

A NEW CONSTRUCTION OF THE KAC JORDAN SUPERALGEBRA

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To Irving Kaplansky

ABSTRACT. We present an elementary construction of the 10-dimensional simple Jordan superalgebra K_{10} of Kac using the 3-dimensional tiny Kaplansky superalgebra. This new realization of K_{10} enables us to derive many of its properties.

INTRODUCTION

The 10-dimensional Jordan superalgebra K_{10} of Kac [K] is the only exceptional finite-dimensional simple Jordan superalgebra over an algebraically closed field of characteristic 0. We give a new, elementary construction of it (over an arbitrary field of characteristic not 2) using the tiny Kaplansky superalgebra K . In our construction, K_{10} appears as a direct sum $F1 \oplus (K \otimes K)$ with unit element 1 and with a suitable multiplication (see (2.1)). In characteristic 3, K_{10} is not simple but possesses a simple ideal K_9 of dimension 9, which is just the tensor product $K \otimes K$ (of superalgebras).

Our realization of K_{10} makes it easy to deduce certain of its known properties: for example, that its Lie superalgebra of superderivations is isomorphic to a direct sum of two copies of the orthosymplectic Lie superalgebra $\mathfrak{osp}(1, 2)$, or that its Lie multiplication superalgebra is $\mathfrak{osp}(2, 4)$ if -1 is a square in the field F .

The Kac superalgebra was originally constructed in [K] by Lie theoretical methods from a 3-grading of the simple Lie superalgebra $F(4)$. No direct proof of the fact that it is a Jordan superalgebra is known. Indeed, in their work on Jordan superalgebras with semisimple even part, Racine and Zelmanov [RZ] remark “That K_{10} , and hence K_9 , is a Jordan superalgebra can be obtained by Lie methods as in [K] but a direct proof would be desirable.” Here we provide such a proof.

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1. THE KAPLANSKY SUPERALGEBRA

Suppose F is a field of characteristic not 2. The tiny **Kaplansky superalgebra** $([\mathbb{K}p, \mathbb{M}])$ is the 3-dimensional Jordan superalgebra $K = K_{\bar{0}} \oplus K_{\bar{1}}$, with $K_{\bar{0}} = Fe$ and $K_{\bar{1}} = Fx \oplus Fy$, and with multiplication given by

$$(1.1) \quad \begin{aligned} e^2 = e, \quad ex = \frac{1}{2}x = xe, \quad ey = \frac{1}{2}y = ye, \\ xy = e = -yx, \quad x^2 = 0 = y^2. \end{aligned}$$

It can be seen using (1.1) that the even supersymmetric bilinear form $(|)$ defined on K by

$$(e | e) = \frac{1}{2}, \quad (x | y) = 1, \quad \text{and} \quad (K_{\bar{0}} | K_{\bar{1}}) = 0$$

is associative $((ab | c) = (a | bc)$ for all $a, b, c \in K$). Thus,

$$(1.2) \quad (ab | c) = (-1)^{\bar{a}(\bar{b}+\bar{c})}(bc | a) = (-1)^{\bar{c}(\bar{a}+\bar{b})}(ca | b)$$

for all $a, b, c \in K_{\bar{0}} \cup K_{\bar{1}}$. (Our convention is that $\bar{a} = i$ whenever $a \in K_i$.) Hence, since the supertrace, $\text{str}(\mathbb{L}_e) = \text{tr}_{K_{\bar{0}}}(\mathbb{L}_e) - \text{tr}_{K_{\bar{1}}}(\mathbb{L}_e)$, of the left multiplication operator \mathbb{L}_e of e on K is 0, it follows that

$$\begin{aligned} \mathbb{L}_{K_i} &= \{ \phi \in \text{End}_F(K)_i \mid (\phi(u) | v) \\ &= (-1)^{i\bar{u}}(u | \phi(v)) \quad \forall u, v \in K, \text{ and } \text{str}(\phi) = 0 \} \end{aligned}$$

for $i = \bar{0}, \bar{1}$. Moreover, $[\mathbb{L}_K, \mathbb{L}_K] = \text{Der } K = \mathfrak{osp}(K) \cong \mathfrak{osp}(1, 2)$.

Proposition 1.3. *For all $u, v, w \in K_{\bar{0}} \cup K_{\bar{1}}$,*

$$(1.4) \quad (uv)w = (u | v)w + \frac{1}{2} \left(u(v | w) + (-1)^{\bar{u}\bar{v}}v(u | w) \right).$$

Proof. Observe first that the expression in (1.4) is supersymmetric in u and v . If $u = v = e$, then (1.4) reduces to $ew = \frac{1}{2}w + (e | w)e$, which is valid for all homogeneous elements w . If $u = e$ and $v \in K_{\bar{1}}$, then (1.4) becomes

$$\frac{1}{2}vw = \frac{1}{2} \left(e(v | w) + v(e | w) \right),$$

which is true. Finally, if $u, v \in K_{\bar{1}}$, then $uv = (u | v)e$, and in this case, (1.4) says

$$(u | v)ew = (u | v)w + \frac{1}{2} \left(u(v | w) + (-1)^{\bar{u}\bar{v}}v(u | w) \right).$$

For $w = e$ this reduces to $(u | v)e = (u | v)e$. If instead $w \in K_{\bar{1}}$, then (1.4) holds because $(u | v)w + (v | w)u + (w | u)v = 0$, as the expression on the left is a skew-symmetric trilinear map on a 2-dimensional vector space (namely $K_{\bar{1}}$). \square

Corollary 1.5. *For all $u, v \in K_{\bar{0}} \cup K_{\bar{1}}$,*

$$(1.6) \quad [L_u, L_v] = \frac{1}{2} \left(u(v | -) - (-1)^{\bar{u}\bar{v}}v(u | -) \right) \quad \text{and}$$

$$(1.7) \quad L_u \circ L_v = (u | v) \mathbb{I} + \frac{3}{2} \left(u(v | -) + (-1)^{\bar{u}\bar{v}}v(u | -) \right),$$

where $\phi \circ \psi = \phi\psi + (-1)^{\bar{\phi}\bar{\psi}}\psi\phi$ for all homogeneous $\phi, \psi \in \text{End}_F(K)$.

Proof. From (1.4) we have

$$\begin{aligned} u(vw) - (-1)^{\bar{u}\bar{v}}v(uw) &= u(v | w) + \frac{1}{2}\left((u | v)w + (-1)^{\bar{u}\bar{v}}v(u | w)\right) \\ &\quad - (-1)^{\bar{u}\bar{v}}\left(v(u | w) + \frac{1}{2}\left((v | u)w + (-1)^{\bar{u}\bar{v}}(v | w)u\right)\right) \\ &= \frac{1}{2}\left(u(v | w) - (-1)^{\bar{u}\bar{v}}v(u | w)\right), \end{aligned}$$

which demonstrates (1.6). The verification of (1.7) is similar. □

2. THE JORDAN SUPERALGEBRA J

In this section we consider the superalgebra

$$J = F1 \oplus (K \otimes K)$$

with unit element 1 and with product defined by

$$(2.1) \quad (a \otimes b)(c \otimes d) = (-1)^{\bar{b}\bar{c}}\left(ac \otimes bd - \frac{3}{4}(a | c)(b | d)1\right)$$

for homogeneous elements $a, b, c, d \in K$. (All tensor products are over F unless specified otherwise.) By its definition J is a (super-)commutative superalgebra. Our aim is to show that J is a Jordan superalgebra. First we establish some properties of the multiplication.

Proposition 2.2. *For homogeneous elements $a, b, c, d \in K$,*

$$(2.3) \quad \begin{aligned} [L_{a \otimes b}, L_{c \otimes d}](1) &= 0, \\ [L_{a \otimes b}, L_{c \otimes d}]|_{K \otimes K} &= (-1)^{\bar{b}\bar{c}}\frac{1}{2}\left([L_a, L_c] \otimes (b | d) I + (a | c) I \otimes [L_b, L_d]\right). \end{aligned}$$

Proof. First let us remark that

$$(\phi \otimes \psi)(u \otimes v) = (-1)^{\bar{\psi}\bar{u}}\phi(u) \otimes \psi(v)$$

for homogeneous $\phi, \psi \in \text{End}_F(K)$, $u, v \in K$. The first equation in (2.3) is clear by supercommutativity. As for the second, note that

$$(2.4) \quad \begin{aligned} [L_{a \otimes b}, L_{c \otimes d}](u \otimes v) &= (-1)^{\bar{d}\bar{u}}(a \otimes b)\left(cu \otimes dv - \frac{3}{4}(c | u)(d | v)1\right) \\ &\quad - (-1)^{(\bar{a}+\bar{b})(\bar{c}+\bar{d})}(-1)^{\bar{b}\bar{u}}(c \otimes d)\left(au \otimes bv - \frac{3}{4}(a | u)(b | v)1\right) \\ &= (-1)^{\bar{d}\bar{u}}(-1)^{\bar{b}(\bar{c}+\bar{u})}\left(a(cu) \otimes b(dv) - \frac{3}{4}(a | cu)(b | dv)1 - \frac{3}{4}a(c | u) \otimes b(d | v) \right. \\ &\quad \left. - (-1)^{\bar{a}\bar{c}+\bar{b}\bar{d}}\left(c(au) \otimes d(bv) - \frac{3}{4}(c | au)(d | bv)1 - \frac{3}{4}c(a | u) \otimes d(b | v)\right)\right). \end{aligned}$$

By associativity $(a \mid cu) = (-1)^{\bar{a}\bar{c}}(c \mid au)$ and $(b \mid dv) = (-1)^{\bar{b}\bar{d}}(d \mid bv)$, so that the $F1$ -component of (2.4) is 0. Moreover,

$$\begin{aligned} a(cu) \otimes b(dv) - (-1)^{\bar{a}\bar{c} + \bar{b}\bar{d}} c(au) \otimes d(bv) \\ = \frac{1}{2} \left([\mathbf{L}_a, \mathbf{L}_c](u) \otimes (\mathbf{L}_b \circ \mathbf{L}_d)(v) + (\mathbf{L}_a \circ \mathbf{L}_c)(u) \otimes [\mathbf{L}_b, \mathbf{L}_d](v) \right), \end{aligned}$$

and by (1.6),

$$\begin{aligned} a(c \mid u) \otimes b(d \mid v) - (-1)^{\bar{a}\bar{c} + \bar{b}\bar{d}} c(a \mid u) \otimes d(b \mid v) \\ = \frac{1}{2} \left(a(c \mid u) - (-1)^{\bar{a}\bar{c}} c(a \mid u) \right) \otimes \left(b(d \mid v) + (-1)^{\bar{b}\bar{d}} d(b \mid v) \right) \\ + \left(a(c \mid u) + (-1)^{\bar{a}\bar{c}} c(a \mid u) \right) \otimes \left(b(d \mid v) - (-1)^{\bar{b}\bar{d}} d(b \mid v) \right) \\ = [\mathbf{L}_a, \mathbf{L}_c](u) \otimes \left(b(d \mid v) + (-1)^{\bar{b}\bar{d}} d(b \mid v) \right) \\ + \left(a(c \mid u) + (-1)^{\bar{a}\bar{c}} c(a \mid v) \right) \otimes [\mathbf{L}_b, \mathbf{L}_d](v). \end{aligned}$$

Therefore, it follows from (1.7) that

$$\begin{aligned} [\mathbf{L}_{a \otimes b}, \mathbf{L}_{c \otimes d}]|_{K \otimes K} \\ = (-1)^{\bar{b}\bar{c}} \left([\mathbf{L}_a, \mathbf{L}_c] \otimes \left(\frac{1}{2} \mathbf{L}_b \circ \mathbf{L}_d - \frac{3}{4} \left(b(d \mid -) + (-1)^{\bar{b}\bar{d}} d(b \mid -) \right) \right) \right. \\ \left. + \left(\frac{1}{2} \mathbf{L}_a \circ \mathbf{L}_c - \frac{3}{4} \left(a(c \mid -) + (-1)^{\bar{a}\bar{c}} c(a \mid -) \right) \right) \otimes [\mathbf{L}_b, \mathbf{L}_d] \right) \\ = (-1)^{\bar{b}\bar{c}} \frac{1}{2} \left([\mathbf{L}_a, \mathbf{L}_c] \otimes (b \mid d) \mathbf{I} + (a \mid c) \mathbf{I} \otimes [\mathbf{L}_b, \mathbf{L}_d] \right), \end{aligned}$$

as desired. \square

Corollary 2.5. For $a, b, c, d \in K_{\bar{0}} \cup K_{\bar{1}}$, $[\mathbf{L}_{a \otimes b}, \mathbf{L}_{c \otimes d}]$ is a superderivation of J .

Proof. Because K is a Jordan superalgebra, for $a, b \in K_{\bar{0}} \cup K_{\bar{1}}$, the mapping $[\mathbf{L}_a, \mathbf{L}_b]$ is a superderivation of K . By the associativity of (\mid) , it is super-skew-symmetric relative to (\mid) . Consequently, from (2.3) we can conclude that $[\mathbf{L}_{a \otimes b}, \mathbf{L}_{c \otimes d}]$ is a superderivation of J . \square

Theorem 2.6. $J = F1 \oplus (K \otimes K)$ is a Jordan superalgebra if $\text{char } F \neq 2$.

Proof. A Jordan algebra over a field F of characteristic not 2 or 3 is characterized by commutativity and the identity $\sum_{\text{cyclic}} [\mathbf{L}_{uv}, \mathbf{L}_w] = 0$. Hence a Jordan superalgebra over F is characterized by supercommutativity and

$$\sum_{\text{supercyclic}} [\mathbf{L}_{uv}, \mathbf{L}_w] = [\mathbf{L}_{uv}, \mathbf{L}_w] + (-1)^{\bar{u}(\bar{v} + \bar{w})} [\mathbf{L}_{vw}, \mathbf{L}_u] + (-1)^{(\bar{u} + \bar{v})\bar{w}} [\mathbf{L}_{wu}, \mathbf{L}_v] = 0.$$

In the supercommutative algebra J , we have from (1.2) and (2.3) that for all homogeneous $a, b, c, d, u, v \in K$,

$$\begin{aligned} & \sum_{\text{supercyclic}} [\mathbf{L}_{(a \otimes b)(c \otimes d)}, \mathbf{L}_{u \otimes v}] \\ &= \sum_{\text{supercyclic}} (-1)^{\bar{b}\bar{c}} [\mathbf{L}_{ac \otimes bd}, \mathbf{L}_{u \otimes v}] \\ &= \sum_{\text{supercyclic}} (-1)^{\bar{b}(\bar{c} + \bar{u}) + \bar{d}\bar{v}} \frac{1}{2} \left([\mathbf{L}_{ac}, \mathbf{L}_u] \otimes (bd \mid v) \mathbf{I} + (ac \mid u) \mathbf{I} \otimes [\mathbf{L}_{bd}, \mathbf{L}_v] \right) \\ &= (-1)^{\bar{b}(\bar{c} + \bar{u}) + \bar{d}\bar{u}} \frac{1}{2} \left\{ \left(\sum_{\text{supercyclic}} [\mathbf{L}_{ac}, \mathbf{L}_u] \right) \otimes (bd \mid v) \mathbf{I} \right. \\ & \qquad \qquad \qquad \left. + (ac \mid u) \mathbf{I} \otimes \left(\sum_{\text{supercyclic}} [\mathbf{L}_{bd}, \mathbf{L}_v] \right) \right\} \\ &= 0. \end{aligned}$$

Thus, J is a Jordan superalgebra whenever the field has characteristic different from 2 or 3.

If R is the ring of fractions $R = S^{-1}\mathbb{Z}$ relative to the multiplicative set of integers $S = \{2^n \mid n = 0, 1, \dots\}$, we could have defined K and J over R as above. (Denote the result by J_R .) Then $J_{\mathbb{Q}} = \mathbb{Q} \otimes_R J_R$ is a Jordan superalgebra over the rationals \mathbb{Q} , and so is J_R over R . But if F is a field having characteristic not 2, then $J = F \otimes_R J_R$ is a Jordan superalgebra over F . □

From the computations and results above, it is easy to draw the following conclusions about the Jordan superalgebra J . Here we use the associator $(u, v, w) = (uv)w - u(vw)$ in J in the statements.

Proposition 2.7. *For the Jordan superalgebra $J = F1 \oplus (K \otimes K)$:*

- (a) $(J, J, J) = K \otimes K$.
- (b) $J = F1 \oplus (J, J, J)$.
- (c) *If char $F = 3$, then $K \otimes K = (J, J, J)$ is an ideal of J .*
- (d) $e_1 := -\frac{1}{2}1 + 2e \otimes e$ and $e_2 := \frac{3}{2}1 - 2e \otimes e$ are orthogonal idempotents in J . Relative to e_1 the Peirce spaces $J^{(\lambda)} := \{u \in J \mid e_1 u = \lambda u\}$ are given by

$$J^{(1)} = Fe_1, \quad J^{(0)} = Fe_2 \oplus (K_{\bar{1}} \otimes K_{\bar{1}}), \quad J^{(\frac{1}{2})} = (e \otimes K_{\bar{1}}) \oplus (K_{\bar{1}} \otimes e).$$

Thus, $J_{\bar{0}} = J^{(1)} \oplus J^{(0)}$ and $J_{\bar{1}} = J^{(\frac{1}{2})}$. The space $J^{(0)}$ is the Jordan algebra of a nondegenerate symmetric bilinear form, and e_2 is its unit element.

Theorem 2.8. (a) *If char $F \neq 2, 3$, $J = F1 \oplus (K \otimes K)$ is a simple Jordan superalgebra.*

- (b) *If char $F = 3$, $(J, J, J) = K \otimes K$ is a simple Jordan superalgebra.*
- (c) *Der $J = \text{Inder } J$ and it is isomorphic to $\mathfrak{osp}(1, 2) \oplus \mathfrak{osp}(1, 2)$.*

Proof. We begin with (c). Any superderivation of J annihilates $F1$ and leaves invariant $(J, J, J) = K \otimes K$. Therefore, using the natural identifications and the fact that $\text{Der } K = \text{Inder } K \cong \mathfrak{osp}(1, 2)$, we have by (2.3) that

$$(2.9) \quad \text{Der } K \oplus \text{Der } K \cong (\text{Der } K \otimes \mathbf{I}) \oplus (\mathbf{I} \otimes \text{Der } K) = \text{Inder } J.$$

Now one may check that $\text{Inder } J = \text{Der } J$ as follows: First for any $D \in (\text{Der } J)_{\bar{0}}$, $D|_{J_{\bar{0}}} \in \text{Der}(J_{\bar{0}})$. Since $J_{\bar{0}}$ is a direct sum of a 1-dimensional ideal $J^{(1)} = Fe_1$ and the Jordan algebra $J^{(0)} = Fe_2 \oplus (K_{\bar{1}} \otimes K_{\bar{1}})$ of a bilinear form, $\text{Der}(J_{\bar{0}}) = \text{Inder } J_{\bar{0}}$. Thus, there is a $\tilde{D} \in (\text{Inder } J)_{\bar{0}}$ with $(D - \tilde{D})(J_{\bar{0}}) = 0$. It follows easily that $D = \tilde{D}$. Now suppose $E \in (\text{Der } J)_{\bar{1}}$. Then there is a $\tilde{E} \in (\text{Inder } J)_{\bar{1}}$ with $E(e_1) = \tilde{E}(e_1)$ because $(\text{Inder } J)_{\bar{1}}e_1 = J_{\bar{1}}$. For any $z \in J_{\bar{1}}$, $e_1z = \frac{1}{2}z$, so $(E - \tilde{E})(z) = 2(E - \tilde{E})(e_1z) = 2e_1(E - \tilde{E})(z)$. Thus, $(E - \tilde{E})(z) \in J_{\bar{0}} \cap J^{(\frac{1}{2})} = 0$, so that $(E - \tilde{E})(J_{\bar{1}}) = 0$. As a consequence, $E = \tilde{E}$ and $(\text{Der } J)_{\bar{1}} = (\text{Inder } J)_{\bar{1}}$.

Since $\text{Der } K$ acts irreducibly on K , so does $\text{Der } J = \text{Inder } J$ on $K \otimes K = (J, J, J)$, which is an ideal of J if and only if the characteristic of F is 3. From this (a) and (b) follow. \square

The Lie multiplication superalgebra of J . To compute the Lie multiplication superalgebra of J we need some preliminaries. Let H be a unital Jordan superalgebra with a normalized trace, that is, a linear map $\mathfrak{t} : H \rightarrow F$ such that $\mathfrak{t}(1) = 1$ and $\mathfrak{t}((H, H, H)) = 0$. Assume that $(H, H, H) \neq 0$. Let $H_0 = \ker \mathfrak{t}$. Then for any $x, y \in H_0$, $xy = \mathfrak{t}(xy)1 + x * y$, where $x * y = xy - \mathfrak{t}(xy)1$. The *Lie multiplication superalgebra* is the Lie superalgebra spanned by \mathbf{L}_H , which is a direct sum of the central ideal $F\mathbf{1}$ and the ideal $\mathfrak{L}_0(H) = \mathbf{L}_{H_0} \oplus [\mathbf{L}_{H_0}, \mathbf{L}_{H_0}]$. For simplicity, let us adopt the notation $S = \text{Inder } H$ and $T = H_0$. Then $\mathfrak{L}_0(H)$ is isomorphic to the Lie superalgebra $\mathfrak{L} = S \oplus T$ with multiplication determined by

- (1) the natural Lie multiplication on S ,
- (2) the natural action of S on T , that is, $[s, t] = s(t)$ for any $s \in S$ and $t \in T$, and
- (3) $[t_1, t_2] = [\mathbf{L}_{t_1}, \mathbf{L}_{t_2}] \in S$, for any $t_1, t_2 \in T$.

Also, given a nonzero scalar $\nu \in F$, let \mathfrak{L}_ν be the Lie superalgebra defined over \mathfrak{L} , but with the new bracket given by

- (i) $[x, y]^\nu = [x, y]$, for x, y both in S or one in S and the other in T ,
- (ii) $[x, y]^\nu = \nu[x, y]$, for $x, y \in T$.

Lemma 2.10. *Under the hypotheses above, if $\text{Hom}_S(T \otimes T, T)$ is spanned by $u \otimes v \mapsto u * v$ and if any automorphism of the superalgebra $(T, *)$ is an isometry of the supersymmetric bilinear form $(u, v) \mapsto \mathfrak{t}(uv)$, then for any $0 \neq \mu, \nu \in F$, \mathfrak{L}_μ is isomorphic to \mathfrak{L}_ν if and only if $\mu^{-1}\nu \in F^2$.*

Proof. It is enough to apply the ideas of ([BDE], proof of Prop. 4.2). We include the argument for completeness.

First, since any isomorphism $\mathfrak{L}_\mu \rightarrow \mathfrak{L}_\nu$ is also an isomorphism $\mathfrak{L} \rightarrow \mathfrak{L}_{\mu^{-1}\nu}$, it is enough to assume $\mu = 1$. Suppose $\Phi : \mathfrak{L} \rightarrow \mathfrak{L}_\nu$ is an isomorphism. Then the map $T \otimes T \rightarrow T$ defined by $u \otimes v \mapsto \Phi^{-1}(\Phi(u) * \Phi(v))$ is S -invariant. Consequently, there exists a nonzero scalar $\alpha \in F$ such that $\alpha\Phi(u) * \Phi(v) = \Phi(u * v)$ for any $u, v \in T$. Then $\psi : T \rightarrow T$ given by $\psi(u) = \alpha\Phi(u)$ is an automorphism of $(T, *)$ which, by hypothesis, extends by means of $\psi(1) = 1$ to an automorphism of the Jordan superalgebra H . Now the map $\Psi : \mathfrak{L} \rightarrow \mathfrak{L}$, such that $\Psi(s) = \psi s \psi^{-1}$ (composition of maps) for $s \in S$, and $\Psi(t) = \psi(t)$ for $t \in T$ is an automorphism of \mathfrak{L} .

Furthermore, $\tilde{\Phi} = \Phi\Psi^{-1} : \mathfrak{L} \rightarrow \mathfrak{L}_\nu$ is an isomorphism such that $\tilde{\Phi}(t) = \Phi(\psi^{-1}(t)) = \alpha^{-1}t$ for $t \in T$. Hence, for any $t_1, t_2, t_3 \in T$,

$$\begin{aligned} \tilde{\Phi}([t_1, t_2]) &= [\tilde{\Phi}(t_1), \tilde{\Phi}(t_2)]^\nu = \alpha^{-2}\nu[t_1, t_2], \\ \alpha^{-1}[[t_1, t_2], t_3] &= \tilde{\Phi}([[t_1, t_2], t_3]) \\ &= [[\tilde{\Phi}(t_1), \tilde{\Phi}(t_2)]^\nu, \tilde{\Phi}(t_3)]^\nu \\ &= \alpha^{-3}\nu[[t_1, t_2], t_3], \end{aligned}$$

so that $\nu = \alpha^{-2}$ as required. (Note $[[T, T], T] \neq 0$ because $(H, H, H) \neq 0$.)

The converse is clear, since the map Φ given by $\Phi(s) = s$ for $s \in S$ and $\Phi(t) = \alpha t$ for $t \in T$ provides an isomorphism $\mathfrak{L} \rightarrow \mathfrak{L}_{\alpha^2}$. \square

In order to apply the last result to our 10-dimensional Jordan superalgebra J , notice first that J is endowed with a normalized trace \mathfrak{t} such that $\mathfrak{t}(1) = 1$ and $\mathfrak{t}((J, J, J)) = 0$, as $(J, J, J) = K \otimes K$ does not intersect $F1$. Moreover, the associated bilinear form is given by $\mathfrak{t}((a \otimes b)(c \otimes d)) = -(-1)^{\bar{b}\bar{c}}\frac{3}{4}(a | c)(b | d)$ for any homogeneous $a, b, c, d \in K$, because of (2.1). Then all we need is the following

Lemma 2.11. *Any automorphism of $(K \otimes K, *)$ is an isometry of the bilinear trace form.*

Proof. Write $T = J_0 = K \otimes K$ and set $\langle a \otimes b | c \otimes d \rangle = (-1)^{\bar{b}\bar{c}}(a | c)(b | d)$ for any homogeneous elements $a, b, c, d \in K$. Then $T_{\bar{0}} = F(e \otimes e) \oplus (K_{\bar{1}} \otimes K_{\bar{1}})$. Suppose $\tilde{e} = e \otimes e$. Then the multiplication in $T_{\bar{0}}$ is given by

$$(2.12) \quad \tilde{e}^{*2} = \tilde{e}, \quad \tilde{e} * v = \frac{1}{4}v, \quad v * w = \langle v | w \rangle \tilde{e}$$

for all $v, w \in K_{\bar{1}} \otimes K_{\bar{1}}$.

Let φ be an automorphism of $(T, *)$ and let us first verify that $\varphi(\tilde{e}) = \tilde{e}$. This is clear if the characteristic is 3, because \tilde{e} is then the unit element of $T_{\bar{0}}$. Otherwise, \tilde{e} is the only idempotent of $T_{\bar{0}}$ such that the corresponding left multiplication operator has eigenvalue $\frac{1}{4}$ with multiplicity 4. This is because the eigenvalues of $L_{\alpha\tilde{e}+v}$ (for $v \in K_{\bar{1}} \otimes K_{\bar{1}}$, $\alpha \in F$) include $\frac{\alpha}{4}$ with multiplicity at least 3, as any $w \in K_{\bar{1}} \otimes K_{\bar{1}}$ which is orthogonal to v relative to $\langle | \rangle$ is a corresponding eigenvector. Moreover, $\tilde{e} + v$ is an idempotent only if $v = 0$.

Now, $\varphi(K_{\bar{1}} \otimes K_{\bar{1}}) \subseteq K_{\bar{1}} \otimes K_{\bar{1}}$ because $K_{\bar{1}} \otimes K_{\bar{1}} = \{v \in T_{\bar{0}} \mid \tilde{e} \notin Fv \text{ and } v*v \in F\tilde{e}\}$. Because of (2.12), $\varphi|_{T_{\bar{0}}}$ is an isometry of the restriction of $\langle | \rangle$ to $T_{\bar{0}}$.

Finally, $(K_{\bar{1}} \otimes e) \cup (e \otimes K_{\bar{1}}) = \{z \in T_{\bar{1}} \mid z*z \in F\tilde{e}\}$, and $K_{\bar{1}} \otimes e$ and $e \otimes K_{\bar{1}}$ are the only 2-dimensional subspaces contained in this union. Therefore, either φ preserves $K_{\bar{1}} \otimes e$ and $e \otimes K_{\bar{1}}$ or it interchanges them. Since $(u \otimes e) * (v \otimes e) = \frac{1}{2}\langle u \otimes e | v \otimes e \rangle \tilde{e}$ and $(e \otimes u) * (e \otimes v) = \frac{1}{2}\langle e \otimes u | e \otimes v \rangle \tilde{e}$ for any $u, v \in K_{\bar{1}}$, it follows that $\varphi|_{T_{\bar{1}}}$ is an isometry too. \square

We are ready to determine the Lie multiplication superalgebra $\mathfrak{L}_0(J)$ for $J = F1 \oplus (K \otimes K)$. When the underlying field F has characteristic 0, the next result is stated in [Sh] but without the assumption of -1 being a square in F .

Theorem 2.13. $\mathfrak{L}_0(J)_{-1} \cong \mathfrak{osp}(K \oplus K) \left(\cong \mathfrak{osp}(2, 4) \right)$. *In particular, $\mathfrak{L}_0(J)$ is a form of $\mathfrak{osp}(2, 4)$ and $\mathfrak{L}_0(J) \cong \mathfrak{osp}(2, 4)$ if and only if -1 is a square in F .*

Proof. On any vector superspace V equipped with a supersymmetric bilinear form $(|)$, the transformation $\gamma_{u,v} = u(v | -) - (-1)^{\bar{u}\bar{v}}v(u | -)$ belongs to $\mathfrak{osp}(V)$ for all homogeneous $u, v \in V$. Because of (2.3) and (1.7), $\text{Der } J$ is embedded in $\mathfrak{osp}(K \oplus K)$ (with the natural supersymmetric bilinear form on $K \oplus K$ that makes this an orthogonal sum) by means of

$$\Phi\left([\mathbb{L}_{a \otimes b}, \mathbb{L}_{c \otimes d}]\right) = (-1)^{\bar{b}\bar{c}} \frac{1}{4} \left(\gamma_{(a,0),(c,0)}(b | d) + (a | c) \gamma_{(0,b),(0,d)} \right) \in \mathfrak{osp}(K \oplus K).$$

This can be extended to a linear bijection $\Phi : \mathfrak{L}_0(J)_{-1} \rightarrow \mathfrak{osp}(K \oplus K)$ by defining

$$\Phi(\mathbb{L}_{a \otimes b}) = \frac{1}{2} \gamma_{(a,0),(0,b)}$$

for $a, b \in K$. Since

$$\begin{aligned} & \left[\gamma_{(a,0),(0,b)}, \gamma_{(c,0),(0,d)} \right] \\ &= \gamma_{\gamma_{(a,0),(0,b)}(c,0),(0,d)} + (-1)^{(\bar{a}+\bar{b})\bar{c}} \gamma_{(c,0),\gamma_{(a,0),(0,b)}(0,d)} \\ &= -(-1)^{\bar{a}\bar{b}} (-1)^{(\bar{a}+\bar{c})\bar{b}} (a | c) \gamma_{(0,b),(0,d)} + (-1)^{(\bar{a}+\bar{b})\bar{c}} \gamma_{(c,0),(a,0)}(b | d) \\ &= -(-1)^{\bar{b}\bar{c}} \left((a | c) \gamma_{(0,b),(0,d)} + \gamma_{(a,0),(c,0)}(b | d) \right), \end{aligned}$$

it follows that Φ is an isomorphism of Lie superalgebras. The last part is a consequence of Lemmata 2.10 and 2.11. \square

3. THE KAC SUPERALGEBRA

We may identify the simple Jordan superalgebra J with the 10-dimensional simple Jordan superalgebra K_{10} discovered by Kac [K] (see also [HK]) when the characteristic is not 2 or 3, and $(J, J, J) = K \otimes K$ with the 9-dimensional degenerate Kac superalgebra K_9 if $\text{char } F = 3$. This realization may be described explicitly as follows. In the notation of [MZ] or [RZ], we suppose $\{\check{e}, v_1, v_2, v_3, v_4, \check{f}, x_1, y_1, x_2, y_2\}$ is a basis of K_{10} with $(K_{10})_{\bar{0}} = \text{span}_F\{\check{e}, v_1, v_2, v_3, v_4, \check{f}\}$ and $(K_{10})_{\bar{1}} = \text{span}_F\{x_1, y_1, x_2, y_2\}$ and with multiplication given by

$$\begin{aligned} (3.1) \quad & \check{e}^2 = \check{e}, \quad \check{e}v_i = v_i, \quad v_1v_2 = 2\check{e} = v_3v_4, \quad \check{f}^2 = \check{f}, \\ & \check{e}x_j = \frac{1}{2}x_j = \check{f}x_j, \quad \check{e}y_j = \frac{1}{2}y_j = \check{f}y_j, \\ & v_1y_1 = x_2, \quad v_1y_2 = -x_1, \quad v_2x_1 = -y_2, \quad v_2x_2 = y_1, \\ & v_3x_2 = x_1, \quad v_3y_1 = y_3, \quad v_4x_1 = x_2, \quad v_4y_2 = y_1, \\ & x_jy_j = \check{e} - 3\check{f}, \quad x_1x_3 = v_1, \quad x_1y_2 = v_3, \quad x_2y_1 = v_4, \quad y_1y_2 = v_2 \end{aligned}$$

for $i = 1, 2, 3, 4$ and $j = 1, 2$. All other products are either 0 or obtained from those in (3.1) by supercommutativity. Then a straightforward computation using (3.1) shows that the assignment

$$\begin{aligned} \check{e} &\leftrightarrow \frac{3}{2}1 - 2e \otimes e = e_2 & x_1 &\leftrightarrow 4x \otimes e \\ \check{f} &\leftrightarrow -\frac{1}{2}1 + 2e \otimes e = e_1 & x_2 &\leftrightarrow -4e \otimes x \\ v_1 &\leftrightarrow -4x \otimes x & y_1 &\leftrightarrow -2y \otimes e \\ v_2 &\leftrightarrow -y \otimes y & y_2 &\leftrightarrow 2e \otimes y \\ v_3 &\leftrightarrow 2x \otimes y \\ v_4 &\leftrightarrow -2y \otimes x \end{aligned}$$

gives an isomorphism between K_{10} and J . Additionally, if $\text{char} F = 3$, then $\check{e} \leftrightarrow -2e \otimes e$, so that the isomorphism restricts to an isomorphism $K_9 \leftrightarrow K \otimes K$.

Concluding remarks. As the reader may have guessed, the actual process for deriving our results was the reverse path of the presentation above. First it was verified that $\mathfrak{g} := \text{Der } K_{10} = \mathfrak{osp}(1, 2) \oplus \mathfrak{osp}(1, 2)$, (compare Shtern [Sh]), and that as module for the Lie superalgebra \mathfrak{g} , $K_{10} = F1 \oplus (V \otimes V)$, where V is the natural 3-dimensional module for $\mathfrak{osp}(1, 2)$. The first copy of $\mathfrak{osp}(1, 2)$ operates on the first tensor slot of $V \otimes V$, and the second copy on the second slot. The spaces $\text{Hom}_{\mathfrak{osp}(1, 2)}(V \otimes V, F)$ and $\text{Hom}_{\mathfrak{osp}(1, 2)}(V \otimes V, \mathfrak{osp}(1, 2))$ are 1-dimensional. By identifying V with the tiny Kaplansky superalgebra K , we may describe the generators of these spaces nicely by the maps $u \otimes v \mapsto (u | v)$ and $u \otimes v \mapsto [\mathbf{L}_u, \mathbf{L}_v] \in \mathfrak{osp}(1, 2)$ for $u, v \in K$. From this, the multiplication in the Kac superalgebra K_{10} is almost uniquely determined in terms of K .

In [S1], Shestakov proposed quite a different realization of the superalgebra K_{10} in characteristic $\neq 2, 3$, but the same description of K_9 ; however, he has not published this work [S2].

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