PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 130, Number 12, Pages 3515–3518
S 0002-9939(02)06518-8
Article electronically published on May 1, 2002

INTEGRAL REPRESENTATIONS FOR THE ALTERNATING GROUPS

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(Communicated by Stephen D. Smith)

Abstract. We show that every complex representation of an alternating group can be realized over the ring of integers of a "small" abelian number field.

1. Introduction

By a well-known theorem of R. Brauer every (irreducible) complex character χ of a finite group G can be written in the g-th cyclotomic field $\mathbb{Q}(\zeta_g)$ where g denotes the exponent of G. It remains a problem as to whether this can be done integrally, i.e. if there exists a matrix representation with entries in the ring of integers $\mathbb{Z}[\zeta_g]$ affording χ.

For solvable groups G this was shown to be true by Cliff, Ritter and Weiss [1]. In general, Clifford theory reduces the question to representations of quasi-simple groups (stable under some automorphisms) (Knapp-Schmid [4]). So far, integral representations for all irreducible characters of simple groups have been constructed for the sporadic groups, some small alternating groups, some groups of Lie-type of small order [4] and for the groups $\mathrm{SL}(2,p)$ over the prime field [6].

Theorem 1. *Every irreducible complex representation of the alternating group A_n can be realized over the ring of integers of the field*

$$\mathbb{Q}(\sqrt{p^*}\,|\,p \text{ odd prime}, p \leq n),$$

where $p^* = (-1)^{\frac{p-1}{2}}p$ for any odd prime p.

Obviously, the field given in the theorem is contained in the g-th cyclotomic field ($g = \exp(A_n)$). The proof of the theorem is based on a capitulation theorem for ideal classes by Terada [8] which enables us to realize all characters by the same argument. Our method also gives partial results for the covering groups $\tilde{A}_n$ of the alternating groups but a complete answer for these groups seems to require a more specific approach.

2. Ambiguous ideal classes

Let K/k be a cyclic Galois extension of an algebraic number field k. We denote by τ a generator of the Galois group. An ideal class $[I]$ in the (finite) class group

Received by the editors May 3, 2001 and, in revised form, July 30, 2001.
2000 *Mathematics Subject Classification.* Primary 20C10, 20C30.

C_K of K is called ambiguous if it is fixed by τ, that is, if it is contained in the kernel of the homomorphism

$$\phi : C_K \to C_K : [I] \mapsto [I]^{1-\tau}.$$

The image of ϕ is the principal genus; the genus field K_Γ of K/k is the class field corresponding to the principal genus. An ideal I in an ambiguous class $[I]$ is an ambiguous class ideal. It is itself ambiguous if $I^{1-\tau} = (1)$. Note that in general not every ambiguous class can be represented by ambiguous ideals, the latter might form a subgroup of index 2 in the group of ambigous class ideals.

Proposition 1 (Terada [8])**.** *Every ambiguous ideal class of a cyclic Galois extension K/k becomes trivial in the genus field K_Γ of K/k.*

3. Lattices over Dedekind domains

Let K be an algebraic number field and let $R = R_K$ be the ring of algebraic integers in K. Let G be a finite group and W an absolutely irreducible KG-module affording the character χ. There exists a (torsion-free, finitely generated) RG-lattice U in W such that $W = KU \cong K \otimes_R U$ (by taking all R-linear combinations of the G-images of a K-basis of W). Every RG-lattice of this type is isomorphic to such a *full* RG-lattice in W. We wish to find a lattice which is R-free (not just R-projective).

Let U be a full RG-lattice in W. As an R-module U is the direct sum of $s = \chi(1)$ (nonzero) ideals J_i of R. By a well known theorem of Steinitz this *rank* s together with the *Steinitz class*

$$[U] = [\prod_i J_i] = \prod_i [J_i] \in C_K$$

in the class group of the Dedekind domain R determine the R-isomorphism type of U. The *genus* $\gamma(U)$ of U consists, in the present situation, of all RG-lattices of the form JU for some fractional ideal J of R. We have

$$[JU] = [J]^s[U],$$

and there are exactly $|C_K|$ different isomorphism types of RG-lattices in $\gamma(U)$ (see [2], (31.26)).

4. Proof of the theorem

Every irreducible (complex) character of a symmetric group S_n has trivial Schur-index over the rationals (cf. [3], Theorem 2.1.12). Furthermore, every character of S_n has its values in $\mathbb{Q}$, thus every irreducible character of an alternating group has trivial Schur-index over the rationals (cf. [7], Example 3). In particular, there are integral representaions over the principal ideal ring $\mathbb{Z}$ for all irreducible characters of S_n.

Lemma 1. *Let $\chi \in Irr(A_n)$ be an irreducible (complex) character. Then either $\mathbb{Q}(\chi) = \mathbb{Q}$ or $\mathbb{Q}(\chi) = \mathbb{Q}(\sqrt{d})$ for some odd square-free integer $d \neq 1$ with $d \equiv 1$ (mod 4).*

Proof. By [3], Theorem 2.5.13, either $\mathbb{Q}(\chi) = \mathbb{Q}$ or $\mathbb{Q}(\chi)$ is a quadratic number field. In fact, if $\mathbb{Q}(\chi) \neq \mathbb{Q}$, then χ belongs to a self-associated partition α of n

(with diagram $[\alpha]$ symmetric with respect to the main diagonal). Furthermore, for $g \in A_n$ either $\chi(g)$ is an integer or

$$\chi(g) = \frac{1}{2}(\varepsilon \pm \sqrt{\varepsilon \prod_i h_{ii}^{\alpha}}),$$

where $\varepsilon = (-1)^{\frac{n-k}{2}}$, k being the length of the main diagonal of the diagram $[\alpha]$, and where the h_{ii}^{α} are the lengths of the main hooks of $[\alpha]$. All the hook lengths h_{ii}^{α} are odd. Now use that $\chi(g)$ is an algebraic integer, and use that odd squares are congruent to 1 mod 4. □

Observe that in the lemma d is the (absolute) discriminant of the quadratic number field $K = \mathbb{Q}(\sqrt{d})$ and that $d \leq n!$. We have a unique *-decomposition (see Leopoldt [5] or Terada [8], Section 2)

$$d = \prod p^*$$

where $p^* = (-1)^{\frac{p-1}{2}} p$ and the product is taken over all (odd) prime divisors p of d (so that $p \leq n$). The field

$$K^* = \mathbb{Q}(\sqrt{p^*} \,|\, p \,\text{divides}\, d)$$

is the so-called genus field in the narrow sense which always contains the genus field K_Γ of $K/\mathbb{Q}$ with degree $[K^* : K_\Gamma] \leq 2$. It follows from Terada's theorem that every ambiguous ideal class in C_K becomes trivial in C_{K^*}.

By Clifford the restriction to A_n of an irreducible character of S_n remains irreducible or decomposes into the sum of two conjugate irreducibles. Since all Schur indices of the characters involved are trivial, the rational valued characters of A_n are realizable over the integers (as every torsion-free, finitely generated $\mathbb{Z}$-module is free). The theorem thus is an immediate consequence of the following:

Proposition 2. *Let χ be an irreducible character of A_n such that $\mathbb{Q}(\chi) = \mathbb{Q}(\sqrt{d}) \neq \mathbb{Q}$ where d is a (odd) square-free integer. Let R^* be the ring of algebraic integers in*

$$K^* = \mathbb{Q}(\sqrt{p^*} \,|\, p \,\textit{divides}\, d).$$

Then there is a R^*-free $R^* A_n$-module V affording the character χ.

Proof. We know that $d \equiv 1 \pmod 4$ is odd (Lemma 1). Let W be an absolutely irreducible $\mathbb{Q}S_n$-module affording the (irreducible) character χ^{S_n}. Then W, as a $\mathbb{Q}A_n$-module, is irreducible but not absolutely irreducible. In fact, since the Schur index $\mathrm{m}(\chi) = 1$,

$$K = \mathrm{End}_{\mathbb{Q}A_n}(W)$$

is a quadratic number field with discriminant $d = p_1^* \cdots p_t^*$. Thus $K \cong \mathbb{Q}(\chi) \subseteq \mathbb{C}$ and we choose this isomorphism in such a way that W, regarded as an absolutely irreducible (right) KA_n-module, affords the character χ.

Let R be the integral closure of $\mathbb{Z}$ in K, and let $U \subseteq W$ be a full (right) RA_n-lattice. Then

$$\mathrm{End}_{RA_n}(U) = R.$$

Let τ denote the element of $\mathrm{End}_{\mathbb{Q}}(W)$ induced by the transposition (12). Then τ induces the non-trivial automorphism on K which we denote by τ as well.

Both U and $U\tau$ are (right) full RA_n-lattices in the KA_n-module W. Suppose $\{w_i\}_{i=1}^s$ is a K-basis of W (so that $s = \chi(1)$). Then

$$U = \bigoplus_{i=1}^{s} w_i J_i$$

for certain fractional ideals J_i of R. Now $\{w_i\tau\}_{i=1}^s$ is also a K-basis of W, and

$$U\tau = \bigoplus_{i=1}^{s} (w_i J_i)\tau = \bigoplus_{i=1}^{s} w_i \tau J_i^\tau.$$

Consequently the Steinitz class $[U\tau] = [\prod J_i^\tau] = [U]^\tau$ where τ acts naturally on the class group C_K (by inversion, $[U]^\tau = [U]^{-1}$).

Now put $V = U + U\tau$. This is again a full RA_n-lattice in W and as $\tau^2 = 1$ it is τ-invariant. Thus $[V] = [V]^\tau$ is an ambiguous ideal class in C_K. By Terada's theorem (Proposition 1), and since $K_\Gamma \subseteq K^*$, the ideal class $[V]$ becomes trivial in C_{K^*}, that is, it is in the kernel of the natural map $C_K \to C_{K^*}$. Here for convenience we identify $K = \mathbb{Q}(\sqrt{d})$. Now $W^* = K^* \otimes_K W$ is an absolutely irreducible K^*A_n-module affording χ, and $V^* = R^* \otimes_R V$ is a full R^*A_n-lattice in W^*. The Steinitz class $[V^*] \in C_{K^*}$ is the image of $[V]$ under the natural map. Thus V^* is R^*-free, as desired. □

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