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CODIMENSION OF POLYNOMIAL SUBSPACE IN $L_2(\mathbb{R}, d\mu)$ FOR DISCRETE INDETERMINATE MEASURE μ

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ABSTRACT. A calculation formula is established for the codimension of the polynomial subspace in $L_2(\mathbb{R}, d\mu)$ with discrete indeterminate measure μ. We clarify how much the masspoint of the n-canonical solution of an indeterminate Hamburger moment problem differs from the masspoint of the corresponding N-extremal solution at a given point of the real axis.

1. INTRODUCTION AND MAIN RESULT

Let $\mathcal{M}^*(\mathbb{R})$ be the set of positive Borel measures on $\mathbb{R}$ having moments of every order and infinite support,

$$\mathcal{N} := \{f \in \mathrm{Hol}\,(\mathbb{C} \setminus \mathbb{R}) \ \mid \ \mathrm{Im} f(z) \,/\, \mathrm{Im} z \ > \ 0 \quad \forall z \in \mathbb{C} \setminus \mathbb{R}\};$$

$\mathfrak{P} := \mathcal{N} \cup \mathbb{R}$, $\mathfrak{P}^* := \mathfrak{P} \cup \{\infty\}$ and $\mathbb{R}^* := \mathbb{R} \cup \{\infty\}$. We write

$$\mathcal{N}_2 := \left\{ \begin{pmatrix} a(z) & c(z) \\ b(z) & d(z) \end{pmatrix} \mid \ a, d, b, c \in \mathcal{E}\ ; \quad a(z)d(z) - b(z)c(z) \equiv 1\ ; \right.$$

$$\left. \frac{a(z)t + c(z)}{b(z)t + d(z)} \in \mathcal{N} \quad \forall\, t \in \mathbb{R}^* \right\}$$

for the set of all Nevanlinna matrices, where $\mathcal{E}$ denotes the set of all entire functions real-valued on the real axis.

A measure $\mu \in \mathcal{M}^*(\mathbb{R})$ is said to be indeterminate if the set V_μ of all measures $\nu \in \mathcal{M}^*(\mathbb{R})$ such that

$$\int_{\mathbb{R}} x^n \ d\mu(x) \ = \ \int_{\mathbb{R}} x^n \ d\nu(x) \ \ \forall\, n \geq 0,$$

contains at least one measure not coincident with μ. In that case the moment problem generated by $\mu \in \mathcal{M}^*(\mathbb{R})$ (or, more precisely, generated by moments of μ) is called an indeterminate Hamburger moment problem, and all measures from V_μ are referred to as its solutions (see [1, II, §1]). If $\{P_n\}_{n\geq 0}$ and $\{Q_n\}_{n\geq 0}$ (see, for example, [1]) denote, corresponding to this indeterminate moment problem,

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sequences of polynomials of the first and of the second kind, respectively, then by the Nevanlinna theorem, one can construct, using the formulas

$$A(z)= z \sum_{k=0}^{\infty} Q_k(0)Q_k(z), \qquad C(z)= 1 + z \sum_{k=0}^{\infty} P_k(0)Q_k(z),$$
$$B(z)= -1 + z \sum_{k=0}^{\infty} Q_k(0)P_k(z), \qquad D(z)= z \sum_{k=0}^{\infty} P_k(0)P_k(z),$$

the Nevanlinna matrix $\begin{pmatrix} -A(z) & C(z) \\ B(z) & -D(z) \end{pmatrix} \in \mathcal{N}_2$ such that the known Nevanlinna formula

$$\int_{\mathbb{R}} \frac{d\nu_\varphi(t)}{t-z} \;=\; -\;\frac{A(z)\varphi(z) - C(z)}{B(z)\varphi(z) - D(z)} \quad \forall\; z \in \mathbb{C} \setminus \mathbb{R} \tag{1.1}$$

establishes the homeomorphism $\mathfrak{P}^* \ni \varphi \to \nu_\varphi \in V(\mu)$ of $\mathfrak{P}^*$ onto $V(\mu)$.

The special solutions in (1.1) corresponding to $\varphi \in \mathfrak{P}^*$ being a real constant or ∞ are called N-*extremal*. All of them are discrete measures. It is known that for each $x \in \mathbb{R}$

$$\max_{\nu \in V(\mu)} \nu(\{x\}) \;=\; \rho(x) \;:=\; \left(\sum_{n=0}^{\infty} P_n(x)^2 \right)^{-1}, \tag{1.2}$$

and this maximum is attained on only one N-extremal measure ν, depending on $x \in \mathbb{R}$ (see [1, Th.3.4.1]). More precisely, every N-extremal measure at any growth point x has a maximal mass $\rho(x)$ in the sense of (1.2), and the function $\rho(x)$ defined in (1.2) is called a *maximal weight function* of the moment problem generated by the measure μ. It is also known that the following equality holds (see, for example, [6, (2.3)]):

$$B'(x)D(x) \;-\; D'(x)B(x) \;=\; \frac{1}{\rho(x)} \quad \forall\; x \in \; \mathbb{R}. \tag{1.3}$$

N-extremal measures were characterized by M. Riesz in 1923. Denote by $\mathcal{P}[\mathbb{C}]$ the set of all algebraic polynomials with arbitrary complex coefficients.

Riesz's Theorem ([6])**.** *Let* $\mu \in \mathcal{M}^*(\mathbb{R})$.

1. *If* μ *is an indeterminate measure and* $\nu \in V_\mu$ *, then* $\mathcal{P}[\mathbb{C}]$ *is dense in* $L_2(\mathbb{R}, d\nu)$ *if and only if* ν *is an* N*-extremal measure.*

2. *If* μ *is a determinate measure (i.e.,* $V_\mu = \{\mu\}$*), then* $\mathcal{P}[\mathbb{C}]$ *is dense in* $L_2(\mathbb{R}, d\mu)$.

If in (1.1) $\varphi \in \mathfrak{P}^*$ is a rational function of degree n, i.e., $\varphi = \frac{p}{q}$, where p and q are polynomials without common zeros and the maximum of the degrees of p and q is equal to n, then ν_φ is called the n-*canonical measure*, and, according to (1.1),

$$\int_{\mathbb{R}} \frac{d\nu_\varphi(t)}{t-z} \;=\; -\;\frac{A(z)p(z) - C(z)q(z)}{B(z)p(z) - D(z)q(z)} \quad \forall\; z \in \mathbb{C} \setminus \mathbb{R},\; \varphi = \frac{p}{q}. \tag{1.4}$$

That is why any n-canonical measure is also discrete with some masspoints at zeros of $B(z)p(z) - D(z)q(z)$, i.e.,

$$d\nu_\varphi(x) \;=\; \sum_{\lambda \in \Lambda_{Bp-Dq}} \nu_\lambda^\varphi \cdot \delta_\lambda(x),$$

where Λ_f denotes the set of all zeros of some entire function f, δ_λ is Dirac's measure at the point λ, and the masses ν_λ^φ are given by the corresponding residues. It is clear that if $n \geq 1$, then, according to (1.2),

$$0 \;<\; \nu_\lambda^\varphi \;<\; \rho(\lambda) \;\; \forall\, \lambda \in \Lambda_{Bp-Dq}. \tag{1.5}$$

The 0-canonical solutions and $\nu_\infty$are the same as the N-extremal measures.

It is well-known that $\nu \in V_\mu$ is n-canonical if and only if the measure

$$(1+x^2)^{-n} d\nu(x)$$

is N-extremal (see [1, Th.3.4.3]). Another characterization of n-canonical measures is given in the following result (1984) of Cassier, which generalizes Riesz's theorem.

Cassier's Theorem ([3], [2])**.** *Let $\mu \in \mathcal{M}^*(\mathbb{R})$ be an indeterminate measure. The measure μ is n-canonical if and only if the closure of the algebraic polynomials $\mathcal{P}[\mathbb{C}]$ in the space $L_2(\mathbb{R}, d\mu)$ is of codimension n.*

In this paper we partially answer the natural question as how much ν_λ^φ (from (1.5)) is less then $\rho(\lambda)$. Besides that, we also calculate the codimension of the closure of $\mathcal{P}[\mathbb{C}]$ in the space $L_2(\mathbb{R}, d\mu)$ for any indeterminate discrete measure μ.

Theorem 1. *Let*

$$d\mu(x) \;=\; \sum_{k\geq 1} \mu_k \cdot \delta_{\lambda_k}(x)$$

be any discrete indeterminate measure from the class $\mathcal{M}^*(\mathbb{R})$. Then the following statements hold.

(A) *If ρ is a maximal weight function of the indeterminate moment problem generated by μ, then*

$$\sum_{k\geq 1} \left(1 - \frac{\mu_k}{\rho(\lambda_k)}\right) \;=\; \operatorname{codim}_{L_2(\mathbb{R},d\mu)} \mathcal{P}[\mathbb{C}], \tag{1.6}$$

where $\operatorname{codim}_{L_2(\mathbb{R},d\mu)} \mathcal{P}[\mathbb{C}] \;\in \{0, 1, 2, \dots\} \cup \{+\infty\}$ *denotes the codimension of the closure of the algebraic polynomials $\mathcal{P}[\mathbb{C}]$ in the space $L_2(\mathbb{R}, d\mu)$.*

(B) *If μ is an n-canonical measure for some nonnegative integer n, then there exist numbers $\theta_k \in [0, 1)$, $k \geq 1$, such that*

$$\begin{cases} \mu_k \;=\; (1-\theta_k)\,\rho(\lambda_k) \quad \forall k \geq 1; \\ \sum\limits_{k\geq 1} \theta_k \;=\; n. \end{cases}$$

2. Auxiliary lemma

It has been proved in [1, III, 1.1.] that $f(z) \in \mathcal{N}$ and

$$\sup_{|y|\geq 1} |y f(iy)| \;<\; \infty \tag{2.1}$$

if and only if there exists a nondecreasing function $\sigma(x)$ of bounded variation on the whole real axis such that

$$f(z) \;=\; \int_{\mathbb{R}} \frac{d\sigma(u)}{u-z} \quad \forall\, z \in \mathbb{C} \setminus \mathbb{R}.$$

In Lemma 1 below we establish a useful corollary of this statement.

Let $\varphi \in \mathcal{N}$ be a meromorphic function with the set of all its zeros $\{b_k\}_{k\geq 1} \subset \mathbb{R}$. Denote by $\{a_k\}_{k\geq 1} \subset \mathbb{R}$ all its nonzero poles. Then by a known theorem (see [5, VII, Th.2]), there exist nonnegative numbers $A_*(\varphi)$, $A_{-1}(\varphi), A_k(\varphi), k \geq 1$, and a real number $A_0(\varphi)$ such that for all $z \in \mathbb{C} \setminus \mathbb{R}$

$$\varphi(z) = A_{-1}(\varphi)z + A_0(\varphi) - \frac{A_*(\varphi)}{z} + \sum_{k\geq 1} A_k(\varphi)\left(\frac{1}{a_k - z} - \frac{1}{a_k}\right), \tag{2.2}$$

where

$$\sum_{k\geq 1} \frac{A_k(\varphi)}{1 + a_k^2} < \infty. \tag{2.3}$$

Lemma 1. *Let $\varphi \in \mathcal{N}$ be a meromorphic function with zeros $\{b_k\}_{k\geq 1} \subset \mathbb{R}$, and assume the coefficient $A_{-1}(\varphi)$ in its representation (2.2) is positive. Then for all $z \in \mathbb{C} \setminus \mathbb{R}$ corresponding to (2.2), the representation of the function $-\frac{1}{\varphi} \in \mathcal{N}$ has the following specific form:*

$$-\frac{1}{\varphi(z)} = \sum_{k\geq 1} \frac{A_k(-\frac{1}{\varphi})}{b_k - z}, \quad \sum_{k\geq 1} A_k\left(-\frac{1}{\varphi}\right) < \infty, \tag{2.4}$$

where $A_k(-\frac{1}{\varphi}) \geq 0 \ \ \forall k \geq 1$.

Proof. It is easy to verify that (2.3) implies

$$\sum_{k\geq 1} A_k(\varphi)\left(\frac{1}{a_k - z} - \frac{1}{a_k}\right) = \sum_{k\geq 1} A_k(\varphi)\frac{z}{a_k(a_k - z)} = \overline{\overline{o}}\,(y), \tag{2.5}$$

where $z = iy$ and $|y| \to \infty$. Since $A_{-1}(\varphi) > 0$, then (2.2) and (2.5) yield

$$\varphi(iy) = A_{-1}(\varphi)iy + \overline{\overline{o}}\,(|y|), \quad |y| \to \infty,$$

and hence as $|y| \to \infty$ we have

$$-\frac{1}{\varphi(iy)} = -\frac{1}{A_{-1}(\varphi)iy + \overline{\overline{o}}\,(|y|)} = \frac{i}{yA_{-1}(\varphi)}\left(1 + \overline{\overline{o}}\,(1)\right). \tag{2.6}$$

The asymptotic representation (2.6) indicates that the function $-1/\varphi \in \mathcal{N}$ satisfies condition (2.1): $\sup_{|y|\geq 1} |y|\,|-1/\varphi(iy)| < \infty$. Applying the fact from [1, III, 1.1] mentioned at the beginning of this section, we obtain the existence of nonnegative numbers $A_k(-\frac{1}{\varphi})$, $k \geq 1$, such that for all $z \in \mathbb{C} \setminus \mathbb{R}$ the relations (2.4) are true. Lemma 1 is proved. □

3. Proof of Theorem 1

3.1. Let us consider some positive integer $n \geq 1$ and any n-canonical measure $\mu \in \mathcal{M}^*(\mathbb{R})$. According to (1.4), for the indeterminate moment problem generated by μ, we have two polynomials p, q, $\max\{\deg p, \deg q\} = n$, and two entire functions $U(z), V(z)$ such that

$$\int_{\mathbb{R}} \frac{d\mu(t)}{t - z} = -\frac{A(z)p(z) - C(z)q(z)}{B(z)p(z) - D(z)q(z)} =: \frac{U(z)}{V(z)} \quad \forall\, z \in \mathbb{C} \setminus \mathbb{R}. \tag{3.1}$$

But then, for some $\alpha \in [0, 2\pi]$,

$$\begin{pmatrix} U(z) \\ V(z) \end{pmatrix} = \begin{pmatrix} a(z) & c(z) \\ b(z) & d(z) \end{pmatrix} \begin{pmatrix} g(z) \\ h(z) \end{pmatrix}, \tag{3.2}$$

where

$$(3.3)\qquad \begin{aligned} \begin{pmatrix} a(z) & c(z) \\ b(z) & d(z) \end{pmatrix} &= \begin{pmatrix} -A(z) & C(z) \\ B(z) & -D(z) \end{pmatrix} \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}, \\ \begin{pmatrix} g(z) \\ h(z) \end{pmatrix} &= \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} p(z) \\ q(z) \end{pmatrix}. \end{aligned}$$

It follows from known properties of the class $\mathcal{N}$ and Nevanlinna matrices (see [5, VII, Th.2], p. 412, (1); p. 414, Theorem) that

$$(3.4)\qquad \begin{pmatrix} a & c \\ b & d \end{pmatrix} \in \mathcal{N}_2, \quad \frac{g}{h} \in \mathcal{N}\,.$$

Moreover, one can easily derive that for almost all $\alpha \in [0, 2\pi]$, the following relations hold:

$$(3.5)\qquad \begin{cases} \deg h \;=\; \deg\left(p(z)\sin\alpha + q(z)\cos\alpha\right) \;=\; \max\{\deg p,\ \deg q\} \;=\; n; \\ 0 \notin \Lambda_\varphi\,,\ \Lambda_{\varphi_1} \cap \Lambda_{\varphi_2} \;=\; \emptyset \ \ \forall \varphi, \varphi_1 \;\neq\; \varphi_2 \in \{U, V, a, b, c, d, g, h\}\,. \end{cases}$$

Equalities (1.3) and (3.1) can be rewritten as follows:

$$(3.6)\qquad d'(x)b(x) \;-\; b'(x)d(x) \;=\; \frac{1}{\rho(x)} \quad \forall\, x \in\ \mathbb{R},$$

$$(3.7)\qquad \frac{U(z)}{V(z)} \;=\; \int_{\mathbb{R}} \frac{d\mu(t)}{t - z} \;=\; \frac{a(z)g(z) + c(z)h(z)}{b(z)g(z) + d(z)h(z)} \quad \forall\, z \in \mathbb{C} \setminus \mathbb{R}.$$

Everywhere below we assume that the number $\alpha \in [0, 2\pi]$, introduced in (3.3), satisfies (3.5).

3.2. It is easy to see from (1.2) that, for every $a \in \mathbb{R}$, the maximal weight function corresponding to the shifted measure $d\mu_a(x) := d\mu(x - a)$ equals $\rho(x - a)$ and that $\operatorname{codim}_{L_2(\mathbb{R}, d\mu_a)} \mathcal{P}[\mathbb{C}] = \operatorname{codim}_{L_2(\mathbb{R}, d\mu)} \mathcal{P}[\mathbb{C}]$. That is why, without loss of generality, we assume that $0 \notin \{\lambda_k\}_{k \geq 1}$ and $V(0) \;=\; 1$. According to (3.7), $U(z)\,/\,V(z) = \sum_{k \geq 1} \mu_k \,/\, (\lambda_k - z)$, and therefore

$$(3.8)\qquad U(\lambda_k) \;=\; -\ \mu_k\ V'(\lambda_k) \quad \forall\ k \geq\ 1.$$

Besides that, it follows from (3.2) and (3.4) that

$$\begin{cases} g(z) = & U(z)d(z) - V(z)c(z), \\ h(z) = & V(z)a(z) - U(z)b(z), \end{cases}$$

from which one can easily obtain

$$(3.9)\qquad \begin{cases} g(\lambda_k) = U(\lambda_k)d(\lambda_k) \overset{(3.8)}{=} -\ \mu_k d(\lambda_k)V'(\lambda_k), \\ h(\lambda_k) = -U(\lambda_k)b(\lambda_k) \overset{(3.8)}{=} \ \mu_k b(\lambda_k)V'(\lambda_k). \end{cases}$$

But the inclusions (3.4), together with known properties of Nevanlinna matrices (see [7]), mean that

$$\frac{V}{bh} = \frac{bg + dh}{bh} = \frac{g}{h} + \frac{d}{b} \in \mathcal{N}\,,$$

and so,

$$-\frac{bh}{V} \;=\; -\ \frac{1}{\dfrac{g}{h} + \dfrac{d}{b}} \;\in\; \mathcal{N}\,.$$

Therefore, by a well-known theorem [5, VII, Th.2], there exist $V_0 \in \mathbb{R}$ and $V_{-1}, V_k \geq 0 \;\forall\; k \geq 1$ such that

$$-\frac{h(z)b(z)}{V(z)} \;=\; V_{-1}z \;+\; V_0 \;+\; \sum_{k\geq 1} \frac{z}{\lambda_k(\lambda_k - z)} V_k \quad \forall\; z \in \mathbb{C} \setminus \mathbb{R}. \tag{3.10}$$

3.3. We prove now that in the expansion (3.10), $V_{-1} = 0$. Assume the contrary: $A_{-1}\left(-\frac{bh}{V}\right) \equiv V_{-1} \;>\; 0$ (here and everywhere below we use the notation from (2.2) for all functions $\varphi \in \mathcal{N}$). Then (2.4) gives

$$\frac{V}{bh} \;=\; \frac{g}{h} \;+\; \frac{d}{b} \;=\; \sum_{k\geq 1} \frac{A_k(\frac{V}{bh})}{\beta_k - z}, \quad \sum_{k\geq 1} A_k\left(\frac{V}{bh}\right) \;<\; \infty, \tag{3.11}$$

where $\{\beta_k\}_{k\geq 1}$ are all the zeros of $b(z)h(z)$. Denoting by $\{c_k\}_{k\geq 1}$ all the zeros of the entire function $b(z)$, we conclude by (3.11) that, if $\beta_k = c_m$ for some positive integers m and k, then

$$A_k\left(\frac{V}{bh}\right) \;=\; -\,\frac{d(c_m)}{b'(c_m)}$$

and, hence,

$$\sum_{k\geq 1} -\,\frac{d(c_k)}{b'(c_k)} \;<\; \infty.$$

But on the other hand, using equality (3.6) we have

$$\infty \;>\; \sum_{k\geq 1} -\,\frac{d(c_k)}{b'(c_k)} \;=\; \sum_{k\geq 1} \frac{-d(c_k)b'(c_k)}{b'(c_k)^2} \;=\; \sum_{k\geq 1} \frac{1}{\rho(c_k)b'(c_k)^2}. \tag{3.12}$$

Due to (3.4) and (3.7), the function $b(z)$ is an element of the Nevanlinna matrix corresponding to the indeterminate moment problem generated by the measure μ. That's why the measure $\sum_{k\geq 1} \rho(c_k)\delta_{c_k}(x)$ is N-extremal (see (1.1)). But now inequality (3.12) gives a contradiction with the second necessary Hamburger condition for N-extremal measures ([4, p. 516, (8.24)], [1, IV, Addenda and exercises, 2, Th. 1, (7)]). This contradiction proves the required equality $V_{-1} = 0$ in (3.10). Thus, for all $z \in \mathbb{C} \setminus \mathbb{R}$ we can rewrite (3.10) as follows:

$$-\frac{h(z)b(z)}{V(z)} \;=\; V_0 \;+\; \sum_{k\geq 1} \frac{z}{\lambda_k(\lambda_k - z)} V_k, \quad V_0 \in \mathbb{R}, \quad V_k \geq 0 \;\; \forall k \geq 1. \tag{3.13}$$

3.4. Differentiating equality (3.13), we get

$$-\frac{(h(z)b(z))'V(z) - h(z)b(z)V'(z)}{V(z)^2} \;=-\left(\frac{h(z)b(z)}{V(z)}\right)' \;=\; \sum_{k\geq 1} \frac{V_k}{(\lambda_k - z)^2}$$

and

$$\sum_{k\geq 1} V_k \left(\frac{V(z)}{\lambda_k - z}\right)^2 \;=\; h(z)b(z)V'(z) \;-\; h'(z)b(z)V(z) \;-\; h(z)b'(z)V(z)\,. \tag{3.14}$$

Denote by $\eta_1,\ \eta_2,\ \dots,\eta_n$ all zeros of the polynomial $h(z)$ and substitute $z=\eta_m$ in (3.14):

$$\sum_{k\geq 1} V_k \left(\frac{V(\eta_m)}{\lambda_k-\eta_m}\right)^2 \ = -\ h'(\eta_m)b(\eta_m)V(\eta_m) \quad \forall\ 1\leq m\leq n.$$

In addition to these equalities, the equality $V=bg+dh$ implies $V(\eta_m)=b(\eta_m)g(\eta_m)$, and, therefore,

$$-h'(\eta_m)b(\eta_m) \ = \ \sum_{k\geq 1} V_k \frac{V(\eta_m)}{(\lambda_k-\eta_m)^2} \ = \ \left(\sum_{k\geq 1}\frac{V_k}{(\lambda_k-\eta_m)^2}\right) b(\eta_m)g(\eta_m),$$

from which we get

$$1 \ = \ \sum_{k\geq 1} V_k \left(-\ \frac{g(\eta_m)}{h'(\eta_m)}\right)\frac{1}{(\lambda_k-\eta_m)^2} \quad \forall\ 1\leq m\leq n. \tag{3.15}$$

Under our condition on the number α, differentiation of the obvious equality

$$\frac{g(z)}{h(z)} \ = \ C_0 \ + \ \sum_{m=1}^{n}\frac{g(\eta_m)}{h'(\eta_m)}\frac{1}{(z-\eta_m)}$$

gives

$$\left(\frac{g(z)}{h(z)}\right)' \ = \ \sum_{m=1}^{n}\left(-\ \frac{g(\eta_m)}{h'(\eta_m)}\right)\frac{1}{(z-\eta_m)^2}. \tag{3.16}$$

Thus, summing (3.15) over all m and taking (3.16) into account, we have

$$n \ = \ \sum_{k\geq 1} V_k \left(\frac{g}{h}\right)'(\lambda_k). \tag{3.17}$$

To finish the proof of our theorem, it remains only to recount the terms in the right side of (3.17).

3.5. Equality (3.13) shows that

$$V_k \ = \ \frac{h(\lambda_k)b(\lambda_k)}{V'(\lambda_k)} \quad \forall k\geq 1,$$

which, together with (3.9), indicates that

$$V_k \ = \ \frac{h(\lambda_k)^2}{\mu_k V'(\lambda_k)^2} \quad \forall k\geq 1,$$

and therefore

$$V_k\left(\frac{g}{h}\right)'(\lambda_k) \ = \ \frac{1}{\mu_k V'(\lambda_k)^2}\left(g'(\lambda_k)h(\lambda_k) \ - \ g(\lambda_k)h'(\lambda_k)\right) \quad \forall k\geq 1. \tag{3.18}$$

For the sake of convenience, we denote

$$\left\|\frac{F}{G}\right\|(z) \ := \ F'(z)G(z) \ - \ F(z)G'(z)$$

for any two entire functions $F(z), G(z)$. That is why equalities (3.18) can be rewritten as follows:

$$V_k\left(\frac{g}{h}\right)'(\lambda_k) \ = \ \frac{\left\|\frac{g}{h}\right\|(\lambda_k)}{\mu_k V'(\lambda_k)^2} \quad \forall k\geq 1. \tag{3.19}$$

3.6. Now we will find an acceptable expression for $\left\| \frac{g}{h} \right\| (\lambda_k)$ from (3.19). Differentiating the equality

$$\frac{V}{bh} = \frac{g}{h} + \frac{d}{b},$$

we get

$$V'(z)b(z)h(z) - V(z)(bh)'(z) = b(z)^2 \left\| \frac{g}{h} \right\| (z) + h(z)^2 \left\| \frac{d}{b} \right\| (z).$$

Setting $z = \lambda_k$ here, we obtain

$$V'(\lambda_k)b(\lambda_k)h(\lambda_k) = b(\lambda_k)^2 \left\| \frac{g}{h} \right\| (\lambda_k) + h(\lambda_k)^2 \left\| \frac{d}{b} \right\| (\lambda_k). \tag{3.20}$$

Replacement of $h(\lambda_k)$ here by its expression from (3.9) gives

$$\mu_k V'(\lambda_k)^2 b(\lambda_k)^2 = b(\lambda_k)^2 \left\| \frac{g}{h} \right\| (\lambda_k) + \mu_k^2 V'(\lambda_k)^2 b(\lambda_k)^2 \left\| \frac{d}{b} \right\| (\lambda_k),$$

or

$$\mu_k V'(\lambda_k)^2 = \left\| \frac{g}{h} \right\| (\lambda_k) + \mu_k^2 V'(\lambda_k)^2 \left\| \frac{d}{b} \right\| (\lambda_k).$$

That is why

$$\left\| \frac{g}{h} \right\| (\lambda_k) = \mu_k V'(\lambda_k)^2 \left(1 - \mu_k \left\| \frac{d}{b} \right\| (\lambda_k) \right) \quad \forall k \geq 1. \tag{3.21}$$

Substituting (3.21) in (3.19) and taking into account the equality $\left\| \frac{d}{b} \right\| (\lambda_k) = \frac{1}{\rho(\lambda_k)}$ evoked by (3.6), we establish the desired relation (1.6) for any n-canonical measure μ with a positive integer n such that

$$n = \sum_{k \geq 1} \left(1 - \mu_k \left\| \frac{d}{b} \right\| (\lambda_k) \right) = \sum_{k \geq 1} \left(1 - \frac{\mu_k}{\rho(\lambda_k)} \right).$$

With the help of an integral representation of the functions from $\mathcal{N}$ (see [1, III, §1, (3)]) and (1.1), it is possible to approximate any non-canonical but discrete measure from V_μ by canonical measures with their orders n increasing to infinity, and, due to equality (1.6) established for canonical measures, to get for such a measure a convergence to infinity of the series in the left side of (1.6). Finally, statement (B) of the theorem represents a simple reformulation of (A) with the help of Cassier's theorem. For $n = 0$ statements (A) and (B) are evident, and this completes the proof of Theorem 1.

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