covering theorem for the Mitchell-Steel core model, which is constructed using total background extenders. Roughly, Theorem 5 states that this core model correctly computes successors of hyper-Woodin cardinals. Within the large cardinal hierarchy, in increasing order we have: measurable Woodin, weakly hyper-Woodin, Shelah, hyper-Woodin, and superstrong cardinals. (The comparison of Shelah versus hyper-Woodin is due to James Cummings.)

We begin by recalling the definition of a Woodin cardinal. If $\kappa < \lambda$ are cardinals and S is a set, then *κ is λ-S-strong* iff there is a transitive class N and an elementary embedding $j : V \longrightarrow N$ with

$$\kappa = \mathrm{crit}(j)\ ,$$

$$j(\kappa) \geq \lambda$$

and

$$j(S) \cap H_\lambda = S \cap H_\lambda\ .$$

If $\kappa < \delta$, then κ is $< \delta$-S-strong iff κ is λ-S-strong for every $\lambda < \delta$. Finally, *δ is a Woodin cardinal* iff δ is a strongly inaccessible cardinal and for every $S \subseteq H_\delta$, there exists $\kappa < \delta$ such that κ is $< \delta$-S-strong. It is well-known that the existence of an embedding that witnesses that κ is λ-S-strong is equivalent to the existence of an extender $E \subset H_\lambda$ which gives rise to such an embedding through an ultrapower construction. Thus *δ is a Woodin cardinal* is a Π^1_1 property of H_δ.

We will see how to express *δ is a Woodin cardinal* in terms of the Mostowski collapses of Skolem hulls. The uniformity with which this can be done is somewhat surprising. In the second part of this paper, we apply this characterization to core model theory.

For each cardinal θ, let $\lhd_\theta$ be a wellordering of H_θ. Whenever we refer to H_θ as a structure, we mean the structure $\langle H_\theta\ ,\ \in\ ,\ \lhd_\theta \rangle$. Suppose that M is transitive and $\pi : M \longrightarrow H_\theta$ is an elementary embedding. Note that, in this case, π is the inverse of the Mostowski collapse isomorphism for the range of π. Let $\kappa = \mathrm{crit}(\pi)$

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and $\lambda < \pi(\kappa)$. Suppose that $j : V \longrightarrow N$ is also an elementary embedding. We say that j *certifies* π *up to* λ iff

$$\mathrm{crit}(j) = \kappa,$$

$$j(\kappa) \geq \lambda$$

and

$$j(A) \cap H_\lambda = \pi(A) \cap H_\lambda$$

for every $A \in \mathcal{P}(H_\kappa) \cap M$. We say that π *is certified* iff for every $\lambda < \pi(\kappa)$, there is an elementary embedding $j : V \longrightarrow N$ that certifies π up to λ.

Proposition 1. *Suppose that M is a transitive set and $\pi : M \longrightarrow H_\theta$ is a non-trivial elementary embedding. Let $\kappa = \mathrm{crit}(\pi)$ and $\lambda < \pi(\kappa)$. Suppose that $j : V \longrightarrow N$ certifies π up to λ and S is an element of the range of π. Then j witnesses that κ is λ-S-strong.*

Proof. Note that $S \cap H_\kappa \in M$ and $\pi(S \cap H_\kappa) = S \cap H_{\pi(\kappa)}$. Therefore

$$j(S) \cap H_\lambda = j(S \cap H_\kappa) \cap H_\lambda = \pi(S \cap H_\kappa) \cap H_\lambda = S \cap H_\lambda \ .$$

□

Proposition 2. *Suppose that $j : V \longrightarrow N$ is an elementary embedding and $\kappa = \mathrm{crit}(j)$. Let $\theta > \kappa$ be a cardinal and $S \in H_\theta$. Suppose that $\pi : M \longrightarrow j(H_\theta)$ is the inverse of the Mostowski collapse of the Skolem hull of $\kappa \cup \{j(S)\}$ in $j(H_\theta)$. Then*

$$\kappa = \mathrm{crit}(\pi),$$

$$\pi(\kappa) \geq j(\kappa)$$

and

$$j(A) = \pi(A) \cap j(H_\kappa)$$

for all $A \in \mathcal{P}(H_\kappa) \cap M$.

Proof. Note that $j^{-1}(\pi(A) \cap j(H_\kappa)) = (\pi(A) \cap j(H_\kappa)) \cap H_\kappa = A$. □

Proposition 3. *Suppose that δ is a Woodin cardinal. Let $\theta > \delta$ be a cardinal and $S \in H_\theta$. Let T be the first order theory of*

$$\delta \times \{S\}$$

in H_θ coded in a natural way as a subset of H_δ. Suppose that κ is $< \delta$-T-strong. Let π be the inverse of the Mostowski collapse of the Skolem hull of $\kappa \cup \{S\}$ in H_θ. Then

$$\mathrm{crit}(\pi) = \kappa$$

and

$$\pi(\kappa) \geq \delta \ .$$

Moreover, if $\lambda < \delta$ and $j : V \longrightarrow N$ witnesses that κ is λ-T-strong, then j certifies π up to λ.

Proof. Fix everything but λ and j as in the hypothesis of the proposition. Say the language of T has a constant symbol $\dot{S}$ which is interpreted as S in H_θ. Let X be the Skolem hull of $\kappa \cup \{S\}$ in H_θ and $\pi : M \simeq X$ with M transitive.

Lemma 3.1. *$X \cap \delta = \kappa$.*

Proof. Consider any $\gamma \in X \cap \delta$. Say $\alpha_1, \ldots, \alpha_n \in \kappa$ and γ is the unique β such that

$$H_\theta \models \varphi(\alpha_1, \ldots, \alpha_n, S, \beta).$$

Let $j : V \longrightarrow N$ witness that κ is γ^+-T-strong. So

$$j(\kappa) > \gamma^+$$

and

$$j(T) \cap H^N_{\gamma^+} = T \cap H_{\gamma^+} \ .$$

By elementarity, $j(T)$ is the first order theory of

$$j(\delta) \times \{j(S)\}$$

in $j(H_\theta)$. In particular, γ is the unique β such that

$$j(H_\theta) \models \varphi(\alpha_1, \ldots, \alpha_n, j(S), \beta).$$

Therefore, γ is in the range of j and $\gamma < j(\kappa)$. Hence $\gamma < \kappa$. □

By Lemma 3.1, $\kappa = \mathrm{crit}(\pi)$ and $\pi(\kappa) \geq \delta$. Now suppose that λ is a cardinal such that $\kappa < \lambda < \delta$. Let $j : V \longrightarrow N$ witness that κ is λ-T-strong.

Lemma 3.2. *j certifies π up to λ.*

Proof. Let $A \in \mathcal{P}(H_\kappa) \cap M$ and $c \in H_\lambda$. Say $\pi(A)$ is the unique B such that

$$H_\theta \models \varphi(\alpha_1, \ldots, \alpha_n, S, B),$$

where $\alpha_1, \ldots, \alpha_n \in \kappa$. In the following string of biconditionals, it is in the second "iff" that we use that j is λ-T-strong:

$$c \in \pi(A)$$

iff

$$\text{“}c \in \text{ the unique } B \text{ such that } \varphi(\alpha_1, \ldots, \alpha_n, \dot{S}, B)\text{”} \in T$$

iff

$$\text{“}c \in \text{ the unique } B \text{ such that } \varphi(\alpha_1, \ldots, \alpha_n, \dot{S}, B)\text{”} \in j(T)$$

iff

$$c \in \text{ the unique } B \text{ such that } \varphi^{j(H_\theta)}(\alpha_1, \ldots, \alpha_n, j(S), B)$$

iff

$$c \in j\left(\text{the unique } B \text{ such that } \varphi^{H_\theta}(\alpha_1, \ldots, \alpha_n, S, B)\right)$$

iff

$$c \in j(\pi(A))$$

iff

$$c \in j(\pi(A) \cap H_\kappa)$$

iff

$$c \in j(A).$$

□

That completes the proof of Proposition 3. □

Theorem 4. *Let δ be an inaccessible cardinal. Then δ is a Woodin cardinal iff for all $S \subseteq \delta$, there exists $\kappa < \delta$ such that for some cardinal $\theta > \delta$, if π is the inverse of the Mostowski collapse of the Skolem hull of $\kappa \cup \{\delta, S\}$ in H_θ, then $\pi(\kappa) = \delta$ and π is certified. Moreover, this remains true if the word "some" is replaced by "every".*

Proof. Immediate from the definition of a Woodin cardinal and Propositions 1 and 3. □

Next, we apply Theorem 4 to core model theory. Let $L[\vec{E}]$ be the maximal Mitchell-Steel core model constructed using total background extenders. Such a model was constructed in [5] under the anti-large cardinal assumption that there is no sharp for a model with a Woodin cardinal. Later, a number of papers appeared in which the anti-large cardinal assumption was weakened. Most recently, Neeman [6] showed that $L[\vec{E}]$ exists if there is no inner model with a Woodin cardinal that is a limit of Woodin cardinals.

Steel [7] takes a different approach in building his countably certified core model, K^c. Instead of total background extenders, which correspond to embeddings of V, he uses certificates that are more like the inverse Mostowski collapses that we have been working with here. Steel works in higher type set theory under the assumption that Ω is the class of ordinals and U is a normal measure on Ω. He shows that under the anti-large cardinal assumption that there is no sharp for a model with a Woodin cardinal, K^c computes κ^+ correctly for U-a.e. $\kappa < \Omega$. (This is called the *cheapo covering lemma* by Steel.) Again, there have been several improvements in which the anti-large cardinal assumption is weaker. Most recently, Andretta-Neeman-Steel [1] weakened the assumption to every premouse being domestic, a hypothesis that is stronger than that of Neeman [6] mentioned in the previous paragraph.

Let us share some pure intuition. First, it is easier to show that $L[\vec{E}]$ is iterable than to show that K^c is iterable. Second, it is easier to prove covering for K^c than for $L[\vec{E}]$; this is because the requirements for adding an extender to the K^c-sequence are less restrictive and, hence, K^c is fatter. In fact, there is no covering result for $L[\vec{E}]$ in the literature. Last, if there are lots of Woodin cardinals, then the difference between these restrictions becomes less pronounced. Building on the last thought, we obtain the following covering result for $L[\vec{E}]$.

It is well-known that if δ is a Woodin cardinal, then for every set S,

$$\{\kappa < \delta \mid \kappa \text{ is } {<}\,\delta\text{-}S\text{-strong}\}$$

is stationary in δ. (See Kanamori [2].) Let us define *U witnesses that δ is a hyper-Woodin cardinal* to mean that U is a normal measure on δ and for every set S,

$$\{\kappa < \delta \mid \kappa \text{ is } {<}\,\delta\text{-}S\text{-strong}\} \in U.$$

Theorem 5. *Suppose that U witnesses that δ is a hyper-Woodin cardinal. Assume that there is no transitive model W for which*

$$\left\{ \kappa < \delta \;\middle|\; \left(\begin{array}{c} \textit{there is a superstrong embedding} \\ \textit{that maps } \kappa \textit{ to } \delta \end{array} \right)^W \right\} \in U.$$

Assume that the maximal $L[\vec{E}]$ construction using total extenders does not break down.[1] *Then*

$$\left\{ \kappa < \delta \;\middle|\; (\kappa^+)^{L[\vec{E}]} = \kappa^+ \right\} \in U.$$

Proof. Say $j : V \longrightarrow N$ is the ultrapower embedding from U,

$$W = L[j(\vec{E})],$$

and

$$\varepsilon = (\delta^+)^W.$$

For contradiction, suppose that $\varepsilon < \delta^+$. Let $S \subseteq H_\delta$ code $j(\vec{E}) \upharpoonright \varepsilon$ in such a way that δ is definable from S. Let θ be a cardinal greater than δ. Let T be the first order theory of $\delta \times \{S\}$ in H_θ. By hyper-Woodiness, δ is $< j(\delta)$-$j(T)$-strong in N. Let $\pi : M \longrightarrow j(H_\theta)$ be the inverse of the Mostowski collapse of the Skolem hull of $\delta \cup \{j(S)\}$ in $j(H_\theta)$. By Proposition 3 as applied in N,

$$\mathrm{crit}(\pi) = \delta,$$

$$\pi(\delta) = j(\delta)$$

and

$$(\pi \text{ is certified})^N .$$

Since $\pi^{-1}(j(S)) = j(S) \cap H_\delta = S$,

$$j(\vec{E}) \upharpoonright \varepsilon \in M.$$

Hence

$$\mathcal{P}(\delta) \cap W \in M.$$

Let E_π be the extender of length $\pi(\delta)$ derived from π and

$$F = E_\pi \cap \left([\pi(\delta)]^{<\omega} \times W\right).$$

Using the fact that F is certified in N, by the calculations carried out in Chapter 11 of Mitchell and Steel [5], we may conclude that for unboundedly many $\alpha < \pi(\delta)$,

$$F \upharpoonright \alpha = \dot{E}^W_\alpha \ .$$

That is, $F \upharpoonright \alpha$ is the extender with index α on the W-sequence.[2] Thus, for U-a.e. $\kappa < \delta$, there is an $L[\vec{E}]$-extender G of length δ such that G coheres with $L[\vec{E}]$, $\mathrm{crit}(G) = \kappa$, G maps κ to δ, and $G \upharpoonright \alpha = E_\alpha$ for unboundedly many $\alpha < \delta$. For any such κ and G,

$$G = j(G) \upharpoonright \delta \in W,$$

and G witnesses that κ is a superstrong cardinal in W. Contradiction! □

[1] By this we mean that the definition on pages 99–100 of Mitchell and Steel [5] *uniquely* determines a *proper class* premouse, which we call $L[\vec{E}]$.

[2] This is already enough to conclude that δ is a Shelah cardinal in W. The definition of *Shelah cardinal* is recalled a few paragraphs below.

It is important to point out that if δ is a measurable Woodin cardinal, then δ is a limit of Woodin cardinals, so the result in Neeman [6] on the existence of $L[\vec{E}]$ does not apply. In particular, hyper-Woodin cardinals are beyond the reach of current core model theory. Seemingly, whatever applications of Theorem 5 there might be await further progress on the iterability problem for $L[\vec{E}]$.

Finally, we compare hyper-Woodin cardinals with other large cardinals using standard methods. (See Kanamori [2].)

If U witnesses that δ is a hyper-Woodin cardinal, then δ is a measurable Woodin cardinal in the ultrapower by U, so δ is not the least measurable Woodin cardinal.

Let $j : V \longrightarrow N$ be an elementary embedding with critical point δ, where N is a transitive class. Let S be a set and $\lambda < j(\delta)$. Suppose that the extender $\{(A, c) \in \mathcal{P}(H_\delta) \times H_\lambda \mid c \in j(A)\}$ is an element of N. Then δ is λ-$j(S)$-strong in N. Thus, if j is a superstrong embedding and U is the normal measure derived from j, then U witnesses that δ is a hyper-Woodin cardinal.

Recall that δ is a *Shelah cardinal* iff for every $f : \delta \longrightarrow \delta$, there is a transitive class N and an elementary embedding $j : V \longrightarrow N$ with critical point δ such that $V_{j(f)(\delta)} \subset N$.

Let us define a cardinal δ to be *weakly hyper-Woodin* iff for every set S, there is a normal measure U on δ such that

$$\{\kappa < \delta \mid \kappa \text{ is } < \delta\text{-}S\text{-strong}\} \in U.$$

Note the dependence of U on S in this definition, in contrast to the definition of *hyper-Woodin*.

We claim that if δ is a Shelah cardinal, then δ is a weakly hyper-Woodin cardinal. To see this, let S be a given set. For $\kappa < \delta$, let $f(\kappa)$ be the least inaccessible cardinal greater than the least $\lambda < \delta$ such that κ is not λ-S-strong. If there is no such λ, then leave $f(\kappa)$ undefined. Let j be as in the above definition of *Shelah cardinal* and let U be the normal measure derived from j. A standard argument shows that $j(f)(\delta)$ is undefined.

James Cummings observed that 1) immediately from the previous result, the least weakly hyper-Woodin cardinal is strictly less than the least Shelah cardinal, and 2) the least Shelah cardinal is strictly less than the least hyper-Woodin cardinal. Towards 2), suppose that U witnesses that δ is a hyper-Woodin cardinal. Let $j : V \longrightarrow N$ be the ultrapower map corresponding to U. Consider a function $f : \delta \longrightarrow \delta$. Using the assumption that δ is hyper-Woodin, a standard argument shows that, in N, there exists an elementary embedding $k : N \longrightarrow P$ with P transitive,

$$\operatorname{crit}(k) = \delta$$

and

$$V^N_{k(j(f))(\delta)} \subseteq P.$$

But

$$k(j(f))(\delta) = k(j(f) \restriction \delta)(\delta) = k(f)(\delta).$$

Therefore, δ is a Shelah cardinal in N.

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