

LOCAL DUAL SPACES OF BANACH SPACES OF VECTOR-VALUED FUNCTIONS

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ABSTRACT. We show that $L_\infty(\mu, X^*)$ is a local dual of $L_1(\mu, X)$, and $L_1(\mu, X^*)$ is a local dual of $L_\infty(\mu, X)$, where X is a Banach space. A local dual space of a Banach space Y is a subspace Z of Y^* so that we have a local representation of Y^* in Z satisfying the properties of the representation of X^{**} in X provided by the principle of local reflexivity.

1. INTRODUCTION

The principle of local reflexivity shows that there is a close relation between a Banach space X and its second dual X^{**} from a finite dimensional point of view. This means that X can be considered “locally” as a dual of X^* .

In [6] we introduced the local dual spaces of X . These subspaces satisfy the thesis of the principle of local reflexivity in full force: X^* is finitely dual representable in any of its local dual spaces by means of ε -isometries that fix points. Moreover, in [5] we studied the polar property for subspaces Z of X^* as a test to check if X^* is finitely dual representable in Z . We observe that being a local dual is strictly stronger than satisfying the polar property.

It was proved in [6] that $\ell_1(X^*)$ is a local dual of $\ell_\infty(X)$, and $\ell_\infty(X)$ is a local dual of $\ell_1(X^*)$. Moreover, assuming the continuum hypothesis $2^\omega = \omega_1$, $C[0, 1]$ is a local dual of $L_1[0, 1]$ and $L_1[0, 1]$ is a local dual of $C[0, 1]$. Also every separable space X with the metric approximation property has a local dual with the metric approximation property.

Here we describe local dual spaces for some spaces of vector-valued functions. Let μ be a finite measure. We show that $L_1(\mu, X^*)$ is a local dual of $L_\infty(\mu, X)$, and $L_\infty(\mu, X^*)$ is a local dual of $L_1(\mu, X)$. Note that $L_1(\mu, X)^*$ can be described as a space of X^* -valued, weak*-measurable functions [2], and $L_\infty(\mu, X)^*$ can be described as a direct sum of $L_1(\mu, X^*)$ and a certain subspace of singular elements [1]. But these descriptions are not always manageable, especially in the latter case. Previously, Díaz [3] studied the duality between $L_\infty(\mu, X)$ and $L_1(\mu, X^*)$, but our results are stronger and our proofs are more natural.

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In the paper X and Y are Banach spaces, B_X the closed unit ball of X , S_X the unit sphere of X , and X^* the dual of X . We identify X with a subspace of X^{**} . By a *subspace* we always mean a closed subspace. We denote by $\mathcal{B}(X, Y)$ the space of all (bounded linear) operators from X into Y . Given $T \in \mathcal{B}(X, Y)$, $R(T)$ is the range of T , and T^* is the conjugate operator of T .

Given a number $0 < \varepsilon < 1$, an operator $T \in \mathcal{B}(X, Y)$ is an ε -isometry if it satisfies $(1 + \varepsilon)^{-1} < \|Tx\| < 1 + \varepsilon$ for all $x \in S_X$. A space X is said to be *finitely representable in Y* if for each $\varepsilon > 0$ and each finite dimensional subspace M of X there is an ε -isometry $T : M \rightarrow Y$.

2. MAIN RESULTS

Local dual spaces of Banach spaces were introduced in [6, Definition 2.1] as follows:

Definition 1. A subspace Z of X^* is said to be a *local dual space of X* if for every couple of finite dimensional subspaces F of X^* and G of X , and every number $0 < \varepsilon < 1$, there is an ε -isometry $L : F \rightarrow Z$ satisfying the following conditions:

- (a) $\langle Lf, x \rangle = \langle f, x \rangle$ for all $x \in G$ and all $f \in F$, and
- (b) $L(f) = f$ for all $f \in F \cap Z$.

Obviously, X^* is a local dual of X . Moreover, the principle of local reflexivity establishes that every isometric predual is a local dual of X^* , and the principle of local reflexivity for ultrapowers [7, Theorem 7.3] establishes that $(X^*)_{\mathfrak{U}}$ is a local dual of $X_{\mathfrak{U}}$. Observe that we do not have a description of $(X_{\mathfrak{U}})^*$.

Given a couple of subspaces Z of X^* and G of X , an operator $L : G \rightarrow X^{**}$ is said to be an *extension operator* if $Lf|_Z = f$, for every $f \in G$. The following characterization of the local dual spaces of a Banach space will be the key to proving our results.

Theorem 2 ([6, Theorem 2.5]). *A subspace Z of X^* is a local dual of X if (and only if) for every couple of finite dimensional subspaces F of X^* and G of X , and every $0 < \varepsilon < 1$, there is an ε -isometry $L : F \rightarrow Z$ such that*

- (a) $|\langle Lf, x \rangle - \langle f, x \rangle| < \varepsilon \|f\| \|x\|$, for all $x \in G$ and all $f \in F$, and
- (b) $\|L(f) - f\| \leq \varepsilon \|f\|$ for all $f \in F \cap Z$.

The relation “being a local dual” is symmetric. Let \hat{x} denote the vector $x \in X$ as an element of X^{**} . We define a map $\Upsilon : X \rightarrow Z^*$ by $\Upsilon(x) = \hat{x}|_Z$.

Proposition 3 ([6, Proposition 2.10(b)]). *Let Z be a local dual of X . Then $\Upsilon(X)$ is a local dual of Z which is isometric to X .*

Here is our main result.

Theorem 4. *Let μ be a finite measure. Then:*

- (a) $L_1(\mu, X^*)$ is a local dual of $L_\infty(\mu, X)$.
- (b) $L_\infty(\mu, X^*)$ is a local dual of $L_1(\mu, X)$.

Proof. The set $\mathcal{S}_1(\mu, X^*)$ of all functions $g = \sum_{n=1}^{\infty} \chi_{A_n} \otimes x_n^*$, where (A_n) is a disjoint sequence of measurable sets and (x_n^*) is a sequence in X^* so that $\sum_n \mu(A_n) \|x_n^*\| < \infty$, is dense in $L_1(\mu, X^*)$. Analogously, since each $f \in L_\infty(\mu, X)$ is essentially valued in a separable subspace of X , the set $\mathcal{S}_\infty(\mu, X)$ of all functions

$f = \sum_{n=1}^{\infty} \chi_{A_n} \otimes x_n$, where (A_n) is a disjoint sequence of measurable sets and (x_n) is a bounded sequence in X , is dense in $L_{\infty}(\mu, X)$.

(a) In order to apply Theorem 2, we fix $0 < \varepsilon < 1$ and finite dimensional subspaces F of $L_{\infty}(\mu, X)^*$ and G of $L_{\infty}(\mu, X)$. Without loss of generality we can assume that G is large enough so that, for every $\phi \in F$,

$$(1) \quad \|\phi\| \leq (1 + \varepsilon) \sup\{|\langle \phi, f \rangle| : f \in S_G\}.$$

Let $\{\phi_1, \dots, \phi_l\}$ be a basis for $F \cap L_1(\mu, X^*)$. We take $\{\phi_1^*, \dots, \phi_l^*\}$ in $L_{\infty}(\mu, X)^{**}$ so that $\langle \phi_i^*, \phi_j \rangle = \delta_{ij}$, for $i, j = 1, \dots, l$. Let $M = \sup \|\phi_i^*\|$. We select f_1, \dots, f_l in $S_1(\mu, X^*)$ so that $\|\phi_j - f_j\| < \varepsilon/(2lM)$, for $j = 1, \dots, l$.

We define an operator T on $L_{\infty}(\mu, X)^*$ by

$$T(\phi) = \phi - \sum_{j=1}^l \langle \phi_j^*, \phi \rangle (\phi_j - f_j).$$

Then $\|I - T\| < \varepsilon/2$ and $T(\phi_j) = f_j$, for $j = 1, \dots, l$.

Analogously, let $\{h_1, \dots, h_k\}$ be a basis for G . We can select g_1, \dots, g_k in $S_{\infty}(\mu, X)$, and define an operator S on $L_{\infty}(\mu, X)$ such that $\|I - S\| < \varepsilon/2$ and $S(h_i) = g_i$, for $i = 1, \dots, k$.

We take a disjoint sequence (C_n) of measurable sets and sequences $(x_{i,n})_{n=1}^{\infty}$ in X and $(x_{j,n}^*)_{n=1}^{\infty}$ in X^* so that, for $i = 1, \dots, k$ and $j = 1, \dots, l$,

$$g_i = \sum_{n=1}^{\infty} \chi_{C_n} \otimes x_{i,n} \quad \text{and} \quad f_j = \sum_{n=1}^{\infty} \chi_{C_n} \otimes x_{j,n}^*.$$

We define a projection P on $L_{\infty}(\mu, X)$ by

$$P(f) = \sum_{n=1}^{\infty} \mu(C_n)^{-1} \chi_{C_n} \otimes \int_{C_n} f d\mu.$$

Since $\|P(f)\| = \sup_n \|\mu(C_n)^{-1} \chi_{C_n} \otimes \int_{C_n} f d\mu\| = \sup_n \|\mu(C_n)^{-1} \int_{C_n} f d\mu\| \leq \|f\|_{\infty}$, we get $\|P\| = 1$. Moreover $P(S(h)) = S(h)$, for every $h \in G$. Let P^* be the conjugate projection acting on $L_{\infty}(\mu, X)^*$. For each $\phi \in L_{\infty}(\mu, X)^*$, we define $(P^*\phi)_n \in X^*$ by

$$\langle (P^*\phi)_n, x \rangle = \langle \phi, \mu(C_n)^{-1} \chi_{C_n} \otimes x \rangle.$$

Then $P^*(\phi) = \sum_{n=1}^{\infty} \chi_{C_n} \otimes (P^*\phi)_n$. Indeed, for every $h \in L_{\infty}(\mu, X)$,

$$\begin{aligned} \langle P^*(\phi), h \rangle &= \langle \phi, P(h) \rangle \\ &= \sum_{n=1}^{\infty} \mu(C_n)^{-1} \langle \phi, \chi_{C_n} \otimes \int_{C_n} h d\mu \rangle \\ &= \langle \sum_{n=1}^{\infty} \chi_{C_n} \otimes (P^*\phi)_n, h \rangle. \end{aligned}$$

Consequently, $R(P^*) \subset L_1(\mu, X^*)$. Moreover, $P^*(f_j) = f_j$, for every $j = 1, \dots, l$.

We define L as the restriction of P^* to the subspace F . Let $\phi \in F$ and $h \in G$. Since $\|L\| \leq 1$ and $\langle L\phi, S(h) \rangle = \langle \phi, P(S(h)) \rangle = \langle \phi, S(h) \rangle$,

$$(2) \quad |\langle L\phi, h \rangle - \langle \phi, h \rangle| \leq |\langle L\phi, (I - S)h \rangle| + |\langle \phi, (I - S)h \rangle| \leq \varepsilon \|\phi\| \|h\|,$$

which is (a) in Theorem 2. Moreover, formulas (1) and (2) imply that, for every $\phi \in F$,

$$\|L\phi\| \geq \|L(\phi)|_G\| \geq \|\phi|_G\| - \varepsilon\|\phi\| \geq (1 + \varepsilon)^{-1}\|\phi\| - \varepsilon\|\phi\| \geq (1 - 2\varepsilon)\|\phi\|.$$

Hence L is a 2ε -isometry. Moreover, since $\|I - T\| < \varepsilon/2$ and $L(T(\phi)) = T(\phi)$, for every $\phi \in F \cap L_1(\mu, X^*)$,

$$\|L(\phi) - \phi\| \leq \|L(\phi - T(\phi))\| + \|T(\phi) - \phi\| < \varepsilon\|\phi\|.$$

Thus, an application of Theorem 2 finishes the proof of this part.

(b) The proof is similar. In this case F and G are finite dimensional subspaces of $L_1(\mu, X)^*$ and $L_1(\mu, X)$, respectively. The projection P on $L_1(\mu, X)$, defined by

$$P(f) = \sum_{n=1}^{\infty} \mu(C_n)^{-1} \chi_{C_n} \otimes \int_{C_n} f d\mu,$$

satisfies $\|P(f)\| \leq \sum_{n=1}^{\infty} \|\int_{C_n} f d\mu\| \leq \|f\|_1$, hence $\|P\| = 1$. The remainder of the proof is similar to that of part (a). \square

Remark 5. We have $L_\infty(\mu, X^*) = L_1(\mu, X)^*$ if and only if every $T \in \mathcal{B}(L_1(\mu, X^*))$ is representable; equivalently, X^* has the Radon-Nikodym property [4].

It was proved in [6] that $L_\infty(\mu) \hat{\otimes}_\varepsilon X^*$ is a local dual of $L_1(\mu, X) \equiv L_1(\mu) \hat{\otimes}_\pi X$. In this case, $L_\infty(\mu) \hat{\otimes}_\varepsilon X^* = L_1(\mu, X)^*$ if and only if every $T \in \mathcal{B}(L_1(\mu, X^*))$ is compact; equivalently, X is finite dimensional.

Let K denote a compact space. It was proved in [6] that $C(K)^* \hat{\otimes}_\pi X^*$ is a local dual of $C(K, X) \equiv C(K) \hat{\otimes}_\varepsilon X$. Note that $C(K)^* \hat{\otimes}_\pi X^* = C(K, X)^*$ if and only if X^* has the Radon-Nikodym property [4].

The following result is a direct consequence of Theorem 4 and Proposition 3.

Corollary 6. (a) $L_\infty(\mu, X)$ is a local dual of $L_1(\mu, X^*)$.

(b) $L_1(\mu, X)$ is a local dual of $L_\infty(\mu, X^*)$.

Remark 7. Corollary 6 is a generalization of the principle of local reflexivity.

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