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A NON-RIBBON PLUMBING OF FIBERED RIBBON KNOTS

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Abstract. A closer look at an example introduced by Livingston and Melvin and later studied by Miyazaki shows that a plumbing of two fibered ribbon knots (along their fiber surfaces) may be algebraically slice yet not ribbon.

Trivially, the connected sum (i.e., 2-gonal Murasugi sum) of ribbon knots is ribbon. Non-trivially [6, 1], any Murasugi sum of fibered knots (along their fiber surfaces) is fibered. In light of these facts, perhaps the following is a bit surprising.

Theorem. *There exist fibered ribbon knots K_1, K_2 and an algebraically slice plumbing* (4*-gonal Murasugi sum*) $K_1 * K_2$*, along fiber surfaces, which is not ribbon.*

The proof uses a slight embellishment of a result from [4]. Following [4], for any knot K, and relatively prime integers m, n with $m \geq 1$, let $K\{m,n\}$ denote any simple closed curve on the boundary $\partial N(K)$ of a tubular neighborhood $N(K)$ of K in S^3 such that $K\{m,n\}$ represents m times the class of K in $H_1(N(K);\mathbb{Z})$ and has linking number n with K. For instance, if O denotes an unknot, then $O\{m,n\}$ is the (fibered) torus knot of type (m,n). Abbreviate $K\{m,n\}\{p,q\}$ to $K\{m,n;p,q\}$. (N.B.: there is no universally accepted standard notation for such iterated cable knots. In particular, instead of $O\{m,n;p,q\}$, Livingston and Melvin [2] write $(q,p;n,m)$, and Miyazaki [3] writes $(p,q;m,n)$.)

Proposition. *For any K, $K\{m,n\} = K\{m, \operatorname{sgn} n\} * O\{m,n\}$ is a $2m$-gonal Murasugi sum, along a suitable Seifert surface $F_{(m,\operatorname{sgn} n)}$ for $K\{m, \operatorname{sgn} n\}$ and a fiber surface $D_{(m,n)}$ for $O\{m,n\}$; for fibered K, $F_{(m,\operatorname{sgn} n)}$ is a fiber surface.*

Proof. The case $m = 1$ is trivial. Take $m > 1$, $\operatorname{sgn} n = \pm 1$. Let $F \subset S^3$ be a Seifert surface for K. Construct a Seifert surface $F_{(m,\pm 1)}$ from m parallel copies of F, each adjacent pair of copies joined, in order, by a 1-handle with a half-turn of sign $\pm$. If F is a fiber surface, then so is $F_{(m,\pm 1)}$ (see [6]); in any case $\partial F_{(m,\pm 1)} = K\{m, \pm 1\}$. The proof is finished by contemplating an appropriate figure (see Figure 1). □

Proof of the Theorem. Livingston and Melvin [2] drew attention to the connected sum of iterated torus knots

$$K := O\{2,3;2,13\} \# O\{2,15\} \# O\{2,-3;2,-15\} \# O\{2,-13\}.$$

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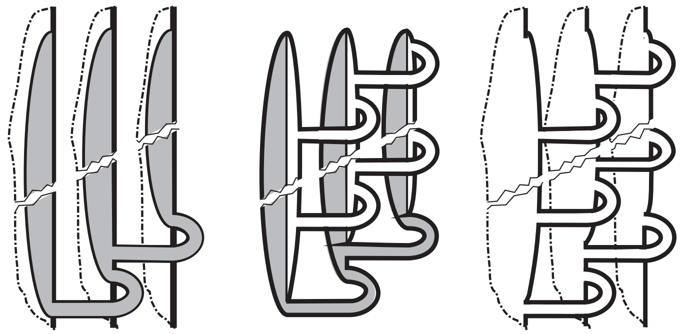

FIGURE 1. A 3-patch N_1 on $F_{(3,1)}$; a 3-patch N_2 on the (braided) fiber surface $D_{(3,n)}$; $F_{(3,n)}$ as a 6-gonal Murasugi sum $F_{(3,1)} *_h D_{(3,n)}$ along the obvious diffeomorphism $h : N_1 \to N_2$.

(Their motivation was a vague question [5] about relations among the concordance classes of the knots associated to complex plane curve singularities, and they gave an answer to one form of the question by observing that K is algebraically slice.) Miyazaki [3] showed that K is not ribbon. By the proposition, and the indifference of connected sums (of knots) to the location of the summation,

$$\begin{aligned} K &= (O\{2,3;2,1\} * O\{2,13\}) \# O\{2,15\} \\ &\qquad \# (O\{2,-3;2,-1\} * O\{2,-15\}) \# O\{2,-13\} \\ &= \big((O\{2,3;2,1\} \# O\{2,-3;2,-1\}) * (O\{2,13\} \# O\{2,-13\})\big) \\ &\qquad * (O\{2,15\} \# O\{2,-15\}) \end{aligned}$$

(where all the Murasugi sums are 4-gonal and along fiber surfaces). The connected sum of a knot and its mirror image is ribbon, so $R_1 := O\{2,3;2,1\} \# O\{2,-3;2,-1\}$, $R_2 := O\{2,13\} \# O\{2,-13\}$, and $R_3 := O\{2,15\} \# O\{2,-15\}$ are ribbon. If $R_1 * R_2$ is ribbon, let $K_1 := R_1 * R_2$, $K_2 := R_3$; if not, let $K_1 := R_1$, $K_2 := R_2$. In either case, K_1 and K_2 are ribbon but $K_1 * K_2$ is not ribbon. On the other hand, since (as mentioned in [4]) the Murasugi sum in the proposition is obviously "direct" (i.e., the Seifert linking between any cycle on $F_{(m,\operatorname{sgn} n)}$ and any cycle on $D_{(m,n)}$, taken in either order, is 0), $K_1 * K_2$ is algebraically slice, for clearly a Murasugi direct sum is algebraically concordant to the connected sum of the same summands. □

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