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A HYPERSURFACE IN $\mathbb{C}^2$ WHOSE STABILITY GROUP IS NOT DETERMINED BY 2-JETS

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ABSTRACT. We give an example of a hypersurface in $\mathbb{C}^2$ through 0 whose stability group at 0 is determined by 3-jets, but not by jets of any lesser order. We also examine some of the properties which the stability group of this infinite type hypersurface shares with the 3-sphere in $\mathbb{C}^2$.

1. STATEMENT OF RESULT

Suppose $M \subset \mathbb{C}^N$ is a real-analytic hypersurface passing through the point p. The *stability group* of M at p, denoted $\mathrm{Aut}(M,p)$, is the group (under composition) of local automorphisms of the germ (M,p). That is, it is the set of all invertible biholomorphic mappings $H : \mathbb{C}^N \to \mathbb{C}^N$, defined in a neighborhood of p, which fix the point p and map M into itself. The stability group of M at p is said to be *determined by ℓ-jets* if for every pair $H_1, H_2 \in \mathrm{Aut}(M,p)$, we have $H_1 = H_2$ (as germs of biholomorphisms at p) whenever

$$\frac{\partial^{|\alpha|} H_1}{\partial Z^\alpha}(p) = \frac{\partial^{|\alpha|} H_2}{\partial Z^\alpha}(p) \quad \forall\, \alpha \in \mathbb{N}^N,\ 0 \le |\alpha| \le \ell.$$

Recall that a hypersurface $M \subset \mathbb{C}^N$ is said to be *minimal* at $p \in M$ if there exists no complex hypersurface contained in M passing through p. If M is real-analytic, then it is well known that this is equivalent to being of *finite type* at p (in the sense of Kohn [Koh72] and Bloom and Graham [BG77]).

In general, if $M \subset \mathbb{C}^N$ is a hypersurface of infinite type at p, then its stability group at p need not be determined by jets of any finite order. For example, the "flat hypersurface" given by

$$M = \left\{(Z_1, \ldots, Z_N) \in \mathbb{C}^N \mid \mathrm{Im}\, Z_N = 0\right\}$$

is of infinite type at the origin. Moreover, any invertible holomorphic mapping of the form

$$H(Z) = \left(F_1(Z), \ldots F_{N-1}(Z), Z_N\right)$$

is a local automorphism of M. This shows that its stability group at 0 is not determined by ℓ-jets for any choice of $\ell \ge 1$.

In some sense, however, this is the most trivial example, and for $\mathbb{C}^2$ in particular, it is the *only* such example. On the other hand, there exists a large body of work

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concerning the jet-determinacy of stability groups of hypersurfaces in $\mathbb{C}^2$ at points of finite type. Poincaré [Poi07] proved that the stability group at any point of the 3-sphere $S^3 \subset \mathbb{C}^2$ is determined by 2-jets. This was extended by Chern and Moser [CM74], who proved that the stability group of a Levi-nondegenerate hypersurface in $\mathbb{C}^N$ is also determined by 2-jets. (For more information, see the survey articles [BER00] and [Vit90].) More recently, Ebenfelt, Lamel, and Zaitsev [ELZ00] have shown that the stability group of any hypersurface of finite type in $\mathbb{C}^2$ is determined by 2-jets.

The purpose of this paper is to present an example which shows that this result cannot be extended to nonflat hypersurfaces in $\mathbb{C}^2$ of infinite type at 0 by presenting a nonflat hypersurface $M \subset \mathbb{C}^2$ of infinite type whose local automorphisms at the origin are determined by their 3-jets, but *not* by their 2-jets.

To state this result more precisely, we make one last definition. Let $M \subset \mathbb{C}^2$ be a hypersurface passing through the origin. A *formal automorphism* of M at 0 is a $\mathbb{C}^2$-valued invertible formal power series H in two indeterminates which vanishes at 0 and formally maps M into itself. That is, for any real-analytic local defining function $\rho(Z, \overline{Z})$ for M, there exists a formal power series a in 4 indeterminates such that the following power series identity holds:

$$\rho\big(H(Z), \overline{H(Z)}\big) \equiv a(Z, \overline{Z})\, \rho(Z, \overline{Z}).$$

The set of all such formal power series (which forms a group under power series composition) is called the *formal stability group* of M at 0, and is denoted $\widehat{\mathrm{Aut}}(M, 0)$. It is easy to see that if a formal automorphism of M converges, then it is a local automorphism of M at 0 as described above, whence it follows that $\mathrm{Aut}(M, 0) \subset \widehat{\mathrm{Aut}}(M, 0)$. We now state our main result.

Theorem 1.1. *For the hypersurface*

$$M := \left\{ (z, w) \in \mathbb{C}^2 \;\middle|\; |z| < 1,\, \mathrm{Im}\, w = (\mathrm{Re}\, w) \frac{|z|^2}{1 + \sqrt{1 - |z|^4}} \right\}, \tag{1.1}$$

every formal automorphism of the germ $(M, 0)$ converges. For $\alpha \in \mathbb{C}$ and $s \in \mathbb{R}$, let $\theta_{\alpha,s}$ be the holomorphic function defined in a neighborhood of $0 \in \mathbb{C}^2$ by

$$\theta_{\alpha,s}(z, w) := \big(1 - 2i\, \overline{\alpha}\, zw - (s + i\, |\alpha|^2) w^2\big)^{1/2},$$

where $\mathbb{C} \ni \zeta \mapsto (\zeta)^{1/2} \in \mathbb{C}$ is the principal branch of the square root function. Then the formal stability group of M at 0 is given explicitly by the following:

$$\begin{aligned} \widehat{\mathrm{Aut}}(M, 0) &= \mathrm{Aut}(M, 0) \\ &= \left\{ H^{\varepsilon, r}_{\alpha, s}(z, w) := \left(\frac{\varepsilon(z + \alpha\, w)}{\theta_{\alpha,s}(z, w)}, \frac{r\, w}{\theta_{\alpha,s}(z, w)} \right) \;\middle|\; \begin{matrix} \varepsilon \in \mathbb{C},\, |\varepsilon| = 1 \\ r \in \mathbb{R} \setminus \{0\} \\ \alpha \in \mathbb{C} \\ s \in \mathbb{R} \end{matrix} \right\}. \end{aligned} \tag{1.2}$$

The proof will be given in the next section. We conclude this section with some remarks.

Remark 1.2. To the author's knowledge, this is the first example of a nonflat hypersurface in $\mathbb{C}^2$ whose stability group (at a point) is not determined by 2-jets, or of any hypersurface in $\mathbb{C}^2$ whose stability group is determined by jets of finite order, but *not* by 2-jets. In fact, it follows from the explicit formula above that if

$$\frac{\partial^{j+k} H^{\varepsilon,r}_{\alpha,s}}{\partial z^j \partial w^k}(0,0) = \frac{\partial^{j+k} H^{\varepsilon',r'}_{\alpha',s'}}{\partial z^j \partial w^k}(0,0) \quad \forall\, j+k \leq 2,$$

then $\varepsilon = \varepsilon'$, $r = r'$, and $\alpha = \alpha'$, but s and s' are arbitrary. Indeed, the mappings

$$H^{1,1}_{0,s}(z,w) = \left(\frac{z}{(1 - s\,w^2)^{1/2}}, \frac{w}{(1 - s\,w^2)^{1/2}} \right), \quad \sigma \in \mathbb{R}, \tag{1.3}$$

form a 1-parameter family of local automorphisms of $(M,0)$ which agree with the identity mapping up to order two, but are distinct for each different value of s.

Remark 1.3. Observe that the hypersurface M given by equation (1.1) is of infinite type at 0, since it contains the nontrivial complex hyperplane $\Sigma = \{w = 0\}$. Hence, the stability groups of hypersurfaces of infinite type need not be determined by 2-jets. However, in [ELZ00] it is shown that the stability group of any nonflat real-analytic hypersurface is determined by jets of *some* predetermined finite order. For the special case of so-called *1-infinite type* hypersurfaces, of which M is an example, the author has shown [Kow01] that the stability group is in fact formally parametrized by such a finite jet.

Remark 1.4. Since the hypersurface M above is of infinite type, it is *not* biholomorphically equivalent to the 3-sphere S^3 in $\mathbb{C}^2$. However, the stability groups of the two hypersurfaces have several traits in common; we point out a few of these.

- It is well known that the 3-sphere in $\mathbb{C}^2$ is locally biholomorphically equivalent to the hypersurface $\{(z,w) \mid \operatorname{Im} w = |z|^2\}$, and in these coordinates, every (formal) local automorphism at 0 is given by

  $$(z,w) \mapsto \left(\frac{r\,\epsilon(z + \alpha\, w)}{1 - 2i\,\overline{\alpha} z - (s + i|\alpha|^2)w}, \frac{r^2\, w}{1 - 2i\,\overline{\alpha} z - (s + i|\alpha|^2)w} \right),$$

  with $r > 0$, $\epsilon \in \mathbb{C}$ with $|\epsilon| = 1$, $\alpha \in \mathbb{C}$, and $s \in \mathbb{R}$. This is similar to the formula given by equation (1.2).
- Like the 3-sphere, the (formal) stability group of M is determined by five real parameters.
- Like the 3-sphere, the elements of the stability group of M do not extend to a common neighborhood of 0 in $\mathbb{C}^2$. That is, there exist automorphisms of the germ $(M,0)$ whose radii of convergence are arbitrarily small. For example, the map $H^{1,1}_{0,s}$ given as in equation (1.3) with $s \neq 0$ converges if and only if $|w| < 1/\sqrt{|s|}$, which can be made arbitrarily small by taking $|s|$ arbitrarily large. In contrast, for Levi-nondegenerate hypersurfaces of $\mathbb{C}^2$ *other than the sphere*, all local automorphisms at a fixed point extend to a common neighborhood.
- The stability group of $(M,0)$ forms a Lie group, which may be identified with the space $(\mathbb{R} \setminus \{0\}) \times S^1 \times \mathbb{C} \times \mathbb{R}$ under the multiplication

  $$(r,\varepsilon,\alpha,s) \cdot (r',\varepsilon',\alpha',s') = \big(rr', \varepsilon\varepsilon', \alpha + r\varepsilon\alpha', s + s' - 2r \operatorname{Im}(\alpha\,\overline{\varepsilon}\,\overline{\alpha'})\big).$$

  In particular, like the 3-sphere, it is noncompact, five-dimensional, and contains a Heisenberg subgroup (namely the subgroup defined by taking $r = \varepsilon = 1$). In contrast, the stability groups of Levi-nondegenerate hypersurfaces in $\mathbb{C}^2$ *other than the sphere* are compact Lie groups of dimension at most four.

2. Proof of Theorem 1.1

We shall denote by $S^1 \subset \mathbb{C}$ the set of unimodular complex numbers. Observe that

$$H^{\varepsilon,r}_{\alpha,s} = \left(H^{\varepsilon,r}_{0,0}\right) \circ \left(H^{1,1}_{\alpha,r}\right) \quad \forall\, (\varepsilon, r, \alpha, s) \in S^1 \times (\mathbb{R} \setminus \{0\}) \times \mathbb{C} \times \mathbb{R},$$

so to prove that $H^{\varepsilon,r}_{\alpha,s}$ is an automorphism of $(M, 0)$, it suffices to show that the mappings $H^{\varepsilon,r}_{0,0}$ and $H^{1,1}_{\alpha,s}$ are local automorphisms of $(M, 0)$. It is obvious that the mappings $H^{\varepsilon,r}_{0,0}$ are *global* automorphisms of M for each unimodular complex number ε and nonzero real number r; we leave it to the diligent reader to show that $H^{1,1}_{\alpha,r}$ is a local automorphism of $(M, 0)$ for each complex number α and real number s.

Thus, it follows that $H^{\varepsilon,r}_{\alpha,s} \in \operatorname{Aut}(M, 0)$ for every $(\varepsilon, r, \alpha, s) \in S^1 \times (\mathbb{R} \setminus \{0\}) \times \mathbb{C} \times \mathbb{R}$. To complete the proof, we must prove that if $H \in \widehat{\operatorname{Aut}}(M, 0)$ is a formal automorphism, then $H = H^{\varepsilon,r}_{\alpha,s}$ for some choice of parameters $(\varepsilon, r, \alpha, s)$. To prove this, we introduce some new notation. Writing $\operatorname{Im} w = (w - \overline{w})/(2i)$ and $\operatorname{Re} w = (w + \overline{w})/2$ in the local defining equation (1.1) for M and solving for w yields the identity

$$M = \left\{ (z, w) \;\middle|\; w = \overline{w}\, S(|z|^2) \right\},$$

where S is the real-analytic, complex-valued function defined by

$$\mathbb{R} \supset (-1, 1) \in t \mapsto S(t) := i\, t + \sqrt{1 - t^2} \in \mathbb{C}.$$

Recall that $H \in \widehat{\operatorname{Aut}}(M, 0)$ means that $H = (H^1, H^2)$ is a $\mathbb{C}^2$-valued formal power series which vanishes at 0, has nonvanishing Jacobian at 0, and satisfies the identity

$$H_2\big(z, \tau\, S(z\chi)\big) \equiv \overline{H_2}(\chi, \tau)\, S\big(H_1\big(z, \tau\, S(z\chi)\big)\, \overline{H_1}(\chi, \tau)\big), \tag{2.1}$$

where $\overline{H_j}$ denotes the power series obtained by replacing the Taylor coefficients of H_j by their complex conjugates. Observe that if we set $\chi = \tau = 0$ in (2.1), we obtain

$$H_2(z, 0) = \overline{H_2}(0, 0)\, S\big(H_2(z, 0)\, \overline{H_2}(0, 0)\big) = 0,$$

since $H(0, 0) = 0$. Hence, we can write

$$H(z, w) = \big(f(z, w), w\, g(z, w)\big) \tag{2.2}$$

with $f(0, 0) = 0$ and $f_z(0, 0) \cdot g(0, 0) \neq 0$. Substituting this into (2.1) and cancelling a common τ from both sides yields the identity

$$S(z\chi) g\big(z, \tau\, S(z\chi)\big) \equiv \overline{g}(\chi, \tau)\, S\big(f\big(z, \tau\, S(z\chi)\big)\, \overline{f}(\chi, \tau)\big). \tag{2.3}$$

Finally, for convenience, we shall formally expand the power series f and g as

$$f(z, w) = \sum_{n=0}^{\infty} \frac{f_n(z)}{n!} w^n, \quad g(z, w) = \sum_{n=0}^{\infty} \frac{g_n(z)}{n!} w^n, \tag{2.4}$$

and shall write

$$a^j_n := \overline{f^{(j)}_n(0)}, \quad b^j_n := \overline{g^{(j)}_n(0)}, \quad n, j \geq 0. \tag{2.5}$$

We now state the main lemma which will complete the proof of Theorem 1.1.

Lemma 2.1. *Let M be the hypersurface defined in Theorem 1.1. Suppose that $H \in \widehat{\mathrm{Aut}}(M,0)$, and write H as in equation (2.2). Then for every $n \geq 0$, there exists a $\mathbb{C}^2$-valued polynomial R_n in eight indeterminates, depending only on M and not the formal map H, such that*

$$\big(f_n(z), g_n(z)\big) = R_n\left(z, \frac{1}{a_0^1}, \frac{1}{b_0^0}, a_0^1, b_0^0, a_1^0, \overline{a_1^0}, \operatorname{Re} b_2^0\right).$$

Moreover, we have $a_0^1 \in S^1$ and $b_0^0 \in \mathbb{R} \setminus \{0\}$.

To see that Lemma 2.1 completes the proof of Theorem 1.1, fix a formal automorphism $H \in \widehat{\mathrm{Aut}}(M,0)$. Lemma 2.1 implies H is *uniquely determined* by its values a_0^1, b_0^0, a_1^0 and $\operatorname{Re} b_2^0$. Define

$$\varepsilon := \overline{a_0^1} \in S^1, \quad r := \overline{b_0^0} \in \mathbb{R} \setminus \{0\}, \quad \alpha := \frac{(\overline{a_1^0})}{(\overline{a_0^1})} \in \mathbb{C}, \quad s := \frac{\operatorname{Re}(\overline{b_2^0})}{(\overline{b_0^0})} \in \mathbb{R},$$

and define $\tilde{H} := H_{\alpha,s}^{\varepsilon,r}$. Define the corresponding (complex conjugated) derivatives $\tilde{a}_n^j$ and $\tilde{b}_n^j$ for $\tilde{H}$ as in equations (2.2), (2.4), and (2.5). It follows from a simple calculation that $a_0^1 = \tilde{a}_0^1$, $b_0^0 = \tilde{b}_0^0$, $a_1^0 = \tilde{a}_0^1$, and $\operatorname{Re} b_2^0 = \operatorname{Re} \tilde{b}_2^0$, whence $H = \tilde{H}$ by uniqueness, and the proof of the theorem is complete. Hence, we need only prove the lemma.

Proof of Lemma 2.1. We proceed by induction. For convenience, we shall set

$$\lambda_0 := \left(\frac{1}{a_0^1}, \frac{1}{b_0^0}, a_0^1, b_0^0, a_1^0, \overline{a_1^0}, \operatorname{Re} b_2^0\right) \in \mathbb{C}^7.$$

For any formal power series H of the form (2.2), define

$$\Phi^H(z,\chi,\tau) := -S(z\chi) g\big(z, \tau\, S(z\chi)\big) + \overline{g}(\chi,\tau)\, S\big(f\big(z, \tau\, S(z\chi)\big)\, \overline{f}(\chi,\tau)\big).$$

By (2.3), it follows that an invertible power series H is a formal automorphism of $(M,0)$ if and only if $\Phi^H \equiv 0$. The basic algorithm of the proof is as follows: given a formal automorphism H, at the n-th step of the induction, we

- Calculate $\Phi^H_{\tau^n}(z,\chi,0)$.
- Solve $\Phi^H_{\tau^n}(z,0,0) = 0$ to obtain an explicit formula for $g_n(z)$ as a polynomial (independent of the mapping H) in $(z, a_n^0, a_n^1, b_n^0, b_n^1, \lambda_0) \in \mathbb{C}^{12}$.
- Solve $\Phi^H_{\chi\,\tau^n}(z,0,0) = 0$ to obtain an explicit formula for $f_n(z)$, similarly expressed.
- Substitute these formulas (and their complex conjugates) into the identity $\Phi^H_{\tau^n}(z,\chi,0) = 0$ and differentiate this repeatedly in z and χ to express $(a_n^0, a_n^1, b_n^0, b_n^1)$ as a polynomial in λ_0.

In the algorithm above, we have used the usual subscript notation to denote partial derivatives, i.e.

$$\Phi^H_{z^j\chi^k\tau^\ell}(z,\chi,\tau) := \frac{\partial^{j+k+\ell}\Phi^H}{\partial z^j \partial\chi^k \partial\tau^\ell}(z,\chi,\tau).$$

We now fill in the details. Fix an automorphism H.

The case $n = 0$. Setting $\Phi^H(z,0,0) = 0$, we obtain $g_0(z) = \overline{g_0}(0) = b_0^0$, from which it follows that b_0^0 is real and, since H is invertible, nonzero. Thus, we have

$$g_0(z) = \overline{g_0}(\chi) = b_0^0 \in \mathbb{R} \setminus \{0\}. \tag{2.6}$$

Setting $\Phi^H_\chi(z,0,0)=0$ and using (2.6), we find $f_0(z)=z/a^1_0$. From this, it follows that $\overline{a^1_0}=1/a^1_0$, so a^1_0 is necessarily unimodular. Thus, we have

$$f_0(z)=\frac{z}{a^1_0},\quad \overline{f_0}(\chi)=a^1_0\,\chi,\quad a^1_0\in S^1, \tag{2.7}$$

which completes the base step of the induction.

The case $n=1$. Using the identity $\Phi^H_\tau(z,\chi,0)=0$ as indicated above and substituting in the formulas (2.6) and (2.7) as needed, we find

$$f_1(z)=\frac{i\,a^0_1}{(a^1_0)^2}\,z^2+\left(\frac{b^0_1}{a^1_0\,b^0_0}-\frac{a^1_1}{(a^1_0)^2}\right)z+\frac{i\,b^1_1}{a^1_0\,b^0_0},\quad g_1(z)=\frac{i\,b^0_0\,a^0_1}{a^1_0}\,z+b^0_1.$$

Conjugating these, we obtain

$$\overline{f_1}(\chi)=\frac{a^1_0\,b^1_1}{b^0_0}\,\chi^2+a^1_1\,\chi+a^0_1,\quad \overline{g_1}(\chi)=b^1_1\,\chi+b^0_1.$$

Using these formulas, it follows that

$$\begin{pmatrix}0\\0\end{pmatrix}=\begin{pmatrix}\Phi^H_{z^2\chi^2\tau}(0,0,0)\\ \Phi^H_{z^3\chi^3\tau}(0,0,0)\end{pmatrix}=\begin{pmatrix}4\frac{b^0_0}{a^1_0} & -2\\ 54i\frac{b^0_0}{a^1_0} & -18i\end{pmatrix}\begin{pmatrix}a^1_1\\ b^0_1\end{pmatrix}. \tag{2.8}$$

Since the 2×2 matrix on the right-hand side of equation (2.8) is invertible, it follows from equation (2.8) that $a^1_1=b^0_1=0$. Moreover, equating $\overline{a^0_1}=f_1(0)$ yields $b^1_1=-i\,a^1_0\,b^0_0\,\overline{a^0_1}$. Hence, we have

$$f_1(z)=\frac{i\,a^0_1}{(a^1_0)^2}\,z^2+\overline{a^0_1},\quad g_1(z)=\frac{i\,b^0_0\,a^0_1}{a^1_0}\,z,$$

which completes the induction at this step.

The case $n=2$. Using the identity $\Phi^H_{\tau^2}(z,\chi,0)=0$ as above, we find

$$\begin{aligned}f_2(z)&=-\frac{3(a^0_1)^2}{(a^1_0)^3}\,z^3+\frac{2i\,a^0_2}{(a^1_0)^2}\,z^2+\left(\frac{2i\,a^0_1\,\overline{a^0_1}}{a^1_0}+\frac{2b^0_2}{a^1_0\,b^0_0}-\frac{a^1_2}{(a^1_0)^2}\right)z+\frac{i\,b^1_2}{a^1_0\,b^0_0},\\ g_2(z)&=-\frac{3b^0_0\,(a^0_1)^2}{(a^1_0)^2}\,z^2+\frac{i\,b^0_0\,a^0_2}{a^1_0}\,z+b^0_2+2i\,b^0_0\,a^0_1\,\overline{a^0_1}.\end{aligned}$$

Conjugating as above and substituting into $\Phi^H_{\tau^2}(z,\chi,0)=0$, the relations

$$\Phi^H_{z^2\chi^2\tau^2}(0,0,0)=0,\quad \Phi^H_{z^3\chi^2\tau^2}(0,0,0)=0,\quad \Phi^H_{z^2\chi^3\tau^2}(0,0,0)=0$$

yield $a^0_2=b^1_2=0$ and $a^1_2=(a^1_0\,b^0_2)/b^0_0-2i\,a^1_0\,a^0_1\,\overline{a^0_1}$. Similarly, equating $\overline{b^0_2}=g_2(0)$, we find

$$\operatorname{Im} b^0_2=\frac{b^0_2-\overline{b^0_2}}{2i}=-\frac{i\,a^0_1\,b^1_1}{a^1_0}=-b^0_0\,a^0_1\,\overline{a^0_1}.$$

Under these substitutions, we have

$$\begin{aligned}f_2(z)&=-\frac{3(a^0_1)^2}{(a^1_0)^2}\,z^3+\left(\frac{\operatorname{Re} b^0_2}{a^1_0\,b^0_0}+\frac{3i\,a^0_1\,\overline{a^0_1}}{a^1_0}\right)z,\\ g_2(z)&=-\frac{3b^0_0(a^0_1)^2}{(a^1_0)^2}\,z^2+\operatorname{Re} b^0_2+i\,b^0_0\,a^0_1\,\overline{a^0_1},\end{aligned}$$

which completes the induction at this step.

The general inductive step. Assume now that the lemma holds up to some $n-1 \geq 2$; we prove it for n. The Chain Rule implies

$$\begin{aligned}(2.9)\quad \Phi^H_{\tau^n}(z,\chi,0) &= -S(z\chi)^{n+1} g_n(z) + S\big(z\chi\big)\overline{g_n}(\chi)\\ &+ b_0^0\, S'\big(z\chi\big)\left(a_0^1\,\chi\, S(z\chi)^n f_n(z) + \frac{z}{a_0^1}\,\overline{f_n}(\chi)\right)\\ &+ P^n\left(\big(S^{(j)}(z\chi)\big)_{j=0}^{n}, \big(f_j(z), g_j(z), \overline{f_j}(\chi), \overline{g_j}(\chi)\big)_{j=0}^{n-1}\right),\end{aligned}$$

where P_n is a complex-valued polynomial (in $5n+1$ indeterminates) which is independent of the mapping H. By the inductive hypothesis (and its conjugation), we may rewrite the last term in equation (2.9) as

$$Q^n\left(z, \chi, \lambda_0, \big(S^{(j)}(z\chi)\big)_{j=0}^{n}\right),$$

where Q^n is complex polynomial in $n+10$ indeterminates, independent of H. Proceeding as above, we find

$$f_n(z) = \frac{i\,n\,a_n^0}{(a_0^1)^2}\,z^2 + \left(\frac{n\,b_n^0}{a_0^1\,b_0^0} - \frac{a_n^1}{(a_0^1)^2}\right)z + \frac{i\,b_n^1}{a_0^1\,b_0^0} + p_n(z,\lambda_0), \tag{2.10}$$

$$g_n(z) = \frac{i\,b_0^0\,a_n^0}{a_0^1}z + b_n^0 + q^n(z,\lambda_0), \tag{2.11}$$

where p_n, q_n are complex polynomials in 8 indeterminates, independent of H. Substituting these and their conjugates into the identity $\Phi^H_{\tau^n}(z,\chi,0) = 0$ and then computing $\Phi^H_{z^j\chi^k\tau^n}(0,0,0)$ for $j,k = 2,3$ yields a 4×4 system of equations of the form

$$A_n \cdot \big(a_n^0, a_n^1, b_n^0, b_n^1\big)^t = B_n(\lambda_0), \tag{2.12}$$

where B_n is a $\mathbb{C}^4$-valued polynomial in λ_0, and A_n is the 4×4 matrix given by

$$\begin{pmatrix} 0 & 4n\frac{b_0^0}{a_0^1} & -2n^2 & 0\\ -6i(n^2-1)\frac{b_0^0}{a_0^1} & 0 & 0 & 0\\ 0 & 0 & 0 & 6(n^2-1)\\ 0 & 18i(n^2+2n)\frac{b_0^0}{a_0^1} & -6i(2n^3+3n^2-2n) & 0 \end{pmatrix}.$$

Observe that

$$\det(A_n) = -\frac{432(b_0^0)^2}{(a_0^1)^2}(n-2)(n-1)^2 n^2 (n+1)^2 (n+2),$$

which is nonzero for $n \geq 3$, whence A_n is invertible. By Cramer's Rule, it follows that A_n^{-1} is a 4×4 matrix whose entries are polynomial in (a_0^1, b_0^0) and their reciprocals (and so in particular are polynomial in λ_0). Thus, equation (2.12) implies that $\big(a_n^0, a_n^1, b_n^0, b_n^1\big)$ is a polynomial in λ_0. Substituting this into equations (2.10) and (2.11) completes the induction. □

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