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TOPOLOGICALLY TRANSVERSAL REVERSIBLE HOMOCLINIC SETS

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ABSTRACT. An R-reversible diffeomorphism on $\mathbb{R}^{2N}$ is studied possessing a hyperbolic fixed point. If the stable manifold of the hyperbolic fixed point and the fixed point set $\mathrm{Fix}\, R$ of R have a nontrivial local topological crossing, then an infinite number of R-symmetric periodic orbits of the diffeomorphism is shown. A perturbed problem is also studied by showing the relationship between a corresponding Melnikov function and the nontriviality of a local topological crossing of the set $\mathrm{Fix}\, R$ and the stable manifold for the perturbed diffeomorphism.

1. INTRODUCTION

Let $R : \mathbb{R}^{2N} \to \mathbb{R}^{2N}$ be a linear involution, i.e. $R^2 = I$, such that $\dim \mathrm{Fix}\ R = N$, where $\mathrm{Fix}\ R = \{x \in \mathbb{R}^{2N} \mid Rx = x\}$. Consider a C^1-smooth diffeomorphism $f : \mathbb{R}^{2N} \to \mathbb{R}^{2N}$ which is R-reversible, i.e. $Rf(x) = f^{-1}(Rx)$, $\forall x \in \mathbb{R}^{2N}$, and possessing an R-symmetric hyperbolic fixed point $p \in \mathrm{Fix}\ R$. Any subset of $\mathbb{R}^{2N}$ invariant under the action of R is called R-symmetric. Reversible diffeomorphisms naturally come from mechanics [5] as the time flow mappings of second order gradient differential equations. Let W_p^s, W_p^u be the global stable and unstable manifolds of p, respectively. Let $\widetilde{W}_p^s$ be an open subset of W_p^s which is a submanifold of $\mathbb{R}^{2N}$, i.e. the immersed and induced topologies on $\widetilde{W}_p^s$ coincide, and such that $\widetilde{W}_p^s \setminus \{p\} \cap \mathrm{Fix}\ R \neq \emptyset$, i.e. there is an R-symmetric point q homoclinic to p [10]. Since $RW_p^s = W_p^u$, we put $\widetilde{W}_p^u = R\widetilde{W}_p^s$. We also suppose the existence of a compact component $K \ni q$ of the set $\widetilde{W}_p^s \cap \mathrm{Fix}\ R$, that is a compact subset $K \subset \widetilde{W}_p^s \setminus \{p\} \cap \mathrm{Fix}\ R$ such that $q \in K$ and there exists an open bounded subset $U \subset \bar{U} \subset \mathbb{R}^{2N} \setminus \{p\}$ satisfying $U \cap \widetilde{W}_p^s \cap \mathrm{Fix}\ R = K$. By shrinking U, we can assume that $\overline{\widetilde{W}_p^s \cap U} = \widetilde{W}_p^s \cap \bar{U}$. We note that $\widetilde{W}_p^s \cap U$ is an orientable submanifold of $\mathbb{R}^{2N}$. Then we can define the local intersection number $\#\big(\widetilde{W}_p^s \cap U, \mathrm{Fix}\ R \cap U\big)$ of the stable manifold W_p^s and the plain $\mathrm{Fix}\ R$ in $U \subset \mathbb{R}^{2N}$ [7]. The main purpose of this note is to prove the following result.

Theorem 1.1. *If $\#\big(\widetilde{W}_p^s \cap U, \mathrm{Fix}\ R \cap U\big) \neq 0$, then there is an $\omega_0 \in \mathbb{N}$ such that for any $\mathbb{N} \ni \omega \geq \omega_0$ the diffeomorphism f possesses a 2ω-periodic orbit $\{x_n^\omega\}_{n\in\mathbb{Z}}$ such*

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that $Rx_n^\omega = x_{-n}^\omega$, $n \in \mathbb{Z}$. *Moreover,* $x_0^\omega \in \mathrm{Fix}\ R$ *is near the set* K*, while* $x_\omega^\omega \in \mathrm{Fix}\ R$ *is near the point* p.

When q is a transversal intersection of W_p^s and Fix R, then Theorem 1.1 is proved in [3], [4], [5], [9] and [10]. Then clearly $\#\big(\widetilde{W}_p^s \cap U, \mathrm{Fix}\ R \cap U\big) \neq 0$ for a small open neighbourhood U of q. Furthermore, we study the case where W_p^s and Fix R intersect on a compact manifold. Then we consider a C^2-smooth R-reversible perturbation of f. Associated to such a perturbation there is a Melnikov function. We show that if the Brouwer degree [6] of this Melnikov function is not zero, and the perturbation is small, then the perturbed stable manifold $W_{p,per}^s$ and the plain Fix R satisfy $\#\big(\widetilde{W}_{p,per}^s \cap U, \mathrm{Fix}\ R \cap U\big) \neq 0$ with the corresponding infinitely many R-symmetric periodic orbits of the perturbed diffeomorphism. Finally, we note that any accumulation point of the set $\{x_0^\omega\}_{\omega \geq \omega_0} \subset \mathrm{Fix}\ R$ from Theorem 1.1 is a starting point of an R-symmetric homoclinic orbit of f to p.

If p is a hyperbolic fixed point of f but not R-symmetric, then Rp is also a hyperbolic fixed point of f. If $q \in W_p^s \cap \mathrm{Fix}\ R$, then $q \in W_p^s \cap W_{Rp}^u$ [5], hence q lies on an R-symmetric heteroclinic orbit connecting p and Rp. Consequently, as for Theorem 1.1, we can prove the following result.

Theorem 1.2. *Suppose* f *has a non-R-symmetric hyperbolic fixed point* p*. If* W_p^s *and* W_p^u *meet* Fix R *locally topologically transversally, then* f *has an infinite number of R-symmetric periodic orbits with periods tending to infinity.*

We note that for the case $N = 1$ it is elementary to show that if p is a hyperbolic fixed point of f and W_p^s (or W_p^u) meets Fix R, then a local intersection number of W_p^s (or W_p^u) with Fix R is nonzero. Then Theorems 1.1 and 1.2 can be applied. Indeed, let $q \in W_p^s \cap \mathrm{Fix}\ R$ be the first intersection starting on W_p^s from p. Then the points $f^{-1}(q), f(q) \in W_p^s$ lie on the opposite half-plains separated by Fix R. Hence an open bounded connected part $\widetilde{W}_p^s$ of W_p^s such that $f^{-1}(q), f(q) \in \widetilde{W}_p^s$ topologically nontrivially crosses Fix R. Similarly for W_p^u.

The paper is finished with an example of a perturbed second order differential equation in $\mathbb{R}^2$ with a topologically transversal, but non-C^1-transversal, intersection of the stable manifold and Fix R.

2. Preliminary results

Let $(\cdot,\cdot)$ be an inner product on $\mathbb{R}^{2N}$. By following [10], we set

$$\langle x, y\rangle = \frac{1}{2}\big((x,y) + (Rx, Ry)\big).$$

Then $\langle Rx, Ry\rangle = \langle x, y\rangle$, $x, y \in \mathbb{R}^{2N}$, and so $||R|| = ||R^{-1}|| = 1$. Since $RK = K$, we can assume that $RU = U$.

For any $\xi \in \widetilde{W}_p^s \cap \bar{U}$ we set $\xi_n = f^n(\xi)$, $n \in \mathbb{Z}_+$, $\eta_n = f^n(\eta)$, $n \in \mathbb{Z}_-$, $\eta = R\xi$, where $\mathbb{Z}_+ = \{0, 1, 2, \cdots\}$ and $\mathbb{Z}_- = \{\cdots, -2, -1, 0\}$. We note that $\eta_{-n} = R\xi_n$, $n \in \mathbb{Z}_+$. Then the linear system

$$v_{n+1} = Df(\xi_n)v_n, \quad n \in \mathbb{Z}_+, \tag{1}$$

has an exponential dichotomy on $\mathbb{Z}_+$ [8], i.e. there are positive constants L, $\delta \in (0,1)$ and the orthogonal projection $P_\xi : \mathbb{R}^{2N} \to T_\xi \widetilde{W}_p^s$ such that the fundamental solution $V_\xi(n)$ of (1) satisfies the following:

$$||V_\xi(n)P_\xi V_\xi(m)^{-1}|| \le L\delta^{n-m}, \quad m \le n, \quad m,n \in \mathbb{Z}_+\,,$$

$$||V_\xi(n)(I-P_\xi)V_\xi(m)^{-1}|| \le L\delta^{m-n}, \quad n \le m, \quad m,n \in \mathbb{Z}_+\,.$$

We note that L and δ can be chosen to be independent of $\xi \in \widetilde{W}_p^s \cap \bar{U}$ [1].

By defining $Rv_n = w_{-n}$ in (1), the reversibility of f implies

$$w_{n+1} = Df(\eta_n)w_n, \quad n \in \mathbb{Z}_-, \quad n \neq 0\,. \tag{2}$$

Hence, the fundamental solution $W_\xi(n)$ of (2) is given by $W_\xi(n) = RV_\xi(-n)R^{-1}$, $n \in \mathbb{Z}_-$, and since $||R|| = ||R^{-1}|| = 1$, then (2) has an exponential dichotomy on $\mathbb{Z}_-$ with the constants L, δ and the orthogonal projection $I - Q_\eta$, where $Q_\eta = RP_\xi R^{-1}$. We note $\eta = R\xi$.

Now we fix $\omega \in \mathbb{N}$ large and put

$$J_\omega = \big\{ -\omega, -\omega+1, \cdots, \omega-1, \omega \big\}\,,$$

$$J_\omega^- = \big\{ -\omega, -\omega+1, \cdots, -1, 0 \big\}, \quad I_\omega^- = \big\{ -\omega, -\omega+1, \cdots, -1, \big\}\,,$$

$$J_\omega^+ = \big\{ 0, 1, \cdots, \omega-1, \omega \big\}, \quad I_\omega^+ = \big\{ 0, 1, \cdots, \omega-2, \omega-1 \big\}\,.$$

We note that the family $\{P_\xi \mid \xi \in \widetilde{W}_p^s \cap U\}$ is continuous on $\widetilde{W}_p^s \cap U$. In this paper, $\mathcal{R}L$ and $\mathcal{N}L$ denote, respectively, the range and the kernel of a linear operator L.

Theorem 2.1 ([1]). *There exist $\omega_0 \in \mathbb{N}$ and a constant $c > 0$ such that, for any $\omega \in \mathbb{N}$, $\omega \ge \omega_0$, and $\xi \in \widetilde{W}_p^s \cap U$, there exist unique $\{x_n^+(\omega,\xi)\}_{n \in J_\omega^+}$ and $\{x_n^-(\omega,\xi)\}_{n \in J_\omega^-}$ which satisfy $x_{n+1} = f(x_n)$ separately on I_ω^+ and I_ω^- such that*

$$P_\xi x_0^+(\omega,\xi) = P_\xi \xi, \quad Q_{R\xi} x_0^-(\omega,\xi) = Q_{R\xi} R\xi\,,$$

$$x_\omega^+(\omega,\xi) = x_{-\omega}^-(\omega,\xi)\,,$$

together with

$$\max_{n \in J_\omega^+} |x_n^+(\omega,\xi) - \xi_n| \le c\delta^\omega\,,$$

$$\max_{n \in J_\omega^-} |x_n^-(\omega,\xi) - \eta_n| \le c\delta^\omega\,.$$

Moreover, $x_n^\pm(\omega,\xi)$ are continuous with respect to ξ.

Proof. We study the nonlinear system

$$x_{n+1} = f(x_n) \tag{3}$$

near $\{\xi_n\}_{n \in J_\omega^+}$ and $\{\eta_n\}_{n \in J_\omega^-}$. By putting $x_n^+ = \xi_n + v_n$, $n \in J_\omega^+$ and $x_n^- = \eta_n + w_n$, $n \in J_\omega^-$, we get the systems

$$\begin{aligned} v_{n+1} &= Df(\xi_n)v_n + f(\xi_n + v_n) - f(\xi_n) - Df(\xi_n)v_n \\ &= Df(\xi_n)v_n + o(|v_n|), \quad n \in I_\omega^+, \end{aligned} \tag{4}$$

and

$$\begin{aligned} w_{n+1} &= Df(\eta_n)w_n + f(\eta_n + w_n) - f(\eta_n) - Df(\eta_n)w_n \\ &= Df(\eta_n)w_n + o(|w_n|), \quad n \in I_\omega^-\,. \end{aligned} \tag{5}$$

Since we are looking for solutions of equation (3) such that $x_\omega^+ = x_{-\omega}^-$, we add the boundary value conditions

$$v_\omega - w_{-\omega} = \eta_{-\omega} - \xi_\omega = O(\delta^\omega), \quad P_\xi v_0 = 0, \quad Q_{R\xi} w_0 = 0\,. \tag{6}$$

Let $v = (v_0, \dots, v_\omega) \in \mathbb{R}^{N(\omega+1)}$, $w = (w_{-\omega}, \dots, w_0) \in \mathbb{R}^{N(\omega+1)}$. To solve equations (4)–(6), we take the mapping $\Gamma_\omega : \widetilde{W}_p^s \cap \bar{U} \times \mathbb{R}^{2N(\omega+1)} \to \mathbb{R}^{2N(\omega+1)}$ defined by

$$\Gamma_\omega(\xi, v, w) = \begin{pmatrix} (v_{n+1} - f(\xi_n + v_n) + f(\xi_n))_{n \in I_\omega^+} \\ (w_{n+1} - f(\eta_n + w_n) + f(\eta_n))_{n \in I_\omega^-} \\ v_\omega - w_{-\omega} - (\eta_{-\omega} - \xi_\omega) \\ P_\xi v_0 \\ Q_{R\xi} w_0 \end{pmatrix}.$$

where $\begin{pmatrix} P_\xi v_0 \\ Q_{R\xi} w_0 \end{pmatrix}$ has to be meant as a vector in $\mathbb{R}^N = \mathcal{R}P_\xi \times \mathcal{R}Q_{R\xi}$. We already know that P_ξ and $Q_{R\xi}$ are continuous. Thus, for any fixed $\omega \geq \omega_0$, Γ_ω is continuous in (ξ, u, v) as well as its derivatives with respect to (v, w) when we take on $\mathbb{R}^{2N(\omega+1)}$ the maximum norm $\max_i\{|v_i|, |w_i|\}$. We have $\Gamma_\omega(\xi, 0, 0) = O(\delta^\omega)$ uniformly with respect to ξ and the linearized map $D_{(v,w)}\Gamma_\omega(\xi, 0, 0)$ has the form

$$D_{(v,w)}\Gamma_\omega(\xi, 0, 0) \begin{pmatrix} v \\ w \end{pmatrix} = \begin{pmatrix} (v_{n+1} - Df(\xi_n)v_n)_{n \in I_\omega^+} \\ (w_{n+1} - Df(\eta_n)w_n)_{n \in I_\omega^-} \\ v_\omega - w_{-\omega} \\ P_\xi v_0 \\ Q_{R\xi} w_0. \end{pmatrix}.$$

Arguing as in Lemma 2.1 of [1] or Lemma 2 of [2], the map $D_{(v,w)}\Gamma_\omega(\xi, 0, 0)$ is invertible and that its inverse is bounded uniformly with respect to ξ. Hence from the implicit function theorem we get that $c > 0$ and $\omega_0 \gg 1$ exist such that for $\omega \geq \omega_0$, the equation $\Gamma_\omega(\xi, v, w) = 0$ can be solved uniquely for (v, w) in a neighborhood of $(0, 0)$ in terms of (ξ, ω). Moreover $\max_i\{|v_i|, |w_i|\} < c\delta^\omega$, and the solution is continuous in ξ, for any fixed $\omega \geq \omega_0$. The proof is finished.

Since $Q_{R\xi} = RP_\xi R^{-1}$, $\eta_{-n} = R\xi_n$, $n \in \mathbb{Z}_+$, we see that the sequences given by $y_n^-(\omega, \xi) = Rx_{-n}^+(\omega, \xi)$, $n \in J_\omega^-$, $y_n^+(\omega, \xi) = Rx_{-n}^-(\omega, \xi)$, $n \in J_\omega^+$ also satisfy the statement of Theorem 2.1. The uniqueness of such orbits implies that

$$\begin{aligned} x_n^+(\omega, \xi) &= y_n^+(\omega, \xi) = Rx_{-n}^-(\omega, \xi), \quad n \in J_\omega^+, \\ x_n^-(\omega, \xi) &= y_n^-(\omega, \xi) = Rx_{-n}^+(\omega, \xi), \quad n \in J_\omega^-\,. \end{aligned}$$

Hence $Rx_n^\pm(\omega, \xi) = x_{-n}^\mp(\omega, \xi)$, $n \in J_\omega^\pm$. So the orbit of f in Theorem 2.1 is R-symmetric.

3. R-symmetric periodic orbits

In this section we prove Theorem 1.1. If $x_0^+(\omega, \xi) \in \operatorname{Fix} R$, then

$$x_0^+(\omega, \xi) = Rx_0^+(\omega, \xi) = x_0^-(\omega, \xi)\,.$$

Consequently, the orbit of Theorem 2.1 becomes an R-symmetric periodic orbit of f. Hence we have to solve the equation

$$(I - R)x_0^+(\omega, \xi) = 0, \quad \xi \in \widetilde{W}_p^s \cap U\,. \tag{7}$$

Let V be an open subset such that $K \subset V \subset \bar{V} \subset U$ and let ω_0 be as in Theorem 2.1. Note that the solution $x_0^+(\omega, \xi)$ is defined for $\xi \in \widetilde{W}_p^s \cap \bar{V}$ and

$$\#(\widetilde{W}_p^s \cap V, \text{Fix } R \cap V) = \#(\widetilde{W}_p^s \cap U, \text{Fix } R \cap U) \neq 0.$$

To solve (7), we put $F_\omega(\xi) = (I-R)x_0^+(\omega, \xi)$, $F_\omega : \widetilde{W}_p^s \cap \bar{V} \to R_- = \mathcal{R}(I-R)$. We note that $\dim R_- = \dim \widetilde{W}_p^s$.

Now we put F_ω into the homotopy

$$H_\omega : \widetilde{W}_p^s \cap \bar{V} \times [0,1] \to R_-,$$
$$H_\omega(\xi, \lambda) = \lambda F_\omega(\xi) + (1-\lambda)(I-R)\xi\,.$$

Theorem 2.1 gives

$$|H_\omega(\xi, \lambda) - (I-R)\xi| = |\lambda(F_\omega(\xi) - (I-R)\xi)| \leq c\delta^\omega\,.$$

Consequently, $H_\omega(\cdot, \lambda) \neq 0$ on the boundary $\partial(\widetilde{W}_p^s \cap V)$ for any $0 \leq \lambda \leq 1$. This gives for the Brouwer degree [6],

$$\deg\left(F_\omega, \widetilde{W}_p^s \cap V, 0\right) = \pm\#\left(\widetilde{W}_p^s \cap V, \text{Fix } R \cap V\right) \neq 0\,.$$

Summarizing, we see that $F_\omega(\xi) = 0$ has a solution $\xi \in \widetilde{W}_p^s \cap V$ for any $\omega \geq \omega_0$, where ω_0 is sufficiently large. This proves Theorem 1.1.

4. Perturbation theory

In this section, we consider a C^2-smooth perturbation $f(x, \varepsilon)$ of f, i.e. we suppose that $f(x,0) = f(x)$ and $Rf(x,\varepsilon) = f^{-1}(Rx, \varepsilon)$, $\forall x \in \mathbb{R}^{2N}$, ε small. Then Theorem 2.1 gives a C^1-mapping $x_0^+(\omega, \xi, \varepsilon)$ and we are led to the equation

$$(I-R)x_0^+(\omega, \xi, \varepsilon) = 0, \quad \xi \in \widetilde{W}_p^s \cap U\,.$$

By taking $\omega \to \infty$ in the above equation, we get $F(\xi, \varepsilon) = (I-R)x_0^+(\infty, \xi, \varepsilon) = 0$ which is precisely the equation of R-symmetric homoclinic solutions to the hyperbolic symmetric fixed point p_ε of $f(x,\varepsilon)$ near p [10]. We assume that this equation has a compact nondegenerate solution manifold, i.e.

(H1) There is an embedded compact C^2-smooth submanifold $\mathcal{M} \subset \widetilde{W}_p^s \backslash \{p\} \cap \text{Fix } R$ for an open subset $\widetilde{W}_p^s$ of W_p^s which is a submanifold of $\mathbb{R}^{2N}$ and such that $\dim \mathcal{N}(I-R)D_\xi x_0^+(\infty, \xi, 0) = \dim \mathcal{M}$ for any $\xi \in \mathcal{M}$. Furthermore, let $\mathcal{O} \subset \widetilde{W}_p^s \setminus \{p\}$ be an open bounded subset such that $\mathcal{M} \subset \mathcal{O}$. Then $\mathcal{O}$ can be oriented. We suppose that $\mathcal{M}$ is orientable embedded into $\mathcal{O}$.

We note that always $T_\xi\mathcal{M} \subset \mathcal{N}(I-R)D_\xi x_0^+(\infty, \xi, 0)$, $\forall \xi \in \mathcal{M}$ and $\dim \mathcal{M} = \dim T_\xi \mathcal{M}$, hence (H1) implies $T_\xi\mathcal{M} = \mathcal{N}(I-R)D_\xi x_0^+(\infty, \xi, 0)$, $\forall \xi \in \mathcal{M}$. Since $x_0^+(\infty, \xi, 0) = \xi$, we have $D_\xi x_0^+(\infty, \xi, 0)v = v$, $v \in T_\xi \widetilde{W}_p^s$. Hence

$$\mathcal{N}(I-R)D_\xi x_0^+(\infty, \xi, 0) = \text{Fix } R \cap T_\xi \widetilde{W}_p^s\,.$$

Now we take a tubular neighbourhood $\mathcal{V}$ of $\mathcal{M}$ in $\widetilde{W}_p^s$, i.e. any $\xi \in \mathcal{V}$ can be uniquely expressed as a pair (τ, v), where $\tau \in \mathcal{M}$ and $v \in T_\tau \widetilde{W}_p^s / T_\tau \mathcal{M} = T_\tau \widetilde{W}_p^s / (\text{Fix } R \cap T_\tau \widetilde{W}_p^s) = N_\tau$—the fiber of the normal vector bundle of $\mathcal{M}$ in $\widetilde{W}_p^s$, and $|v| < \triangle$ for some $\triangle > 0$. Hence we identify $\mathcal{V}$ with an open neighbourhood of the zero section of the normal vector bundle of $\mathcal{M}$ in $\widetilde{W}_p^s$. Let $S_\tau : \text{Fix}(-R) \to \mathcal{R}D_v F(\tau, 0, 0)$ be the orthogonal projection. We note that the assumption (H1) implies the invertibility of the linear mapping $D_vF(\tau,0,0): N_\tau \to \mathcal{R}D_vF(\tau,0,0)$, since $D_vF(\tau,0,0)w = (I-R)w$, $w \in N_\tau$.

From $F(\tau,0,0)=0$, we get $F(\tau,v,\varepsilon) = D_vF(\tau,0,0)v + \varepsilon D_\varepsilon F(\tau,0,0) + o(|v|) + o(\varepsilon)$. We consider the homotopy

$$H(\tau,v,\varepsilon,\lambda) = S_\tau\big(\lambda F(\tau,v,\varepsilon) + (1-\lambda)D_vF(\tau,0,0)v\big) + (I-S_\tau)\big(\lambda F(\tau,v,\varepsilon) + (1-\lambda)\varepsilon D_\varepsilon F(\tau,0,0)\big)\,.$$

Then $H(\tau,v,\varepsilon,0) = S_\tau D_vF(\tau,0,0)v + \varepsilon(I-S_\tau)D_\varepsilon F(\tau,0,0)$ and $H(\tau,v,\varepsilon,1) = F(\tau,v,\varepsilon)$.

Now we suppose

(H2) There is an open connected subset $\Omega \subset \mathcal{M}$ such that $B(\tau) \neq 0$, $\forall \tau \in \partial\Omega$, where $B(\tau) = (I-S_\tau)D_\varepsilon F(\tau,0,0)$.

Since $\mathcal{M}$ is orientable embedded into $\mathcal{O}$ and $\mathcal{O}$ is orientable, the tangent vector bundle $T\mathcal{M}$ and the normal vector bundle $\bigcup_{\tau\in\mathcal{M}} N_\tau$ are both orientable. Hence the vector bundle $\bigcup_{\tau\in\mathcal{M}}(I-R)N_\tau = \bigcup_{\tau\in\mathcal{M}}\mathcal{R}D_vF(\tau,0,0)$ is also orientable, because $D_vF(\tau,0,0) : N_\tau \to \mathcal{R}D_vF(\tau,0,0)$ is invertible. Since $S_\tau : \mathrm{Fix}(-R) \to \mathcal{R}D_vF(\tau,0,0)$ is the orthogonal projection and the vector bundle $\bigcup_{\tau\in\mathcal{M}}\mathrm{Fix}(-R) = \mathcal{M}\times\mathrm{Fix}(-R)$ is orientable, we get the orientability of the vector bundle $\bigcup_{\tau\in\mathcal{M}}\mathcal{R}(I-S_\tau)\mathrm{Fix}(-R)$. Since $B(\tau)$ is a section of this vector bundle, the following assumption makes sense [7].

(H3) $\deg(B(\tau),\Omega,0) \neq 0$.

According to (H2), there is an open connected bounded neighbourhood $U_1 \subset \mathcal{M}$ of $\bar{\Omega}$ such that $B(\lambda) \neq 0$, $\forall \tau \in U_1 \setminus \Omega$. Now we take an open subset $V_\varepsilon = \{(\tau,v) \in V \mid \tau \in U_1 \text{ and } |v| < |\varepsilon| r_1\}$ for a positive constant r_1 and $0 < |\varepsilon| < \triangle/r_1$. Since

$$S_\tau\big(\lambda F(\tau,v,\varepsilon) + (1-\lambda)D_vF(\tau,0,0)v\big) = S_\tau D_vF(\tau,0,0)v + o(|v|) + O(\varepsilon)$$

and $S_\tau D_vF(\tau,0,0) : N_\tau \to \mathcal{R}D_vF(\tau,0,0)$ is invertible, we get that

$$S_\tau\big(\lambda F(\tau,v,\varepsilon) + (1-\lambda)D_vF(\tau,0,0)v\big) \neq 0,$$

$\forall(\tau,v) \in V_\varepsilon$, $|v| = r_1|\varepsilon|$ for a r_1 sufficiently large and fixed. Furthermore, we have

$$(I-S_\tau)\big(\lambda F(\tau,v,\varepsilon) + (1-\lambda)\varepsilon D_\varepsilon F(\tau,0,0)\big) = \varepsilon(I-S_\tau)D_\varepsilon F(\tau,0,0) + o(|v|) + o(\varepsilon)$$

since $(I-S_\tau)D_vF(\tau,0,0) = 0$. Hence $(I-S_\tau)\big(\lambda F(\tau,v,\varepsilon) + (1-\lambda)\varepsilon D_\varepsilon F(\tau,0,0)\big) \neq 0$ for $(\tau,v) \in V_\varepsilon$, $|v| \leq r_1|\varepsilon|$ and $\tau \in U_1 \setminus \Omega$.

Summarizing, we see that $H(\tau,v,\varepsilon,\lambda) \neq 0$ for any $(\tau,v) \in \partial V_\varepsilon$, $\lambda \in [0,1]$ and $\varepsilon \neq 0$ sufficiently small. Consequently [6],

$$\deg(F,V_\varepsilon,0) = \deg(H(\cdot,\varepsilon,0),V_\varepsilon,0)\,,$$

where $H(\tau,v,\varepsilon,0) = S_\tau D_vF(\tau,v(\tau,\varepsilon),\varepsilon)v + \varepsilon(I-S_\tau)D_\varepsilon F(\tau,0,0)$. Since the linear map $S_\tau D_vF(\tau,v(\tau,\varepsilon),\varepsilon) : N_\tau \to \mathcal{R}D_vF(\tau,0,0)$ is invertible and U_1 is connected, we get

$$\deg(H(\cdot,\varepsilon,0),V_\varepsilon,0) = \pm\deg(B(\tau),\Omega,0) \neq 0\,.$$

This implies $\#(\widetilde{W}^s_{p_\varepsilon} \cap V_\varepsilon, \mathrm{Fix}\ R \cap V_\varepsilon) \neq 0$. Summarizing we get the following result.

Theorem 4.1. *Assume* (H1), (H2) *and* (H3). *Then there exists $\varepsilon_0 > 0$ such that for $0 < |\varepsilon| \leq \varepsilon_0$, it is nonzero the local intersection number of the plain* Fix R *and the stable manifold of the hyperbolic fixed point of the map $x_{n+1} = f(x_n,\varepsilon)$ which is located near the fixed point p of the map $x_{n+1} = f(x_n)$.*

Hence Theorem 1.1 and the assumptions of Theorem 4.1 imply an infinite number of R-symmetric periodic orbits of $f(x,\varepsilon)$ accumulating on R-symmetric homoclinic orbits of $f(x,\varepsilon)$ for any $0<|\varepsilon|\le\varepsilon_0$.

By taking $\Omega=\mathcal{M}$, we see [7] that assumption (H3) holds if the Euler characteristic $\chi\big(\bigcup_{\tau\in\mathcal{M}}\mathcal{R}(I-S_\tau)\mathrm{Fix}(-R)\big)$ is nonzero. Then Theorem 4.1 holds under assumption (H1) for any R-reversible C^2-smooth perturbation $f(x,\varepsilon)$.

If $f(x,\varepsilon)$ is C^3-smooth, then F is C^2-smooth. To solve $F(\tau,v,\varepsilon)=0$, we follow the standard way [1] by splitting it as $F(\tau,v,\varepsilon)=S_\tau F(\tau,v,\varepsilon)+(I-S_\tau)F(\tau,v,\varepsilon)$. By using the implicit function theorem, we can solve the equation $S_\tau F(\tau,v,\varepsilon)=0$ in v for ε small and $\tau\in\mathcal{M}$ to get the C^2-smooth solution $v=v(\tau,\varepsilon)=O(\varepsilon)$. Then we consider the bifurcation equation $C(\tau,\varepsilon)=(I-S_\tau)F(\tau,v(\tau,\varepsilon),\varepsilon)=0$. Clearly $C(\tau,\varepsilon)/\varepsilon\to B(\tau)$ in the C^1-topology on $\mathcal{M}$ as $\varepsilon\to 0$. Consequently, a simple zero τ_0 of $B(\tau)$, i.e. $B(\tau_0)=0$ and $DB(\tau_0)$ is nonsingular, implies the solvability of $C(\tau,\varepsilon)=0$ in τ near τ_0 for $\varepsilon\neq 0$ small. Summarizing we see that B is the Melnikov function for this problem, since its simple zero τ_0 ensures the bifurcation of an R-symmetric homoclinic orbit of $f(x,\varepsilon)$ to p_ε for $\varepsilon\neq 0$ small bifurcating from the R-symmetric homoclinic orbit of $f(x,0)=f(x)$ which starts from $\tau_0\in\mathcal{M}$.

5. An example

In this section, we present an example, but first we simplify the formula of $B(\tau)$. If $a\in\mathcal{R}\,(I-S_\tau)$, then $a\in\mathrm{Fix}(-R)$ and $a\perp(I-R)T_\tau\widetilde{W}_p^s$. Hence for any $w\in T_\tau\widetilde{W}_p^s$ we have

$$0=\langle a,(I-R)w\rangle=\langle a,w\rangle-\langle a,Rw\rangle=\langle a-Ra,w\rangle=2\langle a,w\rangle\,.$$

Furthermore, since $RT_\tau\widetilde{W}_p^s=T_{R\tau}\widetilde{W}_p^u=T_\tau\widetilde{W}_p^u$ and for any $w\in T_\tau\widetilde{W}_p^s$, we have

$$\langle a,Rw\rangle=-\langle Ra,Rw\rangle=-\langle a,w\rangle=0\,,$$

we see that $a\in\mathcal{R}\,(I-S_\tau)$ if and only if $a\perp(T_\tau\widetilde{W}_p^s+T_\tau\widetilde{W}_p^u+\mathrm{Fix}\ R)$. We note that $\mathrm{Fix}(-R)=(\mathrm{Fix}\ R)^\perp$, and $\frac{1}{2}(I-R):\mathbb{R}^{2N}\to\mathrm{Fix}(-R)$ and $\frac{1}{2}(I+R):\mathbb{R}^{2N}\to\mathrm{Fix}\ R$ are the orthogonal projections. Consequently, if $a_i(\tau), i=1,2,\cdots,\dim\mathcal{M}$, is a continuous orthonormal vector field such that $a_i(\tau)\perp(T_\tau\widetilde{W}_p^s+T_\tau\widetilde{W}_p^u+\mathrm{Fix}\ R)$, then the components of $B(\tau)$ are given by

$$B_i(\tau)=\langle a_i(\tau),(I-R)D_\epsilon x_0^+(\infty,\tau,0)\rangle=2\langle a_i(\tau),D_\epsilon x_0^+(\infty,\tau,0)\rangle\,.$$

Now we consider a perturbed second order differential equation

$$\ddot z=g(z)+\varepsilon h(z),\quad z\in\mathbb{R}^N\,, \tag{8}$$

where $g,h\in C^3(\mathbb{R}^N,\mathbb{R}^N)$, $g(0)=h(0)=0$. (8) has the form

$$\dot z_1=z_2,\quad \dot z_2=g(z_1)+\varepsilon h(z_1)\,. \tag{9}$$

Let $\phi(t,z_1,z_2,\varepsilon)$ be the flow of (9), then $f(x,\varepsilon)=\phi(T,x,\varepsilon)$, $x=(z_1,z_2)$ for a $T>0$. Here $R(z_1,z_2)=(z_1,-z_2)$ and $\mathrm{Fix}\ R=\{(z_1,0)\mid z_1\in\mathbb{R}^N\}$, $\mathrm{Fix}\,(-R)=\{(0,z_2)\mid z_2\in\mathbb{R}^N\}$. The inner product $\langle\cdot,\cdot\rangle$ is given by $\langle(z_1^1,z_2^1),(z_1^2,z_2^2)\rangle=(z_1^1,z_1^2)+(z_2^1,z_2^2)$, where $(\cdot,\cdot)$ is the usual inner product on $\mathbb{R}^N$. We assume that $p_\varepsilon=(0,0)$ is a hyperbolic equilibrium of (9). Let $\tau\in\mathrm{Fix}\ R\cap\widetilde{W}_p^s$. Then $\phi(t,\tau,0)=(z_1^\tau(t),z_2^\tau(t))$ with $z_1^\tau(t)$ even and $z_2^\tau(t)$ odd. $\phi(t,\tau,0)$ is a homoclinic solution of (9) with $\varepsilon=0$. The linearization of (9) for $\varepsilon=0$ along $\phi(t,\tau,0)$ has the form

$$\dot v=w,\quad \dot w=Dg(z_1^\tau(t))v\,. \tag{10}$$

It is clear that $T_\tau \widetilde{W}_p^{s(u)} = \{(v(0), w(0)) \mid v(t), w(t)$ are bounded solutions of (10) on $\mathbb{R}_{+(-)}\}$, respectively. According to [2], the condition $a \perp (T_\tau \widetilde{W}_p^s + T_\tau \widetilde{W}_p^u + \mathrm{Fix}\, R)$ is now equivalent to $a = (0, a_2)$ and $v_1(0) = 0$, $w_1(0) = a_2$, where $v_1(t)$ and $w_1(t)$ are the bounded solutions on $\mathbb{R}$ of the adjoint system of (10), i.e. w_1 is the even bounded solution on $\mathbb{R}$ of $\ddot{w}_1 = Dg(z_1^\tau(t))^* w_1$, $w_1(0) = a_2$. Furthermore, $D_\varepsilon x_0^+(\infty, \tau, 0) = (v_\tau(0), w_\tau(0)) \perp T_\tau \widetilde{W}_p^s$, where $v_\tau(t)$ and $w_\tau(t)$ are the bounded solutions on $\mathbb{R}_+$ of the equations

$$\dot{v}_\tau = w_\tau, \quad \dot{w}_\tau = Dg(z_1^\tau(t)) v_\tau + h(z_1^\tau(t))\,. \tag{11}$$

Consequently, the corresponding component of $B(\tau)$ to a derived above is given by

$$2\langle a, D_\varepsilon x_0^+(\infty, \tau, 0)\rangle = 2\langle (0, a_2), (v_\tau(0), w_\tau(0))\rangle = 2(a_2, w_\tau(0)) = 2(w_1(0), w_\tau(0))\,.$$

On the other hand, since (11) holds along with $\lim_{t\to+\infty} w_1(t) = 0$, we have

$$\int_0^\infty (h(z_1^\tau(t), w_1(t))\, dt = (w_1(0), w_\tau(0))\,. \tag{12}$$

To be more concrete, we consider the system

$$\ddot{x} = x - 2x(x^2 + y^2),\ \ddot{y} = y - 2y(x^2 + y^2) + \varepsilon x^4, \quad x, y \in \mathbb{R}\,. \tag{13}$$

(13) has for $\varepsilon = 0$ a homoclinic manifold $x_\tau(t) = \sin\tau r(t)$, $y_\tau(t) = \cos\tau r(t)$, $r(t) = \operatorname{sech} t$, which intersects Fix R in the circle $\mathcal{M} = (\sin\tau, \cos\tau, 0, 0)$. It is not difficult to observe that now assumption (H1) holds and $w_1(t) = (y_\tau(t), -x_\tau(t))$. By (12), the function $B(\tau)$ now has the form

$$B(\tau) = -2\int_0^\infty \sin^5\tau r^5(t)\, dt = -\frac{3}{8}\pi \sin^5\tau\,.$$

We note that $x_0(t)$, $y_0(t)$ are the even solutions of (13). The bifurcation equation $C(\tau, \varepsilon) = 0$ from Section 4 is now analytical. Hence $\tau = 0$ is its isolated solution for $\varepsilon \neq 0$ small and fixed. The Brouwer degree of $B(\tau)$ at $\tau = 0$ is -1, so Theorem 4.1 implies the following result.

Theorem 5.1. *The point* $(0, 1, 0, 0)$ *is an isolated topologically transversal intersection of* $W^s_{p_\varepsilon}$ *and* Fix R *for* (13) *with* $\varepsilon \neq 0$ *small. But this point is not a* C^1*-transversal intersection.*

Proof. To prove the non-C^1-transversal intersection, we consider a C^3-perturbation of (13) given by

$$\ddot{x} = x - 2x(x^2 + y^2),\ \ddot{y} = y - 2y(x^2 + y^2) + \varepsilon\phi_\delta(x), \quad x, y \in \mathbb{R}\,, \tag{14}$$

where $\delta > 0$ and $\phi_\delta(x) = 0$ if $|x| \leq \delta$, $\phi_\delta(x) = (x - \delta \operatorname{sgn} x)^4$ if $|x| \geq \delta$. We see that (14) has even homoclinics $x_\tau(t)$, $y_\tau(t)$ for $|\sin\tau| < \delta$ and any ε. Hence $(0, 1, 0, 0)$ is not an isolated reversible homoclinic point for the C^3-perturbation (14) with $\varepsilon \neq 0$, $\delta > 0$ small of the system (13). The proof is finished.

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