

KRULL DIMENSION OF THE ENVELOPING ALGEBRA OF A SEMISIMPLE LIE ALGEBRA

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ABSTRACT. Let \mathfrak{g} be a complex semisimple Lie algebra and $U(\mathfrak{g})$ be its enveloping algebra. We deduce from the work of R. Bezrukavnikov, A. Braverman and L. Positselskii that the Krull-Gabriel-Rentschler dimension of $U(\mathfrak{g})$ is equal to the dimension of a Borel subalgebra of \mathfrak{g} .

1. INTRODUCTION

The Krull(-Gabriel-Rentschler) dimension of a ring R was introduced in [3] and is denoted by $\text{Kdim } R$. Let \mathfrak{g} be a semisimple complex Lie algebra and $U(\mathfrak{g})$ be its enveloping algebra. It has been conjectured that $\text{Kdim } U(\mathfrak{g})$ is equal to $\dim \mathfrak{b}$ where \mathfrak{b} is a Borel subalgebra of \mathfrak{g} . It is easy to see that $\text{Kdim } U(\mathfrak{g}) \geq \dim \mathfrak{b}$; indeed, this follows from the fact $U(\mathfrak{g})$ is a free (left) module over $U(\mathfrak{b})$ and that $\text{Kdim } U(\mathfrak{b}) = \dim \mathfrak{b}$ (see §2). The opposite inequality is therefore the hard part of the conjecture.

P. Smith [10] proved the conjecture for $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$. Let G be a simply connected semisimple complex algebraic group with Lie algebra \mathfrak{g} , U be a maximal unipotent subgroup of G and set $X = G/U$ (the “basic affine space”). In [7] it was shown that the conjecture would follow from $\text{Kdim } \mathcal{D}(X) \leq \dim X$, where $\mathcal{D}(X)$ is the ring of globally defined differential operators on X (in the sense of [5]). This result was established in [7] when \mathfrak{g} is a direct sum of copies of $\mathfrak{sl}(2, \mathbb{C})$, and in [8] when $\mathfrak{g} = \mathfrak{sl}(3, \mathbb{C})$. Up to now, these were the only cases known and no progress was made on the conjecture.

The difficulty in the study of $\mathcal{D}(X)$ comes from the fact that $\mathcal{D}(X) = \mathcal{D}(\overline{X})$ for some singular variety \overline{X} . Recently R. Bezrukavnikov, A. Braverman and L. Positselskii were able to prove, among other things, that $\mathcal{D}(X)$ is a Noetherian ring. This is deduced from the existence of a finite set $\{F_w\}_{w \in W}$ (W being the Weyl group of \mathfrak{g}) of automorphisms of $\mathcal{D}(X)$ such that: for every $\mathcal{D}(X)$ -module $M \neq 0$, there exists a twist M^{F_w} of M such that the localization $\mathcal{O}_X \otimes_{\mathcal{O}(X)} M^{F_w}$ is nonzero. In this note we want to explain how this result easily implies that $\text{Kdim } \mathcal{D}(X) \leq \dim X$, and, consequently, $\text{Kdim } U(\mathfrak{g}) = \dim \mathfrak{b}$.

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2. KRULL DIMENSION

The definitions and general results related to Krull dimension can be found in [9, Chapter 6] and we will simply quote a few facts that we need.

Recall that the deviation of a partially ordered set (poset) (A, \preceq) is defined (when it exists) as follows:

- $\text{dev } \emptyset = -\infty$;
- $\text{dev } A = 0$ if and only if A satisfies the descending chain condition;
- $\text{dev } A = \alpha$ (some ordinal) if $\text{dev } A \neq \beta$ for $\beta < \alpha$, and if $(a_i)_{i \in \mathbb{N}}$ is a descending chain in A , then there exists i_0 such that $\text{dev}\{x \in A : a_i \succ x \succ a_{i+1}\} < \alpha$ for all $i \geq i_0$.

For the proof of the next lemma, see [9, 6.1.5, 6.1.6].

Lemma 2.1. (a) *Let $B \hookrightarrow A$ be a strictly increasing map of posets. Then, $\text{dev } B \leq \text{dev } A$ when $\text{dev } A$ exists.*

(b) *If A satisfies the ascending chain condition, then $\text{dev } A$ exists.* □

If R is a ring we denote by $R\text{-mod}$ the category of finitely generated left R -modules. Let $M \in R\text{-mod}$ and $\mathcal{L}(M)$ be the lattice of submodules of M . Then $(\mathcal{L}(M), \subseteq)$ is a poset; we say that the Krull dimension of M exists if $\mathcal{L}(M)$ has a deviation, in which case we set $\text{Kdim}_R M = \text{Kdim } M = \text{dev } \mathcal{L}(M)$. By Lemma 2.1, $\text{Kdim } M$ exists if R is (left) Noetherian and one has $\text{Kdim } M \leq \text{Kdim } R$ ([9, 6.2.18]).

Examples. 1. Let \mathfrak{m} be a finite dimensional complex Lie algebra and $\mathfrak{l} \subset \mathfrak{m}$ be a subalgebra. Then, $\text{Kdim } U(\mathfrak{l}) \leq \text{Kdim } U(\mathfrak{m}) \leq \dim \mathfrak{m}$. When \mathfrak{m} is solvable an easy induction on $\dim \mathfrak{m}$ (using Lie's Theorem) shows that $\text{Kdim } U(\mathfrak{m}) = \dim \mathfrak{m}$.

2. Let $\mathcal{D}(Z)$ be the ring of differential operators on a smooth affine complex algebraic variety Z . Then $\mathcal{D}(Z)$ is Noetherian and $\text{Kdim } \mathcal{D}(Z) = \dim Z$; see [9, 15.1.20, 15.3.7].

We will use the following easy result:

Lemma 2.2. *Let R_j , $j = 1, \dots, s$, be some rings and $M_j \in R_j\text{-mod}$. Then, if $\text{Kdim } M_j$ exists for all j , we have*

$$\text{Kdim}_{\bigoplus_{j=1}^s R_j} \left(\bigoplus_{j=1}^s M_j \right) = \max\{\text{Kdim } M_j : j = 1, \dots, s\}.$$

Proof. The claim follows from the identification of $\mathcal{L}(\bigoplus_{j=1}^s M_j)$ with $\mathcal{L}(M_1) \times \dots \times \mathcal{L}(M_s)$. □

3. RINGS OF DIFFERENTIAL OPERATORS

If Z is a complex algebraic variety we denote by \mathcal{O}_Z its structural sheaf and by \mathcal{D}_Z the sheaf of differential operators on Z , as defined in [5]. By taking global sections we get the following \mathbb{C} -algebras:

$$\mathcal{O}(Z) = \mathcal{O}_Z(Z), \quad \mathcal{D}(Z) = \mathcal{D}_Z(Z).$$

Assume that Z is smooth and denote by $\mathcal{D}_Z\text{-coh}$ the category of coherent left \mathcal{D}_Z -modules (see [2] for a definition). Recall [2] that when Z is affine, the functor $\mathcal{M} \rightarrow \Gamma(Z, \mathcal{M})$ yields an equivalence of categories $\mathcal{D}_Z\text{-coh} \cong \mathcal{D}(Z)\text{-mod}$.

Notation. Let \overline{X} be an irreducible affine variety and X be a nonempty (dense) open subset of smooth points in \overline{X} . We will work under the following hypothesis:

$$\overline{X} \text{ is normal and } \text{codim}_{\overline{X}}(\overline{X} \setminus X) \geq 2.$$

In this situation one has $\mathcal{O}(X) = \mathcal{O}(\overline{X})$ and it is easy to show that this implies

$$\mathcal{D}(X) = \mathcal{D}(\overline{X});$$

see, e.g., [6, II.2, Proposition 2]. Since X is quasi-compact and open in \overline{X} we can write $X = \bigcup_{i=1}^s U_i$, where each U_i is a principal affine open subset of \overline{X} , i.e., $U_i = \{x \in \overline{X} : f_i(x) \neq 0\}$ for some $f_i \in \mathcal{O}(X)$. Recall that $\{f_i^k\}_{k \in \mathbb{N}}$ is an Ore subset in $\mathcal{D}(\overline{X})$ and that

$$\mathcal{D}(U_i) = \mathcal{D}(\overline{X})[f_i^{-1}] = \mathcal{O}(X)[f_i^{-1}] \otimes_{\mathcal{O}(X)} \mathcal{D}(X).$$

Therefore, if $\mathcal{M} \in \mathcal{D}_X\text{-coh}$, each restriction $\mathcal{M}|_{U_i} \in \mathcal{D}_{U_i}\text{-coh}$ is determined by $\mathcal{M}(U_i) = \Gamma(U_i, \mathcal{M}) \in \mathcal{D}(U_i)\text{-mod}$.

The next lemma is well known, and we include a proof for completeness.

Lemma 3.1. *Let $\mathcal{L}(\mathcal{M})$ be the lattice of \mathcal{D}_X -submodules of $\mathcal{M} \in \mathcal{D}_X\text{-coh}$. Then $\mathcal{L}(\mathcal{M})$ satisfies the ascending chain condition.*

Proof. Let $(\mathcal{M}_j)_{j \in \mathbb{N}}$ be an ascending chain of \mathcal{D}_X -submodules of $\mathcal{M}_0 = \mathcal{M}$. Set $\mathcal{M}_{j,i} = \mathcal{M}_j(U_i)$ for $i = 1, \dots, s$ and $j \in \mathbb{N}$. Since the functor $\Gamma(U_i, -)$ is left exact, $(\mathcal{M}_{j,i})_{j \in \mathbb{N}}$ is an ascending chain of submodules in the finitely generated $\mathcal{D}(U_i)$ -module $\mathcal{M}(U_i)$. Therefore, there exists $j(i) \in \mathbb{N}$ such that $\mathcal{M}_{j,i} = \mathcal{M}_{j(i),i}$ for all $j \geq j(i)$. Set $j_0 = \max\{j(i) : i = 1, \dots, s\}$; then, since $X = \bigcup_{i=1}^s U_i$, we get that $\mathcal{M}_j = \mathcal{M}_{j_0}$ for all $j \geq j_0$. \square

The previous lemma and §2 enable us to define the Krull dimension of $\mathcal{M} \in \mathcal{D}_X\text{-coh}$ by

$$\text{Kdim } \mathcal{M} = \text{dev } \mathcal{L}(\mathcal{M}).$$

Proposition 3.2. *Let $\mathcal{M} \in \mathcal{D}_X\text{-coh}$. Then,*

$$\text{Kdim } \mathcal{M} \leq \max\{\text{Kdim}_{\mathcal{D}(U_i)} \mathcal{M}(U_i) : i = 1, \dots, s\} \leq \dim X.$$

Proof. Observe that $M = \bigoplus_{i=1}^s \mathcal{M}(U_i)$ is a finitely generated module over the ring $R = \bigoplus_{i=1}^s \mathcal{D}(U_i)$. As $\Gamma(U_i, -)$ is left exact and $X = \bigcup_{i=1}^s U_i$, the map $\mathcal{N} \rightarrow \bigoplus_{i=1}^s \mathcal{N}(U_i)$ yields a strictly increasing map from $\mathcal{L}(\mathcal{M})$ to $\mathcal{L}(M)$. Thus, by definition and Lemma 2.2, we obtain

$$\text{Kdim } \mathcal{M} \leq \text{Kdim } M = \max\{\text{Kdim}_{\mathcal{D}(U_i)} \mathcal{M}(U_i) : i = 1, \dots, s\}.$$

Since $\text{Kdim } \mathcal{D}(U_i) = \dim U_i = \dim X$ for all i (cf. §2, Example 2), the assertion is proved. \square

Recall that we have a localization functor $L : \mathcal{D}(X)\text{-mod} \rightarrow \mathcal{D}_X\text{-coh}$ defined by

$$L(M) = \mathcal{D}_X \otimes_{\mathcal{D}(X)} M.$$

Lemma 3.3. *The functor L is exact.*

Proof. Let $V = \overline{X}_f = \{x \in \overline{X} ; f(x) \neq 0\}$, $f \in \mathcal{O}(X)$, be a principal open subset of \overline{X} contained in X . We have already noticed that $\mathcal{D}_X(V) = \mathcal{O}(X)[f^{-1}] \otimes_{\mathcal{O}(X)} \mathcal{D}(X)$, hence

$$\Gamma(V, L(M)) = \mathcal{D}_X(V) \otimes_{\mathcal{D}(X)} M = \mathcal{O}(X)[f^{-1}] \otimes_{\mathcal{O}(X)} M.$$

The lemma then follows from the exactness of the localization functor $M \rightarrow \mathcal{O}_X \otimes_{\mathcal{O}(X)} M$ on the category $\mathcal{O}(X)\text{-mod}$. \square

Suppose that $\tau \in \text{Aut } \mathcal{D}(X)$ is an automorphism of the algebra $\mathcal{D}(X)$. If $M \in \mathcal{D}(X)\text{-mod}$ we denote by $M^\tau \in \mathcal{D}(X)\text{-mod}$ the module defined by: $M^\tau = M$ as an abelian group and $a.v = \tau(a)v$ for all $a \in \mathcal{D}(X)$, $v \in M$. We now make the supplementary hypothesis:

(H) *There exist $\tau_1, \dots, \tau_p \in \text{Aut } \mathcal{D}(X)$ such that, for every $0 \neq M \in \mathcal{D}(X)\text{-mod}$, $L(M^{\tau_j}) \neq 0$ for some $j \in \{1, \dots, p\}$.*

We then define $\lambda(M) \in \mathcal{D}_X\text{-coh}$ for $M \in \mathcal{D}(X)\text{-mod}$ by setting

$$\lambda(M) = \bigoplus_{j=1}^p L(M^{\tau_j}).$$

Theorem 3.4. *One has $\text{Kdim } M \leq \text{Kdim } \lambda(M)$ for all $M \in \mathcal{D}(X)\text{-mod}$. In particular,*

$$\text{Kdim } \mathcal{D}(X) \leq \dim X.$$

Proof. The hypothesis (H) ensures that $N \rightarrow \lambda(N)$ is a strictly increasing map from $\mathcal{L}(M)$ to $\mathcal{L}(\lambda(M))$. Thus, using Proposition 3.2,

$$\text{Kdim } M = \text{dev } \mathcal{L}(M) \leq \text{dev } \mathcal{L}(\lambda(M)) = \text{Kdim } \lambda(M) \leq \dim X,$$

as required. \square

The properties of the map $\lambda : \mathcal{L}(M) \rightarrow \mathcal{L}(\lambda(M))$ imply that $M \in \mathcal{D}(X)\text{-mod}$ is Noetherian; cf. [1, Theorem 1.3].

4. THE KRULL DIMENSION OF $U(\mathfrak{g})$

Let G be a simply connected semisimple complex algebraic group with Lie algebra \mathfrak{g} . Let U be a maximal unipotent subgroup of G and set $X = G/U$.

Theorem 4.1. *The quasi-affine variety X satisfies the hypotheses of §3 (in particular the hypothesis (H)).*

Proof. It is a classical fact that X can be embedded in a normal affine variety \overline{X} such that $\text{codim}_{\overline{X}}(\overline{X} \setminus X) \geq 2$. This can be shown as follows. Let $\varpi_1, \dots, \varpi_\ell$ be the fundamental dominant weights of \mathfrak{g} ; denote by $E(\varpi_j)$, $j = 1, \dots, \ell$, a simple G -module with highest weight ϖ_j and set $E = \bigoplus_{j=1}^\ell E(\varpi_j)$. If $v_j \in E(\varpi_j)$ is a highest weight vector, the orbit $G.(v_1 \oplus \dots \oplus v_\ell) \subset E$ is isomorphic to X and its closure \overline{X} (in E) has the required properties, see [4] and [11].

Thanks to [1], each element w of the Weyl group of \mathfrak{g} yields an automorphism $F_w \in \text{Aut } \mathcal{D}(X)$. By [1, Theorem 3.8], for every nonzero $M \in \mathcal{D}(X)\text{-mod}$ there exists w such that $L(M^{F_w}) \neq 0$. Thus X satisfies the hypothesis (H). \square

Observe that $\dim X$ is the dimension of a Borel subalgebra of \mathfrak{g} .

Corollary 4.2. *One has*

$$\text{Kdim } U(\mathfrak{g}) = \text{Kdim } \mathcal{D}(X) = \dim X.$$

Proof. By Theorem 3.4 we have $\text{Kdim } \mathcal{D}(X) \leq \dim X$. From [7, Proposition 3.2] we know that $\text{Kdim } U(\mathfrak{g}) \leq \text{Kdim } \mathcal{D}(X)$, thus $\text{Kdim } U(\mathfrak{g}) \leq \text{Kdim } \mathcal{D}(X) \leq \dim X$. The result then follows from $\dim X \leq \text{Kdim } U(\mathfrak{g})$ (see §2, Example 1). \square

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